Lecture 16. October 20, 2005

Practice Problems. Course Reader: 3D-1, 3D-3, 3D-7, 3E-3, 3E-4.

1. Dummy variables. Give a Riemann integrable function $f(x)$ defined on an interval $[a, b]$, the notation,

$$\int_a^b f(x)dx,$$

is shorthand for the Riemann integral of $f(x)$ over this interval. In particular, this equals the limit,

$$\lim_{n \to \infty} f(a + (b - a)k/n)\frac{b - a}{n}.$$

Observe, the variable x does not appear in this limit. It is very convenient to include the variable x in the notation for the Riemann integral; for how else are we to express the function integrated? But, since the definition of the Riemann integral does not involve x, x is really a *dummy variable*. Any variable name may be substituted for x, with the same meaning.

$$\int_a^b f(x)dx = \int_a^b f(u)du = \int_a^b f(v)dv = \int_a^b f(t)dt = \ldots$$

This freedom is very useful, particularly when one or both of the limits of integration depend on some parameter. In this case, by convention, the dummy variable is chosen to be a different parameter.

$$\int_a^x f(x)dx \text{ INCORRECT}, \quad \int_a^x f(t)dt \text{ CORRECT}$$

This convention reduces the likelihood of an error.

2. Variable limits of integration. The Riemann integral is often used to define functions, particularly antiderivatives having no simpler expression.

Example. For every angle $0 \leq \theta < \pi/2$, define $f(\theta)$ to be the area above the x-axis, inside the unit circle $x^2 + y^2 = 1$, and bounded by the vertical lines, $-\cos(\theta) \leq x \leq \cos(\theta)$. This is an integral,

$$f(\theta) = \int_{-\cos(\theta)}^{\cos(\theta)} \sqrt{1 - x^2} dx.$$

The problem is to describe the rate-of-change of f, $df/d\theta$.

The integral $f(\theta)$ is beyond our current techniques of integration (though soon we will have techniques to solve it). The simplest solution is indirect. Here, first, is the direct solution. The integral $f(\theta)$ equals the area of 2 triangles and a circular sector. By high-school geometry, the area is,

$$f(\theta) = \frac{\pi - 2\theta}{2} + 2(\frac{1}{2}\sin(\theta)\cos(\theta)) = \pi/2 - \theta + \frac{1}{2}\sin(2\theta).$$

Using standard rules of differentiation, the derivative is,

$$\frac{df}{d\theta} = -1 + \cos(2\theta).$$

Notice, by the double-angle formula for cosine, this equals,

$$-1 + \cos(2\theta) = \boxed{-2\sin^2(\theta).}$$

The hardest step (hidden here) was the geometric computation of $f(\theta)$. However, this is completely unnecessary. Introduce a function,

$$G(t) = \int_0^t \sqrt{1 - x^2} dx.$$

Using symmetry through the y-axis, $f(\theta)$ equals,

$$f(\theta) = 2G(\cos(\theta)).$$

By the chain rule,
$$\frac{df}{d\theta} = 2\frac{dG}{d\theta} = 2\frac{dG}{dt}\frac{dt}{d\theta} = 2\frac{dG}{dt}\frac{d(\cos(\theta))}{d\theta}.$$

By the Fundamental Theorem of Calculus,

$$\frac{dG}{dt} = \sqrt{1 - t^2}.$$

This gives,

$$\frac{df}{d\theta} = 2\sqrt{1 - \cos^2(\theta)}(-\sin(\theta)) = \boxed{-2\sin^2(\theta).}$$

The second method is indirect. The function $G(t)$ has no simple expression. Nonetheless, this method is faster. In many cases this is the only method that works.

The argument above using the chain rule and the Fundamental Theorem of Calculus is quite general. It gives the general equation,

$$\boxed{d/dx \int_{u(x)}^{v(x)} f(t)dt = f(v(x))v'(x) - f(u(x))u'(x).}$$

3. Geometric area and algebraic area. The Riemann integral is the *algebraic area*,

$$\int_a^b f(x)dx = \text{ Area above the } x\text{-axis } - \text{ Area below the } x\text{-axis .}$$

The *geometric area* is the total area, both above and below the x-axis. Although geometric area does not equal algebraic area, it has a simple expression using the Riemann integral,

$$\text{Geometric area } = \int_a^b |f(x)|dx.$$

Example. Find both the algebraic area and the geometric area bounded by the x-axis and the graph of $y = \sin(x)$ over the interval $-\pi < x < \pi$.

Because $\sin(x)$ is an odd function, the area below the x-axis for $-\pi < x < 0$ equals the area above the x-axis for $0 < x < \pi$. In the expression for the algebraic area, these areas cancel to give 0. This is borne out by computation,

$$\int_{-\pi}^{\pi} \sin(x)dx = (-\cos(x)|_{-\pi}^{\pi} = -\cos(\pi) + \cos(-\pi) = -(-1) + (-1) = 0.$$

On the other hand, the absolute value $|\sin(x)|$ equals,

$$|\sin(x)| = \left\{ \begin{array}{ll} -\sin(x), & -\pi < x \leq 0, \\ \sin(x), & 0 < x < \pi. \end{array} \right.$$

Thus the geometric area equals,

$$\int_{-\pi}^{0} -\sin(x)dx + \int_0^{\pi} \sin(x)dx =$$
$$(\cos(x)|_{-\pi}^{0} + (-\cos(x)|_0^{\pi} = (1 - (-1)) + (-(-1) + 1) = \boxed{4.}$$

Thus the geometric area does not equal the algebraic area. But computation of the geometric area reduces to a straightforward Riemann integral.

4. Estimates. For every pair of Riemann integrable functions $f(x), g(x)$ on $[a, b]$ satisfying the inequality $f(x) \leq g(x)$ for every choice of x, the following inequality holds,

$$\int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

This is very useful for estimating integrals.

Example. Determine the following Riemann integral to within $\pm 10^{-4}$,

$$\int_0^{0.1} 1 + \sqrt{\sin(x)}dx.$$

The expression $\sqrt{\sin(x)}$ has no simple antiderivative. The value of the Riemann integral could be approximated well by a Riemann sum. An alternative approach is to use the estimates,

$$(1 - x^2/6)\sqrt{x} \leq \sqrt{\sin(x)} \leq \sqrt{x},$$

for small values of x. This gives,

$$\int_0^{0.1} 1 + x^{1/2} - \frac{1}{6}x^{5/2}dx \leq \int_0^{0.1} 1 + \sqrt{\sin(x)}dx \leq \int_0^{0.1} 1 + x^{1/2}dx.$$

The first and third Riemann integral follow from the Fundamental Theorem of Calculus,

$$\int_0^{0.1} 1 + x^{1/2} - \frac{1}{6}x^{5/2}dx = \left(x + \frac{2}{3}x^{3/2} - \frac{1}{21}x^{7/2} \right)\Big|_0^{0.1} = 0.1 + \frac{2}{3\sqrt{1000}} - \frac{1}{21\sqrt{10000000}} = 0.1210667926 \pm 10^{-10}.$$

Similarly,

$$\int_0^{0.1} 1 + x^{1/2}dx = \left(x + \frac{2}{3}x^{3/2} \right)\Big|_0^{0.1} = 0.1 + \frac{2}{3\sqrt{1000}} = 0.1210818511 \pm 10^{-10}.$$

Since these two integrals agree to within $\pm 10^{-4}$, this gives the original integral,

$$\int_0^{0.1} 1 + \sqrt{\sin(x)}dx = \boxed{0.1210 \pm 10^{-4}.}$$

5. Change of variables. After the Fundamental Theorem of Calculus, the most useful integral rule is the change of variables rule. The rule for Riemann integrals is nearly the same as the rule for antiderivatives. The additional feature for Riemann integrals is the change of the *limits of integration*.

$$\int_{x=a}^{x=b} f(u(x))u'(x)dx = \int_{u=u(a)}^{u=u(b)} f(u)du.$$

Example. Find the Riemann integral,

$$\int_{\pi/4}^{\pi/3} \tan(x)dx.$$

Since $\tan(x)$ is not visibly the derivative of another function, we rewrite the integral and hope for the best.

$$\int_{\pi/4}^{\pi/3} \tan(x)dx = \int_{\pi/4}^{\pi/3} \frac{\sin(x)}{\cos(x)}dx.$$

In this form, the substitution $u = \cos(x)$ is natural,

$$\int_{x=\pi/4}^{x=\pi/3} \frac{\sin(x)}{\cos(x)}dx,$$

$$
\begin{array}{c|c}
u = \cos(x) & u(\pi/3) = \cos(\pi/3) = 1/2, \\
du = -\sin(x)dx & u(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}.
\end{array}
$$

$$\int_{u=1/\sqrt{2}}^{u=1/2} \frac{1}{u}(-du).$$

The new integral can be computed by the Fundamental Theorem of Calculus, since $1/u$ is the derivative of $\ln(u)$.

$$\int_{u=1/\sqrt{2}}^{u=1/2} \frac{-1}{u}du = \left(-\ln(|u|)\right)\big|_{1/\sqrt{2}}^{1/2} = -\ln(1/2) + \ln(1/\sqrt{2}) = \ln(2) - \ln(\sqrt{2}).$$

This simplifies to give,

$$\int_{\pi/4}^{\pi/3} \tan(x)dx = \boxed{\ln(2)/2.}$$

It is only fair to note there is a second method. Make the same substitution to simplify the antiderivative of $\tan(x)$ to $-\ln(|u|) + C$, and then back-substitute to get,

$$\int \tan(x)dx = -\ln(|\cos(x)|) + C.$$

Now use the Fundamental Theorem of Calculus with the original limits of integration. Both methods are correct. Usually the first method is faster and less error-prone; it requires no back-substitution.

6. Integrating backwards. This comes so naturally for most calculus students, it barely warrants mention. Technically, the Riemann integral,

$$\int_a^b f(x)dx,$$

is only defined if $a \leq b$. What if a is larger than b? The only possible answer consistent with the Fundamental Theorem of Calculus is the following,

$$\int_a^b f(x)dx = -\int_b^a f(x)dx, \text{ if } a > b.$$

Because of the central role of the Fundamental Theorem of Calculus, the above equation is true by convention. With this convention, the Fundamental Theorem of Calculus holds whenever a is less than b, equal to b, or greater than b.