Lecture 15. October 18, 2005

Homework. Problem Set 4 Part I: (d) and (e); Part II: Problem 2.

Practice Problems. Course Reader: 3B-6, 3C-2, 3C-3, 3C-4, 3C-6.

1. The Riemann sum for the exponential function. The problem is to compute the Riemann integral,

$$\int_0^b e^x dx,$$

using Riemann sums. Choose the partition of $[0, b]$ into a sequence of n equally-spaced subintervals of length b/n. So the partition numbers are $x_k = kb/n$. Also the length of each partition is $\Delta x_k = b/n$. Because e^x is increasing, the minimum value of e^x on the interval $[x_{k-1}, x_k]$ occurs at the left endpoint,

$$y_{k,\text{min}} = e^{x_{k-1}} = e^{(k-1)b/n}.$$

Similarly, the maximum value occurs at the right endpoint,

$$y_{k,\text{max}} = e^{x_k} = e^{kb/n}.$$

Thus the lower sum is,

$$A_{\text{min}} = \sum_{k=1}^{n} y_{k,\text{min}} \Delta x_k = \sum_{k=1}^{n} e^{(k-1)b/n} \frac{b}{n}.$$

And the upper sum is,

$$A_{\text{max}} = \sum_{k=1}^{n} y_{k,\text{max}} \Delta x_k = \sum_{k=1}^{n} e^{kb/n} \frac{b}{n}.$$

To evaluate each of the sums, make the substitution $c = e^{b/n}$. Then the lower sum is,

$$A_{\text{min}} = \frac{b}{n} \sum_{k=1}^{n} c^{k-1} = \frac{b}{n} \sum_{l=0}^{n-1} c^l.$$

The sum is a geometric sum,

$$(1 + c + c^2 + \cdots + c^{n-2} + c^{n-1}) = \frac{c^n - 1}{c - 1}.$$

Plugging this in gives,

$$A_{\text{min}} = \frac{b}{n} \frac{c^n - 1}{c - 1} = \frac{b}{n} \frac{e^{bn/n} - 1}{e^{b/n} - 1}.$$

This simplifies to,

$$A_{\min} = (e^b - 1)\frac{b/n}{e^{b/n} - 1}.$$

A similar computation gives,

$$A_{\max} = (e^b - 1)e^{b/n}\frac{b/n}{e^{b/n} - 1}.$$

Now make the substitution, $h = b/n$. This gives,

$$A_{\min} = (e^b - 1)\frac{h}{e^h - 1},$$

$$A_{\max} = (e^b - 1)e^h\frac{h}{e^h - 1}.$$

Taking the limit of $A_{\min}$, respectively $A_{\max}$, as n tends to infinity is the same as taking the limit as h tends to 0.

Now observe that,

$$\lim_{h\to 0}\frac{e^h - 1}{h},$$

is the difference quotient limit giving the derivative of e^x at $x = 0$. Since de^x/dx equals e^x, and since e^0 equals 1, this gives,

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1.$$

Inverting gives,

$$\lim_{h\to 0}\frac{h}{e^h - 1} = \left(\lim_{h\to 0}\frac{e^h - 1}{h}\right)^{-1} = (1)^{-1} = 1.$$

Also, because e^x is continuous,

$$\lim_{h\to 0} e^h = e^0 = 1.$$

Putting this together gives,

$$\lim_{n\to\infty} A_{\min} = (e^b - 1)\lim_{h\to 0}\frac{h}{e^h - 1} = (e^b - 1)(1) = e^b - 1.$$

Similarly,

$$\lim_{n\to\infty} A_{\max} = (e^b - 1)(\lim_{h\to 0} e^h)(\lim_{h\to 0}\frac{h}{e^h - 1}) = (e^b - 1)(1)(1) = e^b - 1.$$

Since the limit of $A_{\min}$ and the limit of $A_{\max}$ exist and are equal, the Riemann integral exists and equals,

$$\int_0^b e^x dx = \boxed{e^b - 1.}$$

2. The Riemann sum for x^r. Let $r > 0$ be a positive real number. The problem is to compute the Riemann integral,

$$\int_1^b x^r dx,$$

using Riemann sums. For this particular integral, a different partition than usual is more efficient. Let n be a positive integer, and let q be the real number,

$$q = b^{1/n}.$$

Choose the partition of $[1, b]$ into n subintervals with partition numbers,

$$x_k = q^k.$$

Observe that,

$$1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = (b^{1/n})^n = b.$$

The length of the k^{th} subinterval is,

$$\Delta x_k = x_k - x_{k-1} = q^k - q^{k-1} = q^{k-1}(q-1).$$

Observe this increases as k increases. So this is not the partition of $[1, b]$ into n equal subintervals. The mesh size is,

$$\text{mesh} = \max(\Delta x_1, \ldots, \Delta x_n) = \Delta x_n = (q-1)b^{(n-1)/n} \leq q - 1.$$

As n tends to infinity, the mesh size tends to,

$$\lim_{n\to 0} \text{mesh} = \lim_{n\to 0} q - 1 = \lim_{n\to 0} b^{1/n} - 1 = 0.$$

Thus, even though this isn't the most obvious choice of partition, it can be used to compute the Riemann integral.

Because x^r is increasing, the minimum value of x^r on the interval $[x_{k-1}, x_k]$ occurs at the left endpoint,

$$y_{k,\min} = x_{k-1}^r = q^{(k-1)r}.$$

Similarly, the maximum value occurs at the right endpoint,

$$y_{k,\max} = x_k^r = q^{kr}.$$

Thus the lower sum is,

$$A_{\min} = \sum_{k=1}^n y_{k,\min} \Delta x_k = \sum_{k=1}^n q^{(k-1)r} \cdot q^{(k-1)}(q-1).$$

This simplifies to,

$$A_{\mathrm{min}} = (q-1)\sum_{k=1}^{n} q^{(k-1)(r+1)}.$$

And the upper sum is,

$$A_{\mathrm{max}} = \sum_{k=1}^{n} y_{k,\mathrm{max}}\Delta x_k = \sum_{k=1}^{n} q^{kr}q^{(k-1)}(q-1).$$

This simplifies to,

$$A_{\mathrm{max}} = (q-1)q^r\sum_{k=1}^{n} q^{(k-1)(r+1)}.$$

To evaluate the sum, make the substitution $c = q^{r+1}$. Then the sum is,

$$\sum_{k=1}^{n} c^{k-1} = 1 + c + c^2 + \cdots + c^{n-2} + c^{n-1}.$$

This geometric sum equals,

$$\frac{c^n - 1}{c - 1} = \frac{q^{n(r+1)} - 1}{q^{r+1} - 1}.$$

Thus the upper and lower sums simplify to,

$$A_{\mathrm{min}} = (q-1)(q^{n(r+1)} - 1)/(q^{r+1} - 1),$$

$$A_{\mathrm{max}} = q^r(q-1)(q^{n(r+1)} - 1)/(q^{r+1} - 1).$$

Now back-substitute $q = b^{1/n}$ to get that $q^{n(r+1)} = b^{r+1}$. Simplifying gives,

$$A_{\mathrm{min}} = (b^{r+1} - 1)\frac{1}{(q^{r+1} - 1)/(q-1)},$$

$$A_{\mathrm{max}} = (b^{r+1} - 1)q^r\frac{1}{(q^{r+1} - 1)/(q-1)}.$$

As n tends to infinity, the quantity $q = b^{1/n}$ tends to 1. The fraction,

$$\frac{q^{r+1} - 1}{q - 1},$$

is the difference quotient for $y = x^{r+1}$ for $x = 1$. As q tends to 1, the limit of the difference quotient is the derivative of $y = x^{r+1}$ at $x = 1$,

$$\lim_{q\to 1}\frac{q^{r+1} - 1}{q - 1} = \frac{d(x^{r+1})}{dx}|_{x=1} = ((r+1)x^r|_{x=1} = (r+1).$$

Also, since x^r is continuous,

$$\lim_{q \to 1} q^r = 1^r = 1.$$

Substituting this in gives,

$$\lim_{n \to \infty} A_{\min} = (b^{r+1} - 1) \left(\lim_{q \to 1} \frac{q^{r+1} - 1}{q - 1} \right)^{-1} = \frac{b^{r+1} - 1}{r + 1},$$

$$\lim_{n \to \infty} A_{\max} = (b^{r+1} - 1) \left(\lim_{q \to 1} q^r \right) \left(\lim_{q \to 1} \frac{q^{r+1} - 1}{q - 1} \right)^{-1} = \frac{b^{r+1} - 1}{r + 1},$$

Since the limit of $A_{\min}$ and the limit of $A_{\max}$ exist and are equal, the Riemann integral exists and equals,

$$\int_1^b x^r dx = \boxed{(b^{r+1} - 1)/(r + 1).}$$

3. The Fundamental Theorem of Calculus. There is a single theorem that it is at the heart of almost all applications involving Riemann integrals. The theorem answers two question simultaneously: Which functions are Riemann integrable? What is the Riemann integral of a function? The answer to the first question is: Every function you are likely to encounter is Riemann integrable. Precisely, every continuous function, and every piecewise continuous function is Riemann integrable.

The answer to the second question is more interesting. Assume $f(x)$ is a continuous function. Let $x = a$ be a fixed point where $f(x)$ is defined. Form the function,

$$F(x) = \int_a^x f(t)dt.$$

The function $F(x)$ is defined whenever $f(t)$ is defined on all of $[a, x]$. If $f(x)$ is continuous, the Fundamental Theorem of Calculus asserts $F(x)$ is differentiable and,

$$\frac{dF}{dx}(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x).$$

The proof of the second part is very easy. Consider the increment in F from x to $x + \Delta x$,

$$F(x + \Delta x) - F(x) = \int_a^{x+\Delta x} f(t)dt - \int_a^x f(t)dt = \int_x^{x+\Delta x} f(t)dt.$$

Let $y_{\min}$ be the minimum value of $f(t)$ on the interval $[x, x + \Delta x]$. Let $y_{\max}$ be the maximum value of $f(t)$ on the interval $[x, x + \Delta x]$. Then for every choice of partition $t_0 < t_1 < \cdots < t_n$ of $[x, x + \Delta x]$, and every choice of values y_k^* on the subintervals,

$$y_{\min} \le y_k^* \le y_{\max},$$

for every k. Thus the Riemann sum is squeezed between,

$$\sum_{k=1}^{n} y_{\text{min}} \Delta t_k \leq \sum_{k=1}^{n} y_k^* \Delta t_k \leq \sum_{k=1}^{n} y_{\text{max}} \Delta t_k.$$

Of course the lower bound is,

$$\sum_{k=1}^{n} y_{\text{min}} \Delta t_k = y_{\text{min}} \sum_{k=1}^{n} \Delta t_k = y_{\text{min}} \Delta x,$$

because the total length of the interval $[x, x + \Delta x]$ is Δx. Similarly, the upper bound is,

$$\sum_{k=1}^{n} y_{\text{max}} \Delta t_k = y_{\text{max}} \Delta x.$$

Thus the Riemann sum is squeezed between,

$$y_{\text{min}} \Delta x \leq \sum_{k=1}^{n} y_k^* \Delta x_k \leq y_{\text{max}} \Delta x.$$

Because the Riemann integral is a limit of Riemann sums, it is also squeezed,

$$y_{\text{min}} \Delta x \leq \int_{x}^{x+\Delta x} f(t) dt \leq y_{\text{max}} \Delta x.$$

Substituting in $F(x + \Delta x) - F(x)$ and dividing each term by Δx gives,

$$y_{\text{min}} \leq \frac{F(x + \Delta x) - F(x)}{\Delta x} \leq y_{\text{max}}.$$

The middle term is the difference quotient. Consider what happens as Δx tends to 0. Because $f(t)$ is continuous, both the maximum and minimum values of $f(t)$ on $[x, x + \Delta x]$ simply limit to the value $f(x)$. Thus,

$$\lim_{\Delta x} y_{\text{min}} = \lim_{\Delta x} y_{\text{max}} = f(x).$$

By the Squeezing Lemma for limits, since these two limits exist and are equals, the middle limit also exists and equals $f(x)$,

$$\lim_{\Delta x \to 0} \frac{F(x + \Delta x) - F(x)}{\Delta x} = f(x).$$

This is precisely what the Fundamental Theorem of Calculus asserts,

$$\frac{d}{dx} \int_{a}^{x} f(t) dt = \boxed{f(x)}.$$

4. Algorithm for computing Riemann integrals. The Fundamental Theorem of Calculus has many important applications. The most obvious is to give us a simpler method for computing Riemann integrals, under the hypothesis that we can compute the antiderivative. If $f(x)$ is a continuous function and $G(x)$ is a known antiderivative of $f(x)$, then,

$$\int_a^b f(t)dt = G(b) - G(a).$$

To see this, observe that,

$$F(x) = \int_a^x f(t)dt,$$

is also an antiderivative of $f(t)$ by the Fundamental Theorem of Calculus. Thus, since the general antiderivative is $G(x) + C$, there is a constant C such that $F(x) = G(x) + C$. But also,

$$F(a) = \int_a^a f(t)dt = 0.$$

Thus, $F(x) = G(x) - G(a)$. Now plug in $x = b$ to get,

$$\int_a^b f(t)dt = F(b) = \boxed{G(b) - G(a).}$$