6.642 Continuum Electromechanics
Fall 2008

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Problem Set 2 - Solutions

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Problem 1

Prob. 2.16.5

Gauss' law and $\mathbf{E} = -\nabla\Phi$ requires that if there is no free charge

$$\epsilon\nabla^2\Phi + \nabla\epsilon \cdot \nabla\Phi = 0 \tag{1}$$

For the given exponential dependence of the permittivity, the x dependence of the coefficients in this expression factors out and it again reduces to a constant coefficient expression

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} + 2\eta\frac{\partial\Phi}{\partial x} = 0 \tag{2}$$

In terms of the complex amplitude forms from Table 2.16.1, Eq. 2 requires that

$$\frac{d^2\tilde{\Phi}}{dx^2} + 2\eta\frac{d\tilde{\Phi}}{dx} - k^2\tilde{\Phi} = 0 \tag{3}$$

Thus, solutions have the form $exp(px)$ where $p = -\eta \pm \lambda$, $\lambda = \sqrt{k^2 + \eta^2}$.

The linear combination of these that satisfies the conditions that $\tilde{\Phi}$ be $\tilde{\Phi}^\alpha$ and $\tilde{\Phi}^\beta$ on the upper and lower surfaces respectively is as given in the problem. The displacement vector is then evaluated as

$$\tilde{D} = -\epsilon_\beta\left\{\tilde{\Phi}^\alpha e^{\eta(x+\Delta)}\frac{-\eta\sinh\lambda x + \lambda\cosh\lambda x}{\sinh\lambda\Delta} - \tilde{\Phi}^\beta e^{\eta x}\frac{-\eta\sinh\lambda(x-\Delta) + \lambda\cosh\lambda(x-\Delta)}{\sinh\lambda\Delta}\right\} \tag{4}$$

Evaluation of this expression at the respective surfaces then gives the transfer relations summarized in the problem.

Problem 2

Prob. 4.3.3

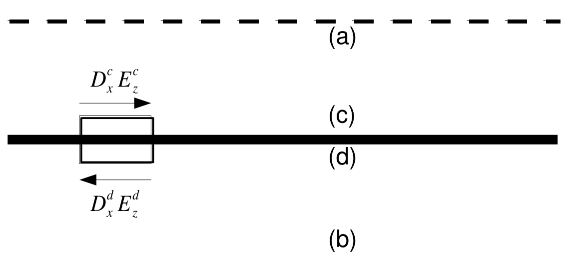

Figure 1: Force per unit area on middle charged surface (c/d) is found using the Maxwell stress tensor (Image by MIT OpenCourseWare.)

With positions as designated in the sketch, the total force per unit area is

$$< f_z >_z = < D_x^c E_z^c - D_x^d E_z^d >_z = \frac{1}{2}\Re(\tilde{D}_x^c \tilde{E}_z^{c*} - \tilde{D}_x^d \tilde{E}_z^{d*}) \tag{5}$$

With the understanding that the surface charge on the sheet is a given quantity, boundary conditions reflecting the continuity of tangential electric field at the three surfaces and that Gauss' law be satisfied through the sheet are

$$\tilde{\Phi}^a \text{ given}; \quad \tilde{\Phi}^b \text{ given}; \quad \tilde{\Phi}^c = \tilde{\Phi}^d; \quad \tilde{D}_x^c - \tilde{D}_x^d = \tilde{\sigma}_f \text{ given} \tag{6}$$

Bulk relations are given by Table 2.16.1. In the upper region

$$\begin{bmatrix} \tilde{D}_x^a \\ \tilde{D}_x^c \end{bmatrix} = \epsilon_0 k \begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ -\frac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^a \\ \tilde{\Phi}^c \end{bmatrix} \tag{7}$$

and in the lower

$$\begin{bmatrix} \tilde{D}_x^d \\ \tilde{D}_x^b \end{bmatrix} = \epsilon_0 k \begin{bmatrix} -\coth kd & \frac{1}{\sinh kd} \\ -\frac{1}{\sinh kd} & \coth kd \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^d \\ \tilde{\Phi}^b \end{bmatrix} \tag{8}$$

In view of Eq. 6, Eq. 5 becomes

$$< f_z >_z = \frac{1}{2}\Re[-jk\tilde{\sigma}_f \tilde{\Phi}^{c*}] \tag{9}$$

so what is now required is the amplitude $\tilde{\Phi}^c$. The surface charge, given by Eq. 6, as the difference $\tilde{D}_x^c - \tilde{D}_x^d$, follows in terms of the potentials from taking the difference of Eqs. 7 (second row) and 8 (first row). The resulting expression is solved for

$$\tilde{\Phi}^c = \frac{\tilde{\sigma}_f}{2\epsilon_0 k \coth kd} + \frac{\tilde{\Phi}^a + \tilde{\Phi}^b}{2\cosh kd} \tag{10}$$

Substituted into Eq. 9 (where the self terms in $\tilde{\sigma}_f \tilde{\sigma}_f^*$ are imaginary and can therefore be dropped) the force is expressed in terms of the given excitations

$$< f_z >_z = \frac{1}{2} k \Re\left[-j\tilde{\sigma}_f \frac{\tilde{\Phi}^{a*} + \tilde{\Phi}^{b*}}{2\cosh kd}\right] \tag{11}$$

b)

Translation of the given excitations into complex amplitudes gives

$$\tilde{\sigma}_f = -\sigma_0 e^{j\omega t} e^{jk\delta}, \quad \tilde{\Phi}^a = V_0 e^{j\omega t}, \quad \tilde{\Phi}^b = \pm V_0 e^{j\omega t} \tag{12}$$

Thus, with the even excitation, where $\Phi^a = \Phi^b$

$$< f_z >_z = -\frac{kV_0\sigma_0}{2\cosh kd} \sin k\delta \tag{13}$$

and with the odd excitation, $< f_z >_z = 0$.

c)

This is a specific case from part (b) with $\omega = 0$ and $\delta = \lambda/4$. Thus,

$$< f_z >_z = -\frac{kV_0\sigma_0}{2\cosh kd} \tag{14}$$

The sign is consistent with the sketch of charge distribution on the sheet and electric field due to the potentials on the walls sketched.

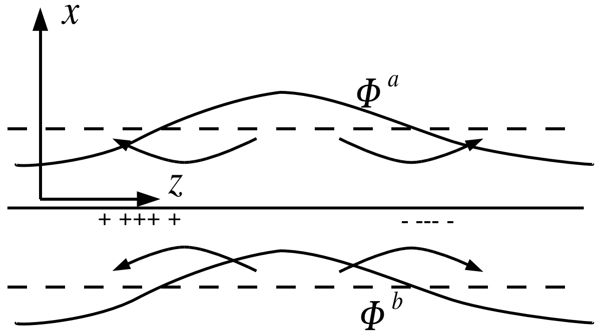

Figure 2: Potentials $\Phi_a = \Phi_b = -V_0 \cos kz$ shown with surface charge $\sigma_f = \sigma_0 \sin kz$ (Image by MIT OpenCourseWare.)

Problem 3 (Zahn, Problem 23, Chapter 1)

a)

Cartesian	Cylindrical	Spherical
$h_x = 1$	$h_r = 1$	$h_r = 1$
$h_y = 1$	$h_\phi = r$	$h_\theta = r$
$h_z = 1$	$h_z = 1$	$h_\phi = r\sin\theta$

b)

$$
\begin{aligned}
df &= \frac{\partial f}{\partial u}du + \frac{\partial f}{\partial v}dv + \frac{\partial f}{\partial w}dw \\
&= \nabla f \cdot \overline{d\ell} \\
&= \nabla f \cdot [h_u du \bar{i}_u + h_v dv \bar{i}_v + h_w dw \bar{i}_w] \\
(\nabla f)_u &= \frac{1}{h_u}\frac{\partial f}{\partial u}; \quad (\nabla f)_v = \frac{1}{h_v}\frac{\partial f}{\partial v}; \quad (\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w} \\
\nabla f &= \frac{1}{h_u}\frac{\partial f}{\partial u}\bar{i}_u + \frac{1}{h_v}\frac{\partial f}{\partial v}\bar{i}_v + \frac{1}{h_w}\frac{\partial f}{\partial w}\bar{i}_w
\end{aligned}
$$

c)

$$
\begin{aligned}
&dS_u = h_v h_w dv dw; \quad dS_v = h_u h_w du dw; \quad dS_w = h_u h_v du dv \\
&dV = h_u h_v h_w du dv dw
\end{aligned}
$$

d)

Divergence

$$
\Phi = \oint_S \bar{A} \cdot \overline{dS} = \int_{1,u} A_u h_v h_w dv dw - \int_{1',u-\Delta u} A_u h_v h_w dv dw
$$

$$
+ \int_{2,v+\Delta v} A_v h_u h_w du dw - \int_{2',v} A_v h_u h_w du dw
$$

$$+\int_{3,w+\Delta w} A_w h_u h_v du dv - \int_{3',w} A_w h_u h_v du dv$$

$$= \left\{ \frac{A_u h_v h_w|_u - A_u h_v h_w|_{u-\Delta u}}{\Delta u} + \frac{A_v h_u h_w|_{v+\Delta v} - A_v h_u h_w|_v}{\Delta v} + \frac{A_w h_u h_v|_{w+\Delta w} - A_w h_u h_v|_w}{\Delta w} \right\} \Delta u \Delta v \Delta w$$

$$\nabla \cdot \bar{A} = \lim_{\Delta u \to 0, \Delta v \to 0, \Delta w \to 0} \frac{\oint_S \bar{A} \cdot \overline{dS}}{\Delta V} = \frac{\oint_S \bar{A} \cdot \overline{dS}}{h_u h_v h_w \Delta u \Delta v \Delta w}$$

$$= \frac{1}{h_u h_v h_w} \left[\frac{\partial (h_v h_w A_u)}{\partial u} + \frac{\partial (h_u h_w A_v)}{\partial v} + \frac{\partial (h_u h_v A_w)}{\partial w} \right]$$

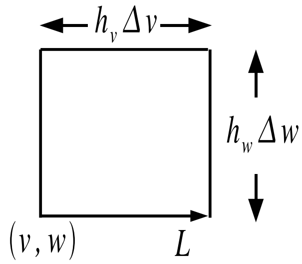

Figure 3: Contour for determining $(\nabla \times \bar{A})_u$ (Image by MIT OpenCourseWare.)

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$$(\nabla \times \bar{A})_u = \lim_{\Delta v \to 0, \Delta w \to 0} \frac{\oint_L \bar{A} \cdot \overline{d\ell}}{h_v h_w \Delta v \Delta w}$$

$$\oint_L \bar{A} \cdot \overline{d\ell} = [A_v h_v \Delta v|_w - A_v h_v \Delta v|_{w+\Delta w}] + [A_w h_w \Delta w|_{v+\Delta v} - A_w h_w \Delta w|_v]$$

$$(\nabla \times \bar{A})_u = \lim_{\Delta v \to 0, \Delta w \to 0} \frac{1}{h_v h_w} \left\{ \frac{A_v h_v|_w - A_v h_v|_{w+\Delta w}}{\Delta w} + \frac{A_w h_w|_{v+\Delta v} - A_w h_w|_v}{\Delta v} \right\}$$

$$= \frac{1}{h_v h_w} \left[\frac{\partial (h_w A_w)}{\partial v} - \frac{\partial (h_v A_v)}{\partial w} \right]$$

Similarly

$$(\nabla \times \bar{A})_v = \frac{1}{h_u h_w} \left[\frac{\partial (h_u A_u)}{\partial w} - \frac{\partial (h_w A_w)}{\partial u} \right]$$

$$(\nabla \times \bar{A})_w = \frac{1}{h_u h_v} \left[\frac{\partial (h_v A_v)}{\partial u} - \frac{\partial (h_u A_u)}{\partial v} \right]$$

e)

$$\nabla^2 f = \nabla \cdot (\nabla f) = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} \left(\frac{h_v h_w}{h_u} \frac{\partial f}{\partial u} \right) + \frac{\partial}{\partial v} \left(\frac{h_u h_w}{h_v} \frac{\partial f}{\partial v} \right) + \frac{\partial}{\partial w} \left(\frac{h_u h_v}{h_w} \frac{\partial f}{\partial w} \right) \right]$$