18.02 Problem Set 7 — Spring 2006

Due Thursday 4/06/06, 12:55 pm

Part A   (15 points)

Hand in the <u>underlined</u> problems only; the others are for more practice.

Lecture 19. Thurs, Mar 23. Change of variables. Probability
   Read: Notes CV. Extra lecture notes on Probability
   Work: 3D/ <u>1</u>, <u>2</u>, 3, <u>4</u> (integrate in the order $dudv$).

Lecture 20. Fri, Mar 24. Triple integrals in rectangular and cylindrical coordinates
   Read: Notes I.3; 18.7 (to 661–662), 20.5, 20.6.
   Work: 18.7/ 1, 2; 20.6/3, <u>5</u>; 5A/ 1, 2a<u>bc</u>d, 3, <u>4</u>, <u>5</u>, 6, 7.

Lecture 21. Tues, Apr 4.   Spherical coordinates. Gravitational attraction.
   Read: Notes I.4, CV.4, G; 18.7 (662–663), 20.7.
   Work: 18.7/3,4; 20.7/14; 5B/ 1a<u>bc</u>, <u>2</u>, <u>3</u>, 4a<u>bc</u>; 5C/ <u>3</u>, 4.

Part B   (47 points + 5 extra credit)

Directions: Attempt to solve *each part* of each problem yourself. If you collaborate, solutions must be written up independently. It is illegal to consult materials from previous semesters. With each problem is the day it can be done.

Problem 0. (not until due date; 5 points) Write the names of all the people you consulted or with whom you collaborated and the resources you used, or say "none" or "no consultation".

Problem 1. (Thursday, 5 points: 2 + 3)

a) Calculate the probability density of r, the distance to the origin, for a point selected uniformly according to area in the disk of radius 10 around the origin. In other words, find $f(r)$ such that if $0 \le a \le b \le 10$,

$$P(a < r < b) = \int_a^b f(r)dr$$

($f(r)$ is called the probability density of r; see the notes on probability.)

b) Calculate the probability density of the distance from the origin, r, for points (x, y) selected according to the density in Problem 4 (PS 6 Part B), that is,

$$P((x, y) \text{ in } R) = \frac{1}{M} \int \int_R \delta(x, y)dxdy,$$

where R is any portion of the half-disk $y > 0$, $x^2 + y^2 < 1$, $\delta(x, y) = y^2$, and M is the total mass of the half disk.

Problem 2. (Thursday, 6 points: 4 + 1 + 1)

a) Find the average distance between a point on the circumference of a circle of radius a and a point inside. Assume the point on the circumference is the origin and the circle

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is centered at $(a, 0)$. Set up the integration both in polar coordinates and in rectangular coordinates, but only carry out the computation using polar coordinates.

b) Find the probability density of the distance from a point in the disk of radius a to a point on the circumference of the disk.

c) Find the probability density of the cube of the distance from a point in the disk of radius a to a point on the circumference of the disk.

Problem 3. (Friday, 6 points: $4 + 2 +$ extra)

a) Suppose that a point (x, y, z) is chosen with uniform probability in the region $0 < x < y < z < 1$ (a tetrahedron). Find the average value and probability density of x.

b) The probability density and average value that you found in part (a) describes the behavior of a number x found by the following recipe: pick three numbers at random between 0 and 1 and then choose the smallest of the three. What is your chance of winning if you bet that the number obtained is less than $1/2$? less than $1/4$?

c) *Extra credit (5 points; added at the end of the term separately from all other scores)* Find the average value of x_1 if the point $(x_1, x_2, \ldots, x_n)$ is chosen with uniform probability in the region $0 < x_1 < x_2 < \cdots < x_n < 1$ in n-dimensional space.

Problem 4. (Friday, 13 points: $4 + 3 + 6$)

a) Change variables to express the integral $\int_0^1 \int_0^1 dx\,dy$ in coordinates $u = x$, $v = 2x - y^2$ in the order $dv\,du$. Draw a picture of the region of integration in the uv-plane. (Get started by finding what the corners and sides of the square in the xy-plane correspond to in the uv-plane.) Check your integral to be sure it has the value 1.

b) Express the same integral in the order $du\,dv$ in three parts. (You can doublecheck your answers by computing each integral directly in uv coordinates and in xy-coodinates.)

c) Suppose that two numbers (x, y) is chosen uniformly according to area in the unit square $0 \le x \le 1$, $0 \le y \le 1$. (See Notes on conditional probability.)

i) What is the conditional probability that $x \le 1/4$, given that $2x - y^2 = -1/2$?

ii) What is the conditional probability that $x \ge 1/2$, given that $2x - y^2 = -1/2$?

iii) Answer the same two questions if $2x - y^2 = 1/2$.

Problem 5. (Friday, 4 points) This problem is a classic. Find the volume of the solid formed by the intersection of two cylinders of radius 1 that meet at right angles. Make the axis of one of the cylinders the x-axis and the other the y-axis.

Problem 6. (Tuesday, 4 points) Find the average distance from a point of a solid ball of radius a to a point on the surface using spherical coordinates. Put the point at South Pole of the ball and use that point as the origin.

Problem 7. (Tuesday, 4 points: $2 + 2$) Set up, **but do not evaluate**, integrals in spherical coordinates for the following.

a) The center of mass of a solid ice cream cone with uniform density δ, vertex at the origin, axis along the z-axis, height h, and a flat top formed by a disk of radius a.

b) The moment of inertia of a ball of radius a with a cylindrical hole of radius b bored through the center (along the z) axis. (See 20.6/5 for the case of cylindrical coordinates).