Problems: The Chain Rule with Constraints

Suppose $w = u^3 - uv^2$, $u = xy$ and $v = u + x$.

1. Find $\left(\dfrac{\partial w}{\partial u}\right)_x$ and $\left(\dfrac{\partial w}{\partial x}\right)_u$ using the chain rule.

Answer: In finding $\left(\frac{\partial w}{\partial u}\right)_x$ we assume v and y are functions of u and that x is a constant.

$$\begin{aligned}\left(\frac{\partial w}{\partial u}\right)_x &= (3u^2 - v^2) - u \cdot 2v \left(\frac{\partial v}{\partial u}\right)_x \\ &= 3u^2 - v^2 - 2uv.\end{aligned}$$

Similarly,

$$\begin{aligned}\left(\frac{\partial w}{\partial x}\right)_u &= 0 - u \cdot 2v \left(\frac{\partial v}{\partial x}\right)_u \\ &= -2uv.\end{aligned}$$

2. Find $\left(\dfrac{\partial w}{\partial u}\right)_x$ and $\left(\dfrac{\partial w}{\partial x}\right)_u$ using differentials.

Answer: We can compute:

$$dw = (3u^2 - v^2)du - 2uv\,dv; \qquad du = x\,dy + y\,dx; \qquad dv = du + dx.$$

We're interested in the independent variables u and x so we substitute $dv = du + dx$ to get:

$$dw = (3u^2 - v^2)du - 2uv(du + dx) = (3u^2 - v^2 - 2uv)du - 2uv\,dx.$$

Using the fact that $dw = \left(\dfrac{\partial w}{\partial u}\right)_x du + \left(\dfrac{\partial w}{\partial x}\right)_u dx$, we get the expected answer.

Note that we did not need the variable y or the equation $u = xy$ in these calculations!