18.02 Supplementary Notes on Lagrange Multipliers

RING ON A STRING.[1] Consider a ring on a string held fixed at two ends at $(0,0)$ and (a,b). The ring is free to slide to any point. Find the position (x,y) of the ring.

Physical principle. The ring settles at the lowest height (lowest potential energy):

$$\text{minimize} \quad f(x,y) = y$$

Constraint. Length L of the string is fixed:

$$g(x,y) = \sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2} = L$$

The curve of all possible positions was drawn on blackboard using string: This is an ellipse, but knowing that does not help.

Lagrange muliplier equation, $\nabla f = \lambda \nabla g$:

$$\langle 0,1\rangle = \lambda \langle g_x, g_y\rangle$$

In particular, $g_x = 0$:

$$g_x = \frac{x}{\sqrt{x^2+y^2}} + \frac{x-a}{\sqrt{(x-a)^2+(y-b)^2}} = 0$$

This equation expresses tangency of the constraint curve $g = L$ with the level curve (straight horizontal line) $y = c$. (The second equation $1 = \lambda g_y$ contains no extra information because one can just choose $\lambda = 1/g_y$.) The equation $g_x = 0$ can be interpreted as the equality of the cosines of two angles.

$$\cos\theta = \frac{x}{\sqrt{x^2+y^2}} = \frac{a-x}{\sqrt{(x-a)^2+(y-b)^2}} = \cos\phi$$

Physical and geometric conclusions: *The angles θ and ϕ are equal.* Using vectors to compute the force exerted by gravity on the two halves of the string, one finds that there is *equal tension* in the two halves of the string – a physical equilibrium. (From another point of view, the equal angle property expresses a geometric property of ellipses: Suppose that the ellipse is a mirror. A ray of light from the focus $(0,0)$ reflects off the mirror according to the rule angle of incidence equals angle of reflection, and therefore the ray goes directly to the other focus at (a,b).)

Formulas for x and y. We did not yet solve the problem all the way for the location of (x,y). We will now show that

$$x = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2-a^2}}\right), \quad y = \frac{1}{2}\left(b - \sqrt{L^2-a^2}\right)$$

Because $\theta = \phi$,

$$x = \sqrt{x^2+y^2}\cos\theta; \quad a - x = \sqrt{(x-a)^2+(y-b)^2}\cos\theta$$

Adding these two equations,

$$a = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\cos\theta = L\cos\theta \implies \cos\theta = a/L$$

The corresponding calculation with y is (noting that $y < 0$)

$$-y = \sqrt{x^2+y^2}\sin\theta; \quad b-y = \sqrt{(x-a)^2+(y-b)^2}\sin\theta$$

Adding these two equations,

$$b-2y = \left(\sqrt{x^2+y^2} + \sqrt{(x-a)^2+(y-b)^2}\right)\sin\theta = L\sin\theta \implies y = \frac{1}{2}(b - L\sin\theta)$$

Use the relation $\cos\theta = a/L$ to express $L\sin\theta = L\sqrt{1-a^2/L^2} = \sqrt{L^2-a^2}$ as a function of L and a. Then the formula for y is

$$y = \frac{1}{2}\left(b - \sqrt{L^2-a^2}\right)$$

Finally, to find the (simplest way to express) the formula for x use the similar right triangles

$$\cot\theta = \frac{x}{-y} = \frac{a-x}{b-y} \implies x(b-y) = (-y)(a-x) \implies (b-2y)x = -ay$$

Therefore,

$$x = \frac{-ay}{b-2y} = \frac{-a\frac{1}{2}\left(b-\sqrt{L^2-a^2}\right)}{b-\left(b-\sqrt{L^2-a^2}\right)} = \frac{a}{2}\left(1 - \frac{b}{\sqrt{L^2-a^2}}\right)$$

Thus we have formulas for x and y in terms of a, b and L. While it is nice to derive the complete solution, we got all of our physical intuition and understanding out of the problem from the balance condition that was the immediate consequence of the critical point computation, which we carried out using the Lagrange multiplier method. This is why I omitted the derivation of the formulas for x and y in lecture.

Final Remark. Any two-variable constrained max/min problem, such as this ring problem can also be solved by one-variable calculus methods using the idea of implicit differentiation. Namely, consider the constraint $g(x,y) = L$ as the implicit definition of y as a function of x. Differentiate with respect to x:

$$g_x + g_y\frac{dy}{dx} = 0$$

We are minimizing y, so the condition $\dfrac{dy}{dx} = 0$ substituted into the equation above yields $g_x = 0$. This is the same as the Lagrange multiplier equation used above.