6.777J/2.751J Design and Fabrication of Microelectromechanical Devices
Spring Term 2007

SOLUTIONS TO PROBLEM SET 6 (10 pts)

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Problem 13.9 (3pts): Calculating fluidic resistances

a). For a parallel-plate Poiseuille flow approximation, the flow rate and the pressure drop is related by

$$\Delta P = \frac{12\eta L}{Wh^3} Q$$

Using the e→V convention, the fluidic resistance of the channel is

$$R_{pois} = \frac{12\eta L}{Wh^3}$$

$$R_{pois} = \frac{12 \times (0.001 \times \frac{1}{60} Pa \cdot \min) \times 0.1 dm}{50 \times 10^{-5} dm \times (50 \times 10^{-5} dm)^3}$$

$$= 320 \times 10^6 \frac{Pa \cdot \min}{L} = 320 \frac{Pa \cdot \min}{\mu L}$$

b). We can express the general relation between flow and pressure to be,

$$Q = (1-A)\frac{Wh^3}{12\eta}\left|\frac{dP}{dx}\right|$$

where we have used A to denote,

$$\text{A}=\left[\frac{192 \cdot h}{\pi^5 \cdot W}\sum_{n=0}^{\infty}\frac{\tanh\left((2 \cdot n+1)\frac{\pi \cdot W}{2 \cdot h}\right)}{(2 \cdot n+1)^5}\right]$$

For Poiseuille flow, pressure drop is linear with length, so we have, $\left|\frac{dP}{dx}\right| = \frac{\Delta P}{L}$, and hence,

$$\Delta P = \frac{12\eta L}{(1-A)Wh^3} Q$$

The fluidic resistance of the channel is

$$R_{poisFull} = \frac{12\eta L}{(1-A)Wh^3}$$

Using Matlab, we can find that A converges at 0.5783, substitute in, we have

$$R_{poisFull} = \frac{12 \times (0.001 \times \frac{1}{60} Pa \cdot \min) \times 0.1 dm}{(1-0.5783) \times 50 \times 10^{-5} dm \times (50 \times 10^{-5} dm)^3}$$

$$= \frac{320}{1-0.5783}\frac{Pa \cdot \min}{\mu L} = 758.83 \frac{Pa \cdot \min}{\mu L}$$

The error is

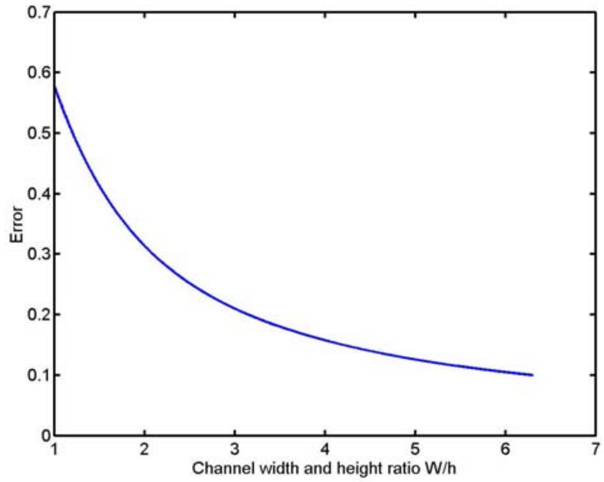
$$Error = \frac{R_{poisFull} - R_{pois}}{R_{poisFull}}$$

$$= \frac{\frac{12\eta L}{(1-A)Wh^3} - \frac{12\eta L}{Wh^3}}{\frac{12\eta L}{(1-A)Wh^3}} = A = 57.83\%$$

c). As we derived from b), the error can be expressed by A, where

$$A = \left[\frac{192 \cdot h}{\pi^5 \cdot W} \sum_{n=0}^{\infty} \frac{\tanh\left((2 \cdot n + 1) \frac{\pi \cdot W}{2 \cdot h} \right)}{(2 \cdot n + 1)^5} \right]$$

Using Matlab, it can be determined that the min W/h ratio for error to be less than 10% is 6.305.



The Matlab script is posted here:

```
clear all;
close all;
format long eng
L=1E-2;
visc=0.001*(1/60);
W=50E-6;
h=linspace(50E-6,5E-6);
h=h';
Wperh=zeros(100,1);
Error=zeros(100,1);
Rpoisfull=zeros(100,1);
Rpois=zeros(100,1);
for m=1:100
    h_now=h(m,1);
    Wperh(m,1)=W/h_now;
    for i=0:24
    Aterm(i+1,1)=((192*h_now)/(pi^5*W))*(tanh((2*i+1)*(pi/2 ...
    )*(W/h_now))/(2*i+1)^5);
    end
    A=sum(Aterm);
    Rpoisfull(m,1)=(12*visc*L*10*1E-6)/((1-A)*(W*10*(h_now*10)^3));
    Rpois(m,1)=(12*visc*L*10*1E-6)/(W*10*(h_now*10)^3);
    Error(m,1)=(Rpoisfull(m,1)-Rpois(m,1))/Rpoisfull(m,1);
end

plot(Wperh,Error)
ylabel('Error')
xlabel('Channel width and height ratio W/h')
```

Problem 13.10 (4 pts): Timescales in microfluidic flows

a). The Navier –Stokes equation is

$$\rho_m \frac{DU}{Dt} = -\nabla P + \rho_m g + \eta \nabla^2 U + \frac{\eta}{3}\nabla(\nabla \bullet U)$$

Assuming incompressible flow and neglecting gravity,

$$\nabla \bullet U = 0 \text{ and } \rho_m g = 0$$

The equation becomes

$$\rho_m \left(\frac{\partial U}{\partial t} + U \bullet \nabla U\right) = -\nabla P + \eta \nabla^2 U$$

b). Substitute the following expressions into the equation:

$$U = U_0 \tilde{u}$$
$$t = \tau \tilde{t}$$
$$P = (\eta U_0 / L)\tilde{P}$$
$$\nabla = \frac{\tilde{\nabla}}{L}$$
$$\nabla^2 = \frac{\tilde{\nabla}^2}{L^2}$$

We have

$$\rho_m \left(\frac{\partial \tilde{u}}{\partial \tilde{t}} \frac{U_0}{\tau} + \tilde{u} \bullet \tilde{\nabla} \tilde{u} \frac{U_0^2}{L}\right) = -\tilde{\nabla}\tilde{P}\frac{\eta U_0}{L^2} + \eta \tilde{\nabla}^2 \tilde{u} \frac{U_0}{L^2}$$

$$\Rightarrow \frac{\rho_m L U_0}{\eta}\left(\frac{1}{\frac{\tau U_0}{L}} \frac{\partial \tilde{u}}{\partial \tilde{t}} + \tilde{u} \bullet \tilde{\nabla}\tilde{u} \right) = -\tilde{\nabla}\tilde{P} + \tilde{\nabla}^2 \tilde{u}$$

$$R_e = \frac{\rho_m L U_0}{\eta}$$
$$S_r = \frac{\tau U_0}{L}$$

c).

$$S_r = \frac{\tau}{\tau_c}$$
$$\Rightarrow \tau_c = \frac{L}{U_0}$$
$$R_e = \frac{\tau_v}{\tau_c}$$
$$\Rightarrow \tau_v = R_e \tau_c = \frac{\rho_m L U_0}{\eta} \frac{L}{U_0}$$
$$= \frac{\rho_m L^2}{\eta}$$

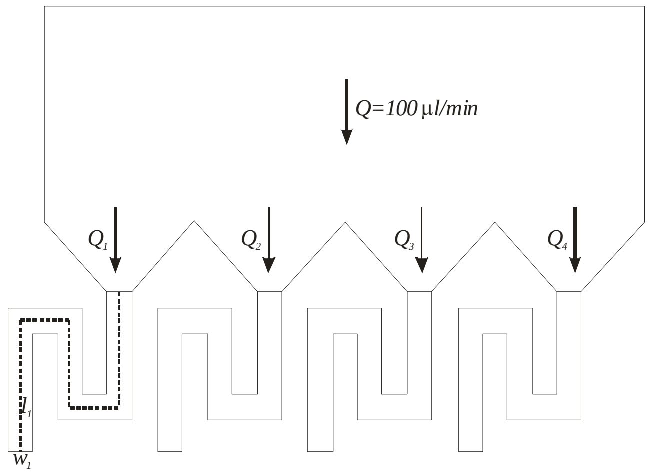
τ_c is the time scale of convective flow and τ_v is the time scale of viscous flow.

d). Applying a step input of pressure, the flow near the wall can not violate the no-slip condition. The time scale to establish steady flow is determined by the visous flow time scale. Using a length scale on the order of the hydraulic diameter, we have

$$L = \frac{2Wh}{W+h} = \frac{2 \times 100 \times 100}{100 + 100} = 100 \mu m$$

$$\tau_v = \frac{(100 \times 10^{-6})^2 m^2}{10^{-6} (m^2 / s)} = 0.01 \text{sec}$$

Problem 13.11 (3pts): Microfluidic networks and fabrication variations



a). First let us summarize the constraints of the design of the fluidic channels:

1. Total volumetric flow rate is 100 μL/min;
2. Flow ratio across the four channels are $Q_1 : Q_2 : Q_3 : Q_4 = 1:3:9:27$;
3. Channel height *h* is fixed at 50 μm; width $w \geq 150\ um$ and total area $A \leq 25\ mm^2$;
4. The maximum pressure at the inlet must be ≤1 psi.

Assuming parallel-plate Poiseuille flow, the flow resistance is,

$$R = \frac{\Delta P}{Q} = \frac{12 \eta l}{wh^3}$$

The volumetric flow rate across each channel can be found to be,

$$Q_1 = 2.5, Q_2 = 7.5, Q_3 = 22.5, Q_4 = 67.5\ \mu L / \min$$

Since the pressure drop across the channels is the same (both the inlets and outlets are connected), we can derive that,

$$R_1 : R_2 : R_3 : R_4 = 27:9:3:1$$

$$\Rightarrow \frac{l_1}{w_1} : \frac{l_2}{w_2} : \frac{l_3}{w_3} : \frac{l_4}{w_4} = 27:9:3:1$$

From total area constraint, we can write,

$$2(l_1 w_1 + l_2 w_2 + l_3 w_3 + l_4 w_4) < A_{\max}$$

To minimize the area, we can set the width at its minimum, $w = 150\ um$, and we can reduce the above expression in terms of l_1,

$$2w\left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27}\right) l_1 < A_{\max}$$

$$\Rightarrow l_1 < \frac{27 A_{\max}}{80w} = \frac{27 \times 25\ mm^2}{80 \times 150 \times 10^{-3} mm} = 56.25\ mm$$

We can choose a set of channel dimensions, such that, $l_1 = 54\ mm, l_2 = 18\ mm, l_3 = 6\ mm, l_4 = 2\ mm$. The pressure drop across the channel is,

$$\Delta P = \frac{12 \eta l_1}{wh^3} Q_1 = \frac{12 \times 0.001\ Pa \cdot s \times 54\ mm}{150 \times 10^{-3}\ mm \times \left(50 \times 10^{-6} m\right)^3} \times \frac{2.5 \times 10^{-6} \times 10^{-3} m^3 / \min}{60\ s / \min} = 1.44 \times 10^{-3}\ Pa = 0.21\ psi < 1\ psi$$

So the constraint on the inlet pressure is met.

b). Now we assume that the channel height varies 10% across the chip in stepwise fashion and we will examine two cases: case 1, the height increases from 50 μm for channel 1, to 55 μm for channel 4, and case 2, the height decreases from 50 μm to 45 μm.

Since all the channels are designed to have the same width and same pressure drop, the flow rate is only a function of length and height,

$$Q \propto \frac{h^3}{l}$$

Therefore, the flow resistance ratios for both cases are,

$$case\,1 \quad Q_1 : Q_2 : Q_3 : Q_4 = \frac{50^3}{27} : \frac{51.67^3}{9} : \frac{53.33^3}{3} : \frac{55^3}{1} = 1 : 3.3 : 10.9 : 35.9$$

$$case\,2 \quad Q_1 : Q_2 : Q_3 : Q_4 = \frac{50^3}{27} : \frac{48.33^3}{9} : \frac{46.67^3}{3} : \frac{45^3}{1} = 1 : 2.7 : 7.3 : 19.7$$

As expected, since Q is proportional to the cube of the height, the flow rate is very sensitive to any height variation. And a 10% height variation across the chip can cause the flow rate to vary from –26% to 33% in this case for the longest channel.