MOVING AVERAGE CHART

- A goal is to detect small shifts as rapidly as possible, without increasing sample size
- Approach: Combine results from multiple samples

Average of a moving window that contains w samples (each sample still of size n)

$$M_t = \frac{\bar{x}_t + \bar{x}_{t-1} + \cdots + \bar{x}_{t-w+1}}{w}$$

$$\left.\begin{aligned} \mu_M &= \mu \\ \sigma^2_M &= \frac{\sigma^2}{nw} \end{aligned}\right\} \Rightarrow \begin{aligned} UCL &= \bar{\bar{x}} + \frac{3\sigma}{\sqrt{nw}} \\ CL &= \bar{\bar{x}} \\ LCL &= \bar{\bar{x}} - \frac{3\sigma}{\sqrt{nw}} \end{aligned}$$

EXPONENTIALLY WEIGHTED MOVING AVERAGE (EWMA)

- If our goal is to catch shifts rapidly, we might consider weighting more recent samples more heavily than older samples:

$$z_t = \lambda \bar{x}_t + (1-\lambda) z_{t-1}, \quad 0 < \lambda \le 1, \quad z_0 = \bar{\bar{x}}$$

$$= \lambda \sum_{j=0}^{t-1} (1-\lambda)^j \bar{x}_{t-j} + (1-\lambda)^t z_0$$

$$UCL = \bar{\bar{x}} + 3\sigma \sqrt{\frac{\lambda}{(2-\lambda)n}}$$

$$LCL = \bar{\bar{x}} - 3\sigma \sqrt{\frac{\lambda}{(2-\lambda)n}}$$

For large t

$$\sigma^2_{z_t} = \frac{\sigma^2}{n} \frac{\lambda}{2-\lambda} \left(1 - (1-\lambda)^{2t}\right)$$

In general (use for small t)

- Smaller $\lambda \Rightarrow$ to detect smaller shifts ("accumulate" more history)

- EWMA Chart Design

  - Choices: (1) Other alternatives than $\pm 3\sigma$
    (2) EWMA filter coefficient λ

    Approach: Choose based on ARL to detect shift of desired size

  - often used with individuals chart: utilize many past runs to detect shift.

  - typical values: $\lambda = 0.1$
    $\pm 2.7\sigma_z$

    $ARL_0 \doteq 500$ (for false alarm)
    $ARL_1 \doteq 10.3$ (for detecting a shift of 1σ)

    Similar to Shewhart/WECO:
    $\lambda = 0.4$
    $\pm 3.054\sigma_z$

    $ARL_0 \doteq 500$ (for false alarm)
    $ARL_1 \doteq 14.3$ (for detecting a shift of 1σ)

Average Run Lengths for Several EWMA Control Schemes [Adapted from Lucas and Saccucci (1990)]

Shift or Mean (multiple of S)	$L = 3.054$ $\lambda = 0.40$	2.998 0.25	2.962 0.20	2.814 0.10	2.615 0.05
0	500	500	500	500	500
0.25	224	170	150	106	84.1
0.50	71.2	48.2	41.8	31.3	28.8
0.75	28.4	20.1	18.2	15.9	16.4
1.00	14.3	11.1	10.5	10.3	11.4
1.50	5.9	5.5	5.5	6.1	7.1
2.00	3.5	3.6	3.7	4.4	5.2
2.50	2.5	2.7	2.9	3.4	4.2
3.00	2.0	2.3	2.4	2.9	3.5
4.00	1.4	1.7	1.9	2.2	2.7

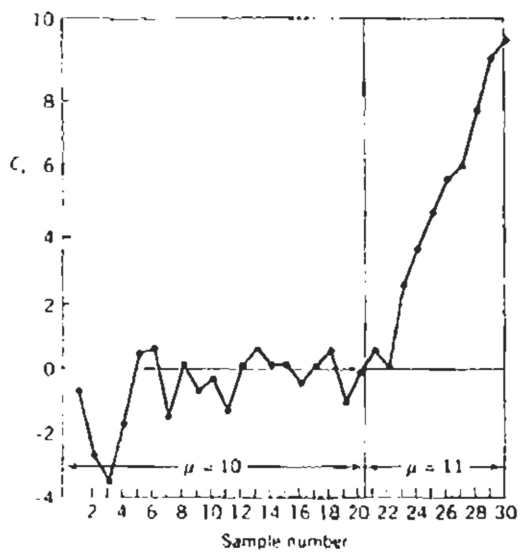
  - EWMV (exponentially weighted moving variance) extensions are also available

CUSUM CHARTS

Principle: Again, use multiple past history points to accelerate detection of a shift.

⇒ Plot cumulative sums of deviations of sample from a target value (with some slack)

- Very useful for $n=1$ sampling



$$C_i = \sum_{j=1}^{i} (x_i - \mu_0)$$

$$C_i = (x_i - \mu_0) + C_{i-1}$$

- Control chart design approaches
  - (1) "tabular" or "algorithmic" cusum
    - \+ some modifications to summation
    - \+ looks similar to normal chart with upper/lower control limits
  - (2) "V-mask" design
    - \+ approach often seen
    - − hard to read and interpret

Tabular Cusum

$$C_i^+ = \max\{0,\ x_i - (\mu_0 + k) + C^+_{i-1}\}$$

$$C_i^- = \max\{0,\ (\mu_0 - k) - x_i + C^-_{i-1}\}$$

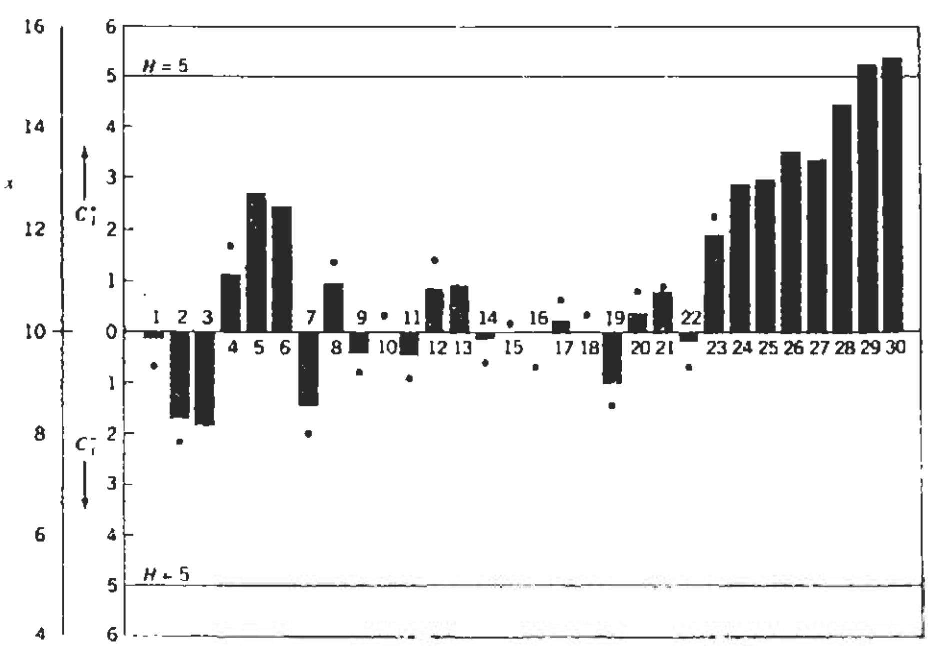
with $C_0^+ = C_0^- = 0$

k = "slack" value ... x_i must be outside this allowance to grow C^+ or C^-

$= \frac{\delta}{2}\sigma$ typically to detect a $\delta\sigma$ shift

$$\mu_0 + H = \mu_0 + h\sigma = UCL$$
$$\mu_0 = CL$$
$$\mu_0 - H = \mu_0 - h\sigma = LCL$$

- Typical value $H = 5\sigma$



Sample Cusum Status Chart.

- CUSUM CHART DESIGN:

$K = k\sigma$ ... pick based on ARL performance
$H = h\sigma$ (many studies exist, see Montgomery)

Typically $k = 1/2$ (close to minimizes ARL_1)
$h = 4$
$ARL_1 = 8.38$ samples to detect 1σ shift

$k = 1/2$
$h = 5$
$ARL_1 = 10.4$

Note: $ARL_0 = 168$ for $h=4$

$ARL_0 = 465$ for $h=5$

False alarm performance

$\frac{1}{1-\beta} = ARL_0$

ARL Performance of the Tabular Cusum with $k = \frac{1}{2}$ and $h = 4$ or $h = 5$

Shift in Mean (multiple of σ)	$h = 4$	$h = 5$
0	168	465
0.25	74.2	139
0.50	26.6	38.0
0.75	13.3	17.0
1.00	8.38	10.4
1.50	4.75	5.75
2.00	3.34	4.01
2.50	2.62	3.11
3.00	2.19	2.57
4.00	1.71	2.01

Values of k and the Corresponding Values of h That Give $ARL_0 = 370$ for the Two-Sided Tabular Cusum [from Hawkins (1993a)]

k	0.25	0.5	0.75	1.0	1.25	1.5
h	8.01	4.77	3.34	2.52	1.99	1.61

MULTI-POINT CHARTS and DETECTION of SHIFTS

- The principle underlying the CUSUM and other charts that look at multiple historical points is the

  DETECTION of a SHIFT

- when we have MULTIPLE points, the principles of MAXIMUM LIKELIHOOD ESTIMATION (MLE) underlie our statistical inferences

MLE

The correct choice of the pdf moments maximizes the collective likelihood of the observations.

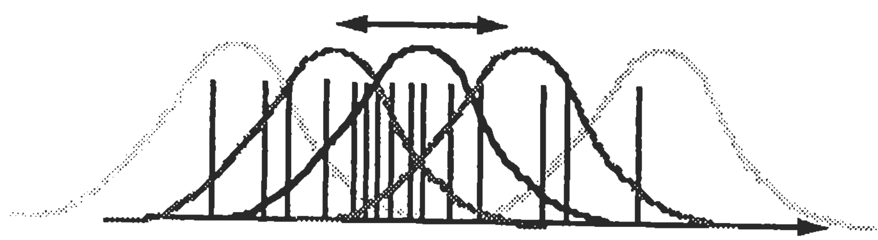
If $x \sim f(x, \theta)$ with unknown θ,
Then θ estimated by solving

$$\max_{\theta} \left[\prod_{i=1}^{m} \text{pdf}(x_i, \theta) \right]$$

$\Rightarrow$ good for both estimation and detection (comparison)

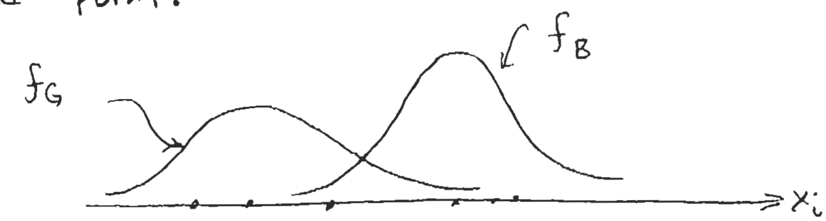
Example: Estimate mean of a normal distribution given observations x_i, $i = 1, 2 \ldots, m$

$$\max_{\mu_{est}} \left\{ \prod_{i=1}^{m} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu_{est}}{\sigma}\right)^2} \right\}$$



MLE CONTROL CHARTS

The goal is to DETECT if the process state has changed from some "GOOD" state to some bad point:



Calculate $\sum_{i=1}^{m} \log \frac{f_B(x_i)}{f_G(x_i)}$ $\Rightarrow$ large when process in the f_B distribution/state

Use "summing" statistic S_m

$$S_m = \sum_{i=1}^{m} \log \frac{f_B(x_i)}{f_G(x_i)} > L \quad \text{as signal}$$

That is, when "enough" evidence mounts that we are in the bad state (determined by L threshold), signal alarm.

Cusum: When θ is mean of a Normal distribution, then S_m simplifies to

$$S_m = \sum_{i=1}^{m} (\bar{x}_i - \mu_0) > L,$$

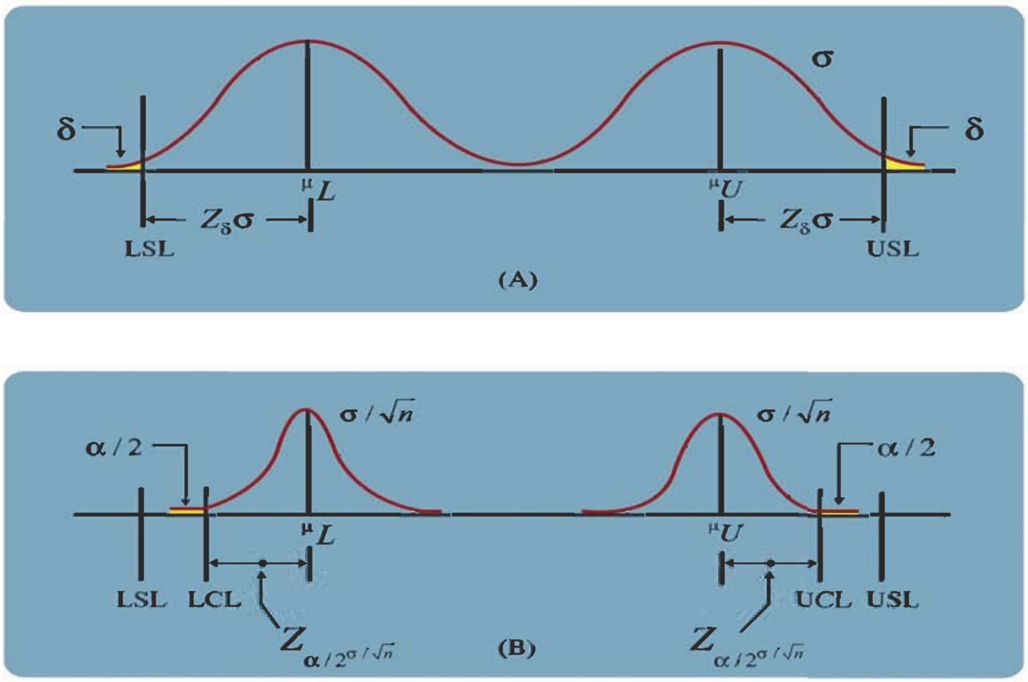
L depends on shift to be detected!

Key Point: use of cusum and related charts requires knowledge about f_B!

- MLE approaches are very general, and can be applied to deviations in means and variances, and also to multivariate cases (means and covariances).

ACCEPTANCE CHARTS

- Issue: what if we have an extremely capable process, $C_{pk} >> 1$? $\Rightarrow$ may only want to be concerned when the process gets "close" to our spec limits.



Control limits on a modified control chart. (A) Distribution of process output. (B) Distribution of the sample mean $\overline{x}$.

- Approach: Build chart equivalent to hypothesis test:

$$H_0: \quad \mu_L \le \mu \le \mu_U$$

where we set μ_L and μ_U so that some δ fraction nonconforming is okay:

$$\mu_L = LSL + Z_\delta \sigma$$

$$\mu_U = UCL - Z_\delta \sigma$$

where Z_δ is upper $100(1-\delta)$ percentage pt.

- Chart design: For α type I error:

$$UCL = \mu_U + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

$$= USL - Z_\delta \sigma + Z_{\alpha/2} \frac{\sigma}{\sqrt{n}} = USL - \left(Z_\delta - \frac{Z_{\alpha/2}}{\sqrt{n}} \right) \sigma$$

$$LCL = \mu_L - Z_{\alpha/2}\, \sigma/\sqrt{n}$$

$$= LSL + Z_\delta \sigma - Z_{\alpha/2}\, \sigma/\sqrt{n} = LSL + \left(Z_\delta - \frac{Z_{\alpha/2}}{\sqrt{n}} \right) \sigma$$

* Can use 3 for $z_{\alpha/2}$ if desired

- Caveat: depends on well-controlled σ

$\Rightarrow$ should also monitor using R or S chart!