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PROFESSOR:

So we're through with techniques of integration, which is really the most technical thing that we're going to be doing. And now we're just clearing up a few loose ends about calculus. And the one we're going to talk about today will allow us to deal with infinity. And it's what's known as L'Hôpital's Rule. Here's L'Hôpital's Rule. And that's what we're going to do today. L'Hôpital's Rule it's also known as L'Hospital's Rule. That's the same name, since the circumflex is what you put in French to omit the s. So it's the same thing, and it's still pronounced L'Hôpital, even if it's got an s in it. Alright, so that's the first thing you need to know about it.

And what this method does is, it's a convenient way to calculate limits including some new ones. So it'll be convenient for the old ones. There are going to be some new ones and, as an example, you can calculate $x \ln x$ as x goes to infinity. You could, whoops, that's not a very interesting one, let's try x goes to 0 from the positive side. And you can calculate, for example, $x e^{-x}$, as x goes to infinity. And, well, maybe I should include a few others. Maybe something like $\ln x / x$ as x goes to infinity. So these are some examples of things which, in fact, if you plug into your calculator, you can see what's happening with these. But if you want to understand them systematically, it's much better to have this tool of L'Hôpital's Rule. And certainly there isn't a proof just based on a calculation in a calculator.

So now here's the idea. I'll illustrate the idea first with an example. And then we'll make it systematic. And then we're going to generalize it. We'll make it much more-- So when it includes these new limits, there are some little pieces of trickiness that you have to understand. So, let's just take an example that you could have done in the very first unit of this class. The limit as x goes to 1 of $(x^{10} - 1) / (x^2 - 1)$. So that's a limit that we could've handled. And the thing that's interesting, I mean, if you like this is in this category, that we mentioned at the beginning of the course, of interesting limits. What's interesting about it is that if you do this silly thing, which is just plug in $x = 1$, at $x = 1$ you're going to get $0 / 0$. And that's what we call an indeterminate form. It's just unclear what it is. From that plugging, in you just can't get it.

Now, on the other hand, there's a trick for doing this. And this is the trick that we did at the beginning of the class. And the idea is I can divide in the numerator and denominator by $x - 1$. So this limit is unchanged, if I try to cancel the hidden factor $x - 1$ in the numerator and denominator. Now, we can actually carry out these ratios of polynomials and calculate them by long division in algebra. That's very, very long. We want to do this with calculus. And we already have. We already know that this ratio is what's called a difference quotient. And then in the limit, it tends to the derivative of this function. So the idea is that this is actually equal to, in the limit, now let's just study one piece of it. So if I have a function $f(x)$, which is $x^{10} - 1$, and the value at 1 happens to be equal to 0, then this expression that we have, which is in disguise, this is in disguise the difference quotient, tends to, as x goes to 1, the derivative, which is $f'(1)$. That's what it is.

So we know what the numerator goes to, and similarly we'll know what the denominator goes to. But what is that? Well, $f'(x) = 10x^9$. So we know what the answer is. In the numerator it's $10x^9$. In the denominator, it's going to be $2x$, that's the derivative of $x^2 - 1$. And then were going to have to evaluate that at $x = 1$. And so it's going to be $10/2$, which is 5. So the answer is 5. And it's pretty easy to get from our techniques and knowledge of derivatives, using this rather clever algebraic trick. This business of dividing by $x - 1$.

What I want to do now is just carry this method out systematically. And that's going to give us the approach to what's known as L'Hôpital's Rule, what-- my main subject for today. So here's the idea. Suppose we're considering, in general, a limit as x goes to some number a of $f(x) / g(x)$. And suppose it's the bad case where we can't decide. So it's indeterminate. $f(a) = g(a) = 0$. So it would be $0 / 0$. Now we're just going to do exactly the same thing we did over here. Namely, we're going to divide the numerator and denominator, and we're going to repeat that argument. So we have here $f(x) / (x-a)$. And $g(x)$, divided by $x - a$ also. I haven't changed anything yet.

And now I'm going to write it in this suggestive form. Namely, I'm going to take separately the limit in the numerator and the denominator. And I'm going to make one more shift. So I'm going to take the limit, as x goes to a in the numerator, but I'm going to write it as $(f(x) - f(a)) / (x - a)$. So that's the way I'm going to write the numerator, and I've got to draw a much longer line here. So why am I allowed to do that? That's because $f(a) = 0$. So I didn't change this numerator of the numerator any by subtracting that. $f(a) = 0$. And I'll do the same thing to the denominator. Again, $g(a) = 0$, so this is OK. And lo and behold, I know what these limits are.

This is $f'(a) / g'(a)$. So that's it. That's the technique and this evaluates the limit. And it's not so difficult. The formula's pretty straightforward here. And it works, provided that $g'(a)$ is not 0. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, is there a more intuitive way of understanding this procedure. And I think the short answer is that there are other, similar, ways. I don't consider them to be more intuitive. I will be mentioning one of them, which is the idea of linearization, which goes back to what we did in Unit 2. I think it's very important to understand all of these, more or less, at once. But I wouldn't claim that any of these methods is a more intuitive one than the other. But basically what's happening is, we're looking at the linear approximation to f , at a . And the linear approximation to g at a . That's what underlies this.

So now I get to formulate for you L'Hôpital's Rule at least in what I would call the easy version or, if you like, Version 1. So here's L'Hôpital's Rule. Version 1. It's not going to be quite the same as what we just did. It's going to be much, much better. And more useful. And what is going to take care of is this problem that the denominator is not 0. So now here's what we're going to do. We're going to say that it turns out that the limit as x goes to a of $f(x) / g(x)$ is equal to the limit as x goes to a of $f'(x) / g'(x)$. Now, that looks practically the same as what we said before. And I have to make sure that you understand when it works. So it works provided this is one of these undefined expressions. In other words, $g(a) = 0$. So we have a $0 / 0$ expression, indeterminate. And, also, we need one more assumption. And the right-hand side, the right-hand limit exists.

Now, this is practically the same thing as what I said over here. Namely, I took the ratio of these functions, $x^{10} - 1$ and $x^2 - 1$. I took their derivatives, which is what I did right here, right. I just differentiated them and I took the ratio. This is way easier than the quotient rule, and is nothing like the quotient rule. Don't think quotient rule. Don't think quotient rule. So we differentiate the numerator and denominator separately. And then I take the limit as x goes to 1 and I get 5. So that's what I'm claiming over here. I take these functions, I replace them with this ratio of derivatives, and then I take the limit instead, over here. And it turned out that the functions got much simpler when I differentiated them. I started with this messy object and I got this much easier object that I could easily evaluate. So that's the big game that's happening here.

It works, if this limit makes sense and this limit exists. Now, notice I didn't claim that g , that the denominator had to be nonzero. So that's what's going to help us a little bit in a few examples. So let me give you a couple of examples and then we'll go further. Now, this is only Version 1. But first we have to understand how this one works. So here's another example. Take the limit as x goes to 0, of $\sin(5x) / \sin(2x)$. This is another kind of example of a limit that we discussed in the first part of the course. Unfortunately, now we're reviewing stuff. So this should reinforce what you did there. This will be an easier way of thinking about it. So by L'Hôpital's Rule, so here's the step. We're going to take one of these steps. This is the limit, as x goes to 0, of the derivatives here. So that's $5 \cos(5x) / (2 \cos(2x))$.

The limit was 1 over there, but now it's 0. a is 0 in this case. This is the number a . Thank you. So the limit as x goes to 0 is the same as the limit of the derivatives. And that's easy to evaluate. Cosine of 0 is 1, right. This is equal to $5 \cos(5 \cdot 0)$ -- And that's a multiplication sign. Maybe I should just write this as 0. Divided by $2 \cos 0$. But you know that that's $5/2$. So this is how L'Hopital's method works. It's pretty painless.

I'm going to give you another example, which shows that it works a little better than the method that I started out with. Here's what happens if we consider the function $(\cos x - 1) / x^2$. That was a little harder to deal with. And again, this is one of these $0 / 0$ things near $x = 0$. As x tends to 0, this goes to an indeterminate form here. Now, according to our method, this is equivalent to, now I'm going to use this little wiggle because I don't want to write limit, limit, limit, limit a million times. So I'm going to use a little wiggle here. So as x goes to 0, this is going to behave the same way as differentiating numerator and denominator. So again this is going to be $-\sin x$ in the numerator. In the denominator, it's going to be $2x$.

Now, notice that we still haven't won yet. Because this is still of $0 / 0$ type. When you plug in $x = 0$ you still get 0. But that doesn't damage the method. That doesn't make the method fail. This $0 / 0$, we can apply L'Hôpital's Rule a second time. And as x goes to 0 this is the same thing as, again, differentiating the numerator and denominator. So here I get $-\cos x$ in the numerator, and I get 2 in the denominator. Again this is way easier than differentiating ratios of functions. We're only differentiating the numerator and the denominator separately.

And now this is the end. As x goes to 0, this is $-\cos 0 / 2$, which is $-1/2$. Now, the justification for this comes only when you win in the end and get the limit. Because what the theorem says is that if one of these limits exists, then the preceding one exists. And once the preceding one exists, then the one before it exists. So once we know that this one exists, that works

backwards. It applies to the preceding limit, which then applies to the very first one. And the logical structure here is a little subtle, which is that if the right side exists, then the left side will also exist. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: Why does the right-hand limit have to exist, isn't it just the derivative that has to exist? No. The derivative of the numerator has to exist. The derivative of the denominator has to exist. And this limit has to exist. What doesn't have to exist, by the way, I never said that f' of a has to exist. In fact, it's much, much more subtle. I'm not claiming that $f'(a)$ exists, because in order to evaluate this limit, $f'(a)$ need not exist. What has to happen is that nearby, for x not equal to a , these things exist. And then the limit has to exist. So there's no requirements that the limits exist. In fact, that's exactly going to be the point when we evaluate these limits here. Is we don't have to evaluate it right at the end.

STUDENT: [INAUDIBLE]

PROFESSOR: So the question that you're asking is, why is this the hypothesis of the theorem? In other words, why does this work? Well, the answer is that this is a theorem that's true. If you drop this hypothesis, it's totally false. And if you don't have this hypothesis, you can't use the theorem and you will get the wrong answer. I mean, it's hard to express it any further than that. So look, in many cases we tell you formulas. And in many cases it's so obvious when they're true that we don't have to worry about what we say. And indeed, there's something implicit here. I'm saying well, you know, if I wrote this symbol down, it must mean that the thing exists. So that's a subtle point. But what I'm emphasizing is that you don't need to know in advance that this one exists. You do need to know in advance that that one exists. Essentially, yeah. So that's the direction that it goes. You can't get away with not having this exist and still have the statement be true.

Alright, another question. Thank you.

STUDENT: [INAUDIBLE]

PROFESSOR: So I'm getting a little ahead of myself, but let me just say. In these situations here, when x is going to 0 and x is going to infinity. For instance, here when x goes to 0, the logarithm is undefined at $x = 0$. Nevertheless, this theorem applies. And we'll be able to use it. Over here, as x goes to infinity, neither of these-- well, actually, come to think of it, e^{-x} , if you like, it's

equal to 0 at infinity. If you want to say that it has a value. But in fact, these expressions don't necessarily have values, at the ends. And nevertheless, the theorem applies. I mean, it can exist. It's perfectly OK for it to exist. It's no problem. It just doesn't need to exist. It isn't forced to exist.

So here's a calculation which we just did. And we evaluated this. Now, I want to make a comparison with the method of approximation. In the method of approximations, this Example 2, which was the example with the sine function, we would use the following property. We would use $\sin u$ is approximately u . We would use that linear approximation. And then what we would have here is that $\sin(5x) / \sin(2x)$ is approximately $(5x)/(2x)$, which is of course $5/2$. And this is true when u is approximately 0, and this is true certainly as x goes to 0, it's going to be a valid limit.

So that's very similar to Example 2. In Example 3, we managed to look at this expression $(\cos x - 1) / x^2$. And for this one, you have to remember the approximation near $x = 0$ to the cosine function. And that's $1 - x^2 / 2$. So that was the approximation, the quadratic approximation to the cosine function. And now, sure enough, this simplifies. This becomes $(-x^2 / 2) / x^2$, which is $-1/2$. So we get the same answer, which is a good thing. Because both of these methods are valid. They're consistent. You can see that neither of them is particularly a lot longer. You may have trouble remembering this property. But in fact it's something that you can easily derive. And, indeed, it's related to the second derivative of the cosine, as is this calculation here. They're almost the same amount of numerical content to them.

So now what I'd like to do is explain to you why L'Hôpital's Rule works better in some cases. And the real value that it has is in handling these other more exotic limits. So now we're going to do L'Hôpital's Rule over again. And I'll handle these functions. But I'll have to rewrite them, but we'll just do that. So here's the property. That the limit as x goes to a of $f(x) / g(x)$ is equal to the limit as x goes to a of $f'(x) / g'(x)$. That's the property. And this is what we'll always be using. Very convenient thing. And remember it was true provided that $f(a) = g(a) = 0$. And that the right-hand side exists.

But I claim that it works better, and I'll get rid of these. But I'll write them again to show you that it works for these. So there are other cases. And the other cases that are allowed are this. First of all, as indicated by what I just erased, you can allow a to be equal to plus or minus infinity. It's also OK. So you can take the limit going to the far ends of the universe. Both left and right. And then the other thing that you can do is, you can allow $f(a)$ and $g(a)$ to be plus or

minus infinity. Is OK. So now, the point is that we can handle not just the $0 / 0$ case, but also the infinity / infinity case. That's a very powerful tool, and quite different from the other cases.

And the third thing is that the right-hand side doesn't really quite have to exist, in the ordinary sense. Or, it could be plus or minus infinity. That's also OK. That's still information. So if we can see where it goes, then we're still good. If it goes to plus infinity, if it goes to 0, if it goes to a finite number, if it goes to minus infinity, all of that will be OK. It just if it oscillates wildly that we'll be lost. And those calculations we'll never encounter. So this basically handles everything that you could possibly hope for. And it's a very convenient process.

So let me carry out a few examples. And, let's see, I guess the first one that I wanted to do was $x \ln x$. So what example are we up to. Example 3, so Example 4 is coming up. Example 4, this is one of the ones that I wrote at the beginning of the lecture, $x \ln x$. This one was on our homework problem. In the limits of some calculation. But so this one, you have to look at it first to think about what it's doing. It's an indeterminate form, but it sort of looks like it's the wrong type. So why is it in an indeterminate form. This one goes to 0, and this one goes to minus infinity. So, excuse me, this is a product. It's 0 times minus infinity. So that's an indeterminate form, because we don't know whether the 0 wins or the infinity this could keep getting smaller and smaller and smaller, and this could be getting bigger and bigger bigger. The product could be anything in between. We just don't know.

So the first step is to write this as a ratio of things, rather than a product of things. And it turns out that the way to do that is to use the logarithm in the numerator, and the $1 / x$ in the denominator. So this is a choice that I'm making here. Now, I've just converted it to a limit of the type minus infinity divided by infinity. Because the numerator is going to minus infinity as x goes to 0^+ and the denominator $1 / x$ is going to plus infinity. Again, there's a competition, but now it's one of the forms to which L'Hôpital's Rule applies.

Now I'm just going to apply L'Hôpital's Rule. And what it says is that I differentiate here. So I just differentiate the numerator and denominator. Applying L'Hôpital's Rule is a breeze. You just differentiate, differentiate. And now it just simplifies and we're done. This is the limit as x goes to 0^+ of, well, the x^2 's cancel. This is the same as just $-x$. x factors cancel. And so that's 0. The answer is that it's 0. So x goes to 0 faster than $\ln x$ goes to minus infinity. This 0 was the winner. Something you can't necessarily predict in advance.

So let's do the other two examples that I wrote down. I'm going to do them in slightly more

generality, because they're the most fundamental rate properties that you're going to need to know for the next section. Which is improper integrals. And also they're just very important for physical math, and any other kind of thing, basically. So here, let's just do these. So let's see, which one do I want to do first. So I wrote down the limit of $x e^{-x}$, but I'm going to make it even more general. I'm going to make it any negative power here, where p is some positive constant. Now again, this is a product of functions, not a quotient, a ratio, of functions. And so I need to rewrite it. I'm going to write it as $x / e^{(px)}$.

And now I'm going to apply, well, so it's of this form infinity / infinity. And now that's the same as the limit as x goes to infinity of $1 / (p e^{(px)})$. So where does that go? As x goes to infinity. Now we can decide. The 1 stays where it is. And this, as x goes to infinity, goes to infinity. So the answer is 0. And the conclusion is that x grows more slowly than $e^{(px)}$. As x goes to infinity. Remember, p is positive here, of course. It's the increasing exponentials. Not the decreasing ones.

Let's do a variant of this. I'll do it the opposite way. So I'm going to call this Example 5'. It really doesn't give us any more information, but it gives you just a little bit more practice. So suppose I look at things the other way. $e^{(px)}$ divided by, say, x^{100} . Now, this is an infinity / infinity example, again. And you can work out what it's doing. But there are two ways of thinking about this. There's the slow way and the fast way. The slow way is to differentiate this 100 times. That is, right? Apply L'Hôpital's Rule over and over and over and over again. All the way. It's clear that you could do it, but it's kind of a nuisance. So there's a much cleverer trick here. Which is to change this to the limit, as x goes to infinity, of the $e^{(px/100)} / x$, to the 100th power.

So if you do that, then we just have one L'Hôpital's Rule step here. And that one is that this is the same as, ...as x goes to infinity of, well it's $p/100 e^{(px/100)}$ divided by 1, all to the 100th power. That's our L'Hôpital step. And of course, that's $(\text{infinity} / 1)^{100}$. Which is infinity.

Now, again I did this in a slightly different way to show you that it works with infinity as well. So that was this other case. The right-hand side can exist, or it can be plus or minus infinity. And that applies to this limit. And therefore, to the original limit. And the conclusion here is that $e^{(px)}$, $p > 0$, grows faster than any power of x . I picked x^{100} , but obviously it didn't matter what power I picked. The exponents beat all the powers.

So we have one more of the ones that I gave at the beginning to take care of. And that one is

the logarithm. And its behavior at infinity. So I'll do a slight variant on that one, too. So we have Example 6, which is $\ln x$, and instead of dividing by x , I'm going to divide by $x^{1/3}$. I could divide by any positive power of x , we'll just do this example here. So now this, as x goes to infinity, is of the form infinity / infinity. And so it's equivalent to what happens when I differentiate numerator and denominator separately. And that's $1/x$, and here I have $1/3 x^{-2/3}$. $1/x$, and then $1/3 x^{-2/3}$. Now, when the dust settles here and you get your exponents right, we have an x^{-1} , and this is an $x^{-2/3}$, and that's a $1/3$ becomes a 3 . So this is what it is. And that's equal to $3x^{-1/3}$. Which we can decide. It goes to 0 . As x goes to infinity.

And so the conclusion is that $\ln x$ grows more slowly as x goes to infinity, than $x^{1/3}$ or any positive power of x . So any x^p , p positive, will work. So log is really slow, going to infinity. It's very, very gradual. Yeah, question.

STUDENT: [INAUDIBLE]

PROFESSOR: The question is, how many hypotheses do you need here? So I said that, and I think what you were asking is, if I have this hypothesis, can I also have this hypothesis. That's OK. I can have this hypothesis combined with this one. I need something about $f(a)$ and $g(a)$. I can't assume nothing about $f(a)$ and $g(a)$. So in other words, I have to be faced with either an infinity / infinity, or a $0/0$ situation.

So let's see. The rule applies in the $0/0$, or infinity / infinity case. These are the only two cases that it applies in. And a can be anything. Including infinity. Plus or minus infinity. The rule applies in these two cases. So in other words, this is what $f(a)/g(a)$ is. Either one of these. And in fact, it can be plus or minus.

STUDENT: [INAUDIBLE]

PROFESSOR: And the right-hand side has to be something. It has to be either finite or plus or minus infinity. So you need something. You need a specific value of a , you need to decide whether it's an indeterminate form. And you need the right-hand limit to exist. It's not hard to impose this. Because when you look at the right-hand side, you'll want to be calculating it. So you want to know what it is. So you'll never have problems confirming this hypothesis.

Alright. Let me give you one more example here. Which is just slightly trickier. Which involves, so here's another indeterminate form. That's going to be 0^0 . So there are lots of these things where you just don't know what to do. And they come out in various different ways. The

simplest example of this is the limit as x goes to 0 from above of x^x .

In order to work out what's happening with this one, we have to use a trick. And the trick is this is a moving exponent. And so it's appropriate to use base e . This is something that we did way back in the first unit. So, since we have a moving exponent, we're going to use base e . That's the good base to use whenever you have a moving exponent. And so rewrite this as $x^x = e^{(x \ln x)}$. And now, in order to figure out what's happening, we really only have to know what's going on with the exponent. So remember, actually we already did this. But I'm going to do it once more for you. This is $\ln x / (1/x)$. And that's equivalent, as x goes to 0, to using L'Hôpital's Rule to $1/x$, and this is $-1/x^2$, which is $-x$, which goes to 0. As x goes to 0.

And so what we have here is that this one is going to be equivalent to, well, it's going to tend to what we got over here. It's e^0 . That exponent is what we want. As x goes to 0. So that's the answer. This limit happens to be 1. That's actually relatively easy to do, given all of the power that we have at our hands.

Now, let me give you one more example. Suppose you're trying to understand the limit of $\sin x / x^2$. If you apply L'Hôpital's Rule, as x goes to 0, you're going to get $\cos x / (2x)$. And if you apply L'Hôpital's Rule again, as x goes to 0, you're going to get $-\sin x / 2$. And this, as x goes to 0, goes to 0. On the other hand, if you look at the linear approximation method, linear approximation says that $\sin x$ is approximately x near 0. So that should be x / x^2 . Which is $1 / x$, which goes to infinity. As x goes to 0, at least from one side, minus infinity to the other side.

So there's something fishy going on here, right? So this is fishy. Or maybe this is fishy, I don't know. So, tell me what's wrong here. Yeah.

STUDENT:

[INAUDIBLE] PROFESSOR: OK. So the claim is that the second application of L'Hôpital's Rule, this one, is wrong. And that's correct. And this is where you have to watch out, with L'Hôpital's Rule. This is exactly where you have to watch out. You have to apply the test. Here it's an indeterminate form. It's $0 / 0$ before I applied the rule. But in order to apply the rule the second time, it still has to be $0 / 0$. But this one isn't. This one is $1 / 0$. It's no longer an indeterminate form. It's actually infinite. Either plus or minus, depending on the sign of the denominator. Which is just what this answer is. So the linear approximation is safe. And we just applied L'Hôpital's Rule wrong. So the moral of the story here is look before you L'Hôpital.

Alright. Now, let me say one more thing. I need to pile it on just a little bit, sorry. So don't use it as a crutch. We don't want to just get ourselves so weak, after being in the hospital for all this

time, that we can't use, I'm sorry. So remember that you shouldn't have lost your senses. If you have something like this, so we'll do this one here. Suppose you're trying to understand what this does as x goes to infinity. Now, you could apply L'Hôpital's Rule five times, or four times. And get the answer here. But really, you should realize that the main terms are sitting there right in front of you. And that there's some algebra that you can do to simplify this. Namely, it's the same as $1 + 2/x + 1/x^5$. And then in the denominator, well, let's see. It's x . So this would be dividing by $1/x^5$ in both numerator and denominator. And here you have $1/x$ plus 2 over, sorry I overshot. But that's OK. $2/x^5$ here. So these are the main terms, if you like. And it's the same as $1 / (1/x)$, which is the same as x , and it goes to infinity. As x goes to infinity. Or, if you like, much more simply, just x^5 / x^4 is the main term. Which is x . Which goes to infinity. So don't forget your basic algebra when you're doing this kind of stuff. Use these things and don't use L'Hôpital's Rule. OK, see you next time.