

6.641 Electromagnetic Fields, Forces, and Motion
Spring 2009

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Lecture 17: Transient Waves on Transmission Lines

I. Wave Equation (Loss Less)

$$\frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t}$$

$$\Rightarrow \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t} \quad c^2 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon\mu}}$$

Solution: $v(z, t) = V_+ \left(t - \frac{z}{c} \right) + V_- \left(t + \frac{z}{c} \right)$

Proof: Let $\alpha = t - \frac{z}{c} \Rightarrow \frac{\partial \alpha}{\partial t} = 1, \frac{\partial \alpha}{\partial z} = -\frac{1}{c}$ (Positive z directed waves)

Superposition: $v_+(z, t) = V_+ \left(t - \frac{z}{c} \right) = V_+(\alpha)$

$$\frac{\partial v_+}{\partial t} = \frac{dv_+}{d\alpha} \frac{\partial \alpha}{\partial t} = \frac{dv_+}{d\alpha}$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 v_+}{d\alpha^2} \frac{\partial \alpha}{\partial t} = \frac{d^2 v_+}{d\alpha^2}$$

$$\frac{\partial v_+}{\partial z} = \frac{dv_+}{d\alpha} \frac{\partial \alpha}{\partial z} = -\frac{1}{c} \frac{dv_+}{d\alpha}$$

$$\frac{\partial^2 v_+}{\partial z^2} = -\frac{1}{c} \frac{d^2 v_+}{d\alpha^2} \frac{\partial \alpha}{\partial z} = +\frac{1}{c^2} \frac{d^2 v_+}{d\alpha^2}$$

$$\frac{\partial^2 v_+}{\partial t^2} = \frac{d^2 v_+}{d\alpha^2} = c^2 \frac{\partial^2 v}{\partial z^2} = c^2 \left(\frac{1}{c^2} \frac{d^2 v_+}{d\alpha^2} \right) = \frac{d^2 v_+}{d\alpha^2}$$

Negative z directed waves: Let $\beta = t + \frac{z}{c} \Rightarrow \frac{\partial \beta}{\partial t} = 1, \frac{\partial \beta}{\partial z} = \frac{1}{c}$

$$\frac{\partial v_-}{\partial t} = \frac{dv_-}{d\beta} \frac{\partial \beta}{\partial t} = \frac{dv_-}{d\beta}$$

$$\frac{\partial^2 v_-}{\partial t^2} = \frac{d^2 v_-}{d\beta^2} \frac{\partial \beta}{\partial t} = \frac{d^2 v_-}{d\beta^2}$$

$$\frac{\partial v_-}{\partial z} = \frac{dv_-}{d\beta} \frac{\partial \beta}{\partial z} = \frac{1}{c} \frac{dv_-}{d\beta}$$

$$\frac{\partial^2 v_-}{\partial z^2} = \frac{1}{c} \frac{d^2 v_-}{d\beta^2} \frac{\partial \beta}{\partial z} = \frac{1}{c^2} \frac{d^2 v_-}{d\beta^2}$$

$$\frac{\partial^2 v_-}{\partial t^2} = \frac{d^2 v_-}{d\beta^2} = c^2 \frac{\partial^2 v_-}{\partial z^2} = c^2 \left(\frac{1}{c^2} \frac{d^2 v_-}{d\beta^2} \right) = \frac{d^2 v_-}{d\beta^2}$$

II. Solution for current $i(z, t)$

$$\frac{\partial v}{\partial z} = -L \frac{\partial i}{\partial t}$$

$$\Rightarrow \frac{\partial^2 i}{\partial t^2} = c^2 \frac{\partial^2 i}{\partial z^2}$$

$$\frac{\partial i}{\partial z} = -C \frac{\partial v}{\partial t}$$

$$\text{Solution: } i(z, t) = I_+ \left(t - \frac{z}{c} \right) + I_- \left(t + \frac{z}{c} \right)$$

$$v(z, t) = V_+ \left(t - \frac{z}{c} \right) + V_- \left(t + \frac{z}{c} \right)$$

$$+z \text{ Solution: } \alpha = t - \frac{z}{c}, \frac{\partial \alpha}{\partial t} = 1, \frac{\partial \alpha}{\partial z} = -\frac{1}{c}$$

$$\begin{aligned} \frac{\partial v_+}{\partial z} &= -L \frac{\partial i_+}{\partial t} \Rightarrow \frac{dv_+}{d\alpha} \frac{\partial \alpha}{\partial z} = -\frac{1}{c} \frac{dv_+}{d\alpha} = -L \frac{di_+}{d\alpha} \frac{\partial \alpha}{\partial t} \\ &= -L \frac{di_+}{d\alpha} \end{aligned}$$

$$\frac{dv_+}{d\alpha} = Lc \frac{di_+}{d\alpha} = \frac{L}{\sqrt{LC}} \frac{di_+}{d\alpha} = \sqrt{\frac{L}{C}} \frac{di_+}{d\alpha}$$

$$v_+ = i_+ Z_0 \Rightarrow I_+ \left(t - \frac{z}{c} \right) = Y_0 V_+ \left(t - \frac{z}{c} \right)$$

$$Y_0 = \sqrt{\frac{C}{L}} = \frac{1}{Z_0}$$

-z Solution: $\beta = t + \frac{z}{c}$, $\frac{\partial \beta}{\partial t} = 1$, $\frac{\partial \beta}{\partial z} = \frac{1}{c}$

$$\begin{aligned} \frac{\partial v_-}{\partial z} &= -L \frac{\partial i_-}{\partial t} \Rightarrow \frac{dv_-}{d\beta} \frac{\partial \beta}{\partial z} = \frac{1}{c} \frac{dv_-}{d\beta} = -L \frac{di_-}{d\beta} \frac{\partial \beta}{\partial t} \\ &= -L \frac{di_-}{d\beta} \end{aligned}$$

$$\frac{dv_-}{d\beta} = -Lc \frac{di_-}{d\beta} = \frac{-L}{\sqrt{LC}} \frac{di_-}{d\beta} = -\sqrt{\frac{L}{C}} \frac{di_-}{d\beta}$$

$$v_- = -Z_0 i_- \Rightarrow I_- \left(t + \frac{z}{c} \right) = -Y_0 V_- \left(t + \frac{z}{c} \right)$$

$$v(z, t) = V_+ \left(t - \frac{z}{c} \right) + V_- \left(t + \frac{z}{c} \right)$$

$$i(z, t) = Y_0 \left[V_+ \left(t - \frac{z}{c} \right) - V_- \left(t + \frac{z}{c} \right) \right]$$

III. Transmission Line Transient Waves

A. Transients on Infinitely Long Transmission Lines

1. Initial Conditions

$$v(z, t = 0) = 0 = V_+ \left(-\frac{z}{c} \right) + V_- \left(\frac{z}{c} \right) = 0$$

$$i(z, t = 0) = 0 = Y_0 \left[V_+ \left(-\frac{z}{c} \right) - V_- \left(\frac{z}{c} \right) \right] = 0$$

$$V_+ \left(-\frac{z}{c} \right) = 0, \quad V_- \left(\frac{z}{c} \right) = 0$$

$$z > 0, \quad t > 0 \Rightarrow t + \frac{z}{c} > 0 \Rightarrow V_- \left(t + \frac{z}{c} \right) = 0$$

$$t - \frac{z}{c} > 0 \quad \text{if} \quad t > \frac{z}{c} \quad \text{to allow} \quad V_+ \left(t - \frac{z}{c} \right) \neq 0$$

$$\text{With } V_- \left(t + \frac{z}{c} \right) = 0 \Rightarrow v(z, t) = V_+ \left(t - \frac{z}{c} \right)$$

$$i(z, t) = Y_0 V_+ \left(t - \frac{z}{c} \right) \Rightarrow \frac{v(z, t)}{i(z, t)} = Z_0$$

2. Traveling Wave Solution with Source Resistance R_s

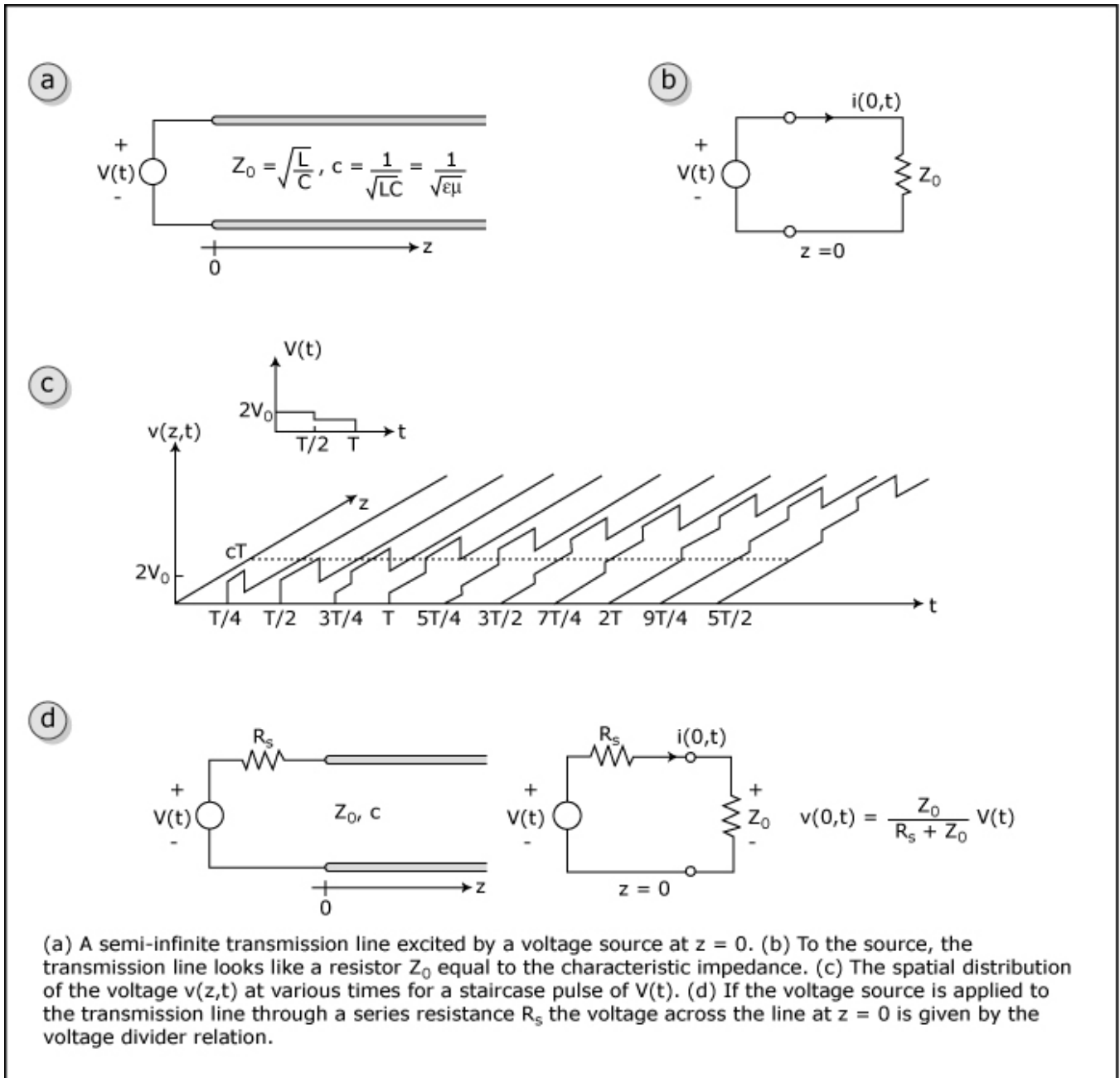
$$v(z = 0, t) = V(t) = V_+(t)$$

$$v(z = 0, t) = \frac{Z_0}{Z_0 + R_s} V(t) = V_+(t)$$

$$i(z = 0, t) = Y_0 V_+(t) = \frac{V(t)}{R_s + Z_0}$$

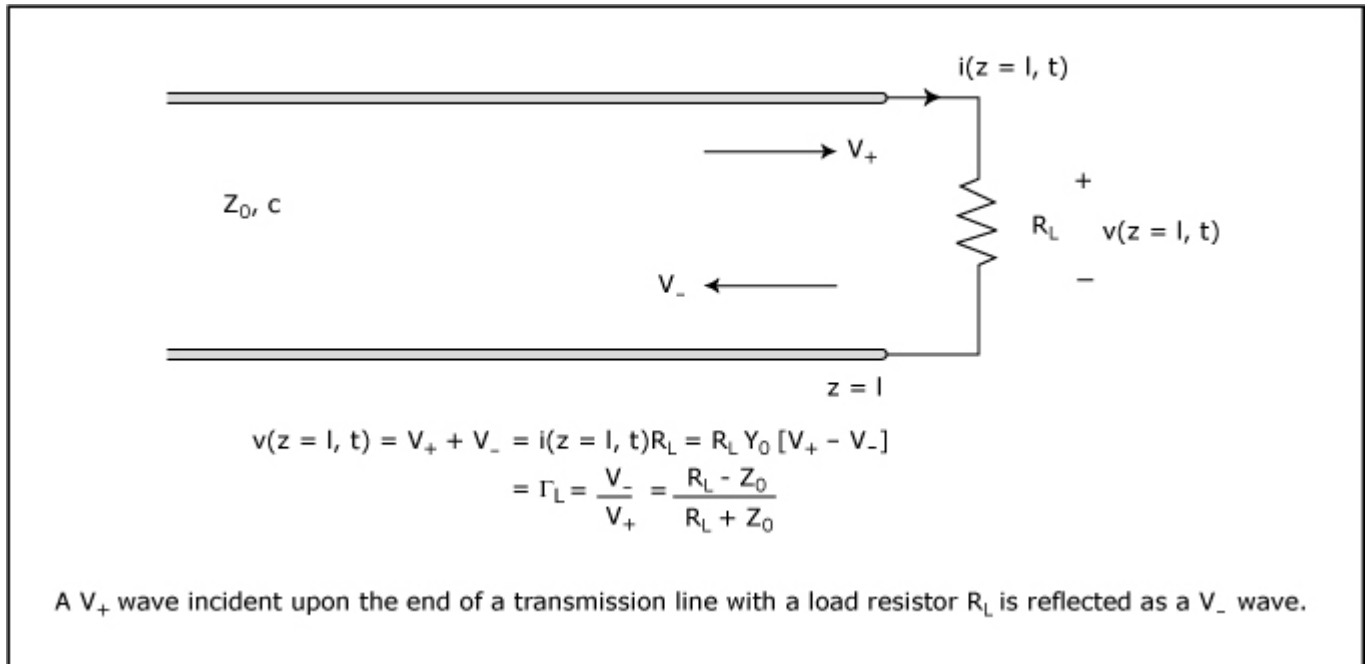
$$v(z, t) = \frac{Z_0}{Z_0 + R_s} V\left(t - \frac{z}{c}\right)$$

$$i(z, t) = \frac{V\left(t - \frac{z}{c}\right)}{R_s + Z_0}$$



B. Reflections from Resistive Terminations

1. Reflection Coefficient



$$\text{at } z = l : v(l, t) = V_+(t - l/c) + V_-(t + l/c)$$

$$= i(l, t)R_L$$

$$= Y_0 R_L \left[V_+ \left(t - \frac{l}{c} \right) - V_-(t + l/c) \right]$$

$$\Gamma_L = \frac{V_-(t + l/c)}{V_+(t - l/c)} = \frac{R_L - Z_0}{R_L + Z_0}$$

Special Cases

i) $R_L = Z_0 \Rightarrow \Gamma_L = 0$ (matched line)

ii) $R_L = 0 \Rightarrow \Gamma_L = -1$ (short circuited line)

If $R_L < Z_0$, $-1 < \Gamma_L < 0$

iii) $R_L = \infty \Rightarrow \Gamma_L = +1$ (open circuited line)

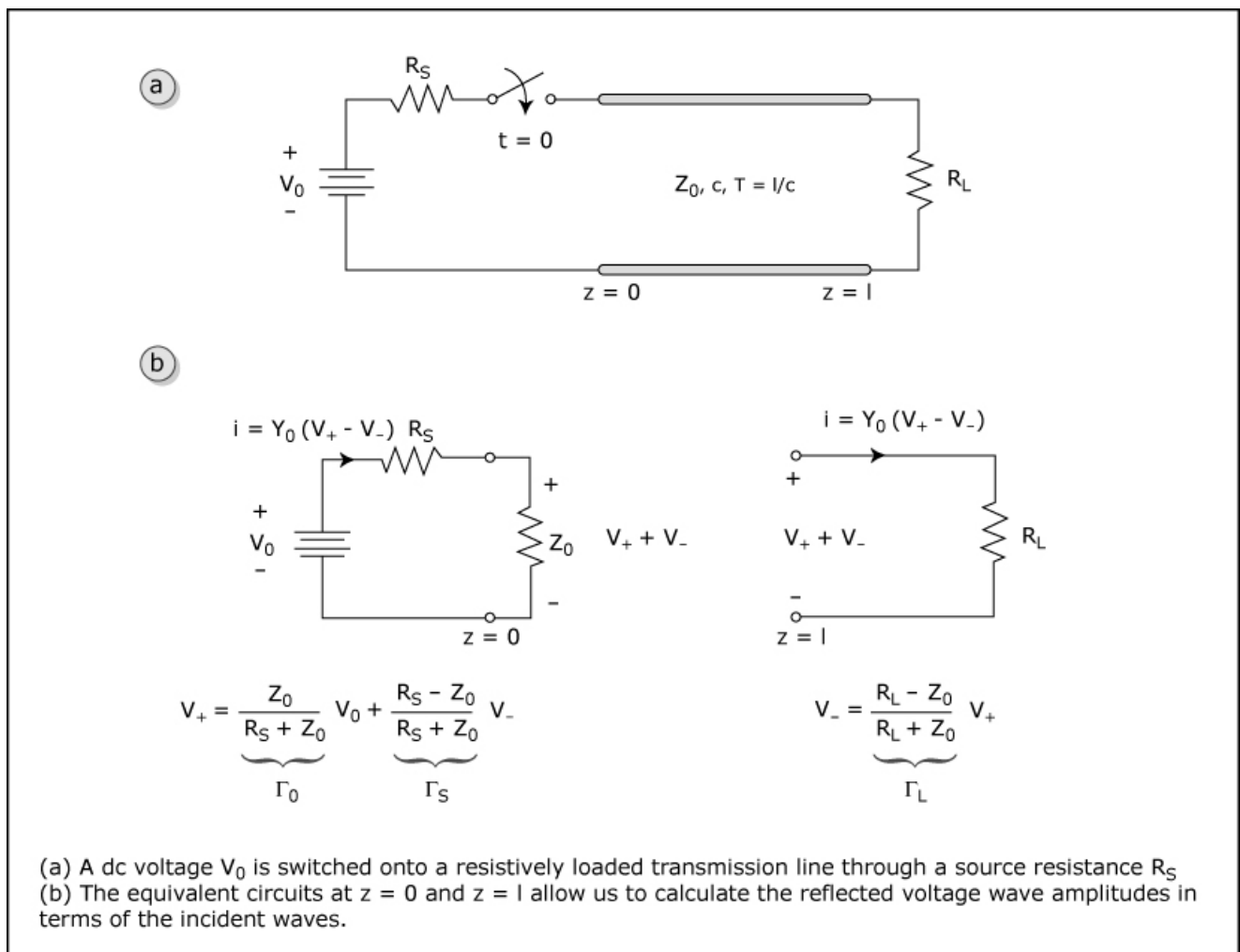
If $R_L > Z_0$, $0 < \Gamma_L < 1$

2. Step Voltage

At $z = 0$: $v(z = 0, t) + i(0, t)R_S = V_0$

$V_+(z = 0, t) + V_-(z = 0, t) + Y_0 R_S [V_+(z = 0, t) - V_-(z = 0, t)] = V_0$

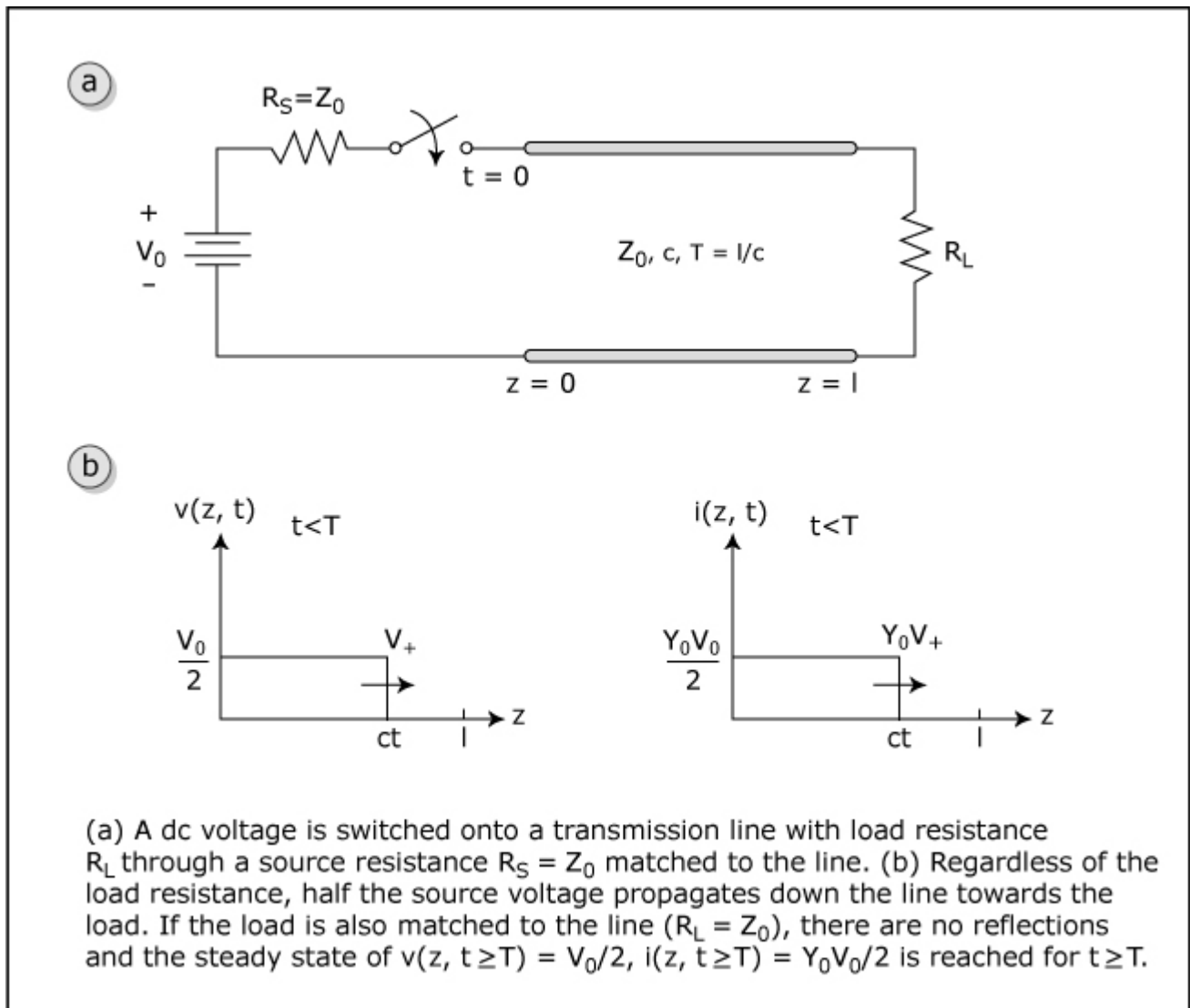
$V_+(z = 0, t) = \Gamma_S V_-(z = 0, t) + \frac{Z_0 V_0}{Z_0 + R_S}$, $\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0}$



a. Matched Line: $R_L = Z_0$, $\Gamma_L = 0$; $R_S = Z_0$, $\Gamma_S = 0$

$$\Gamma_L = 0 \Rightarrow V_-\left(t + \frac{z}{c}\right) = 0$$

$$V_+(z=0, t) = \frac{V_0}{2}, \text{ In steady state after time } T = \frac{l}{c}$$

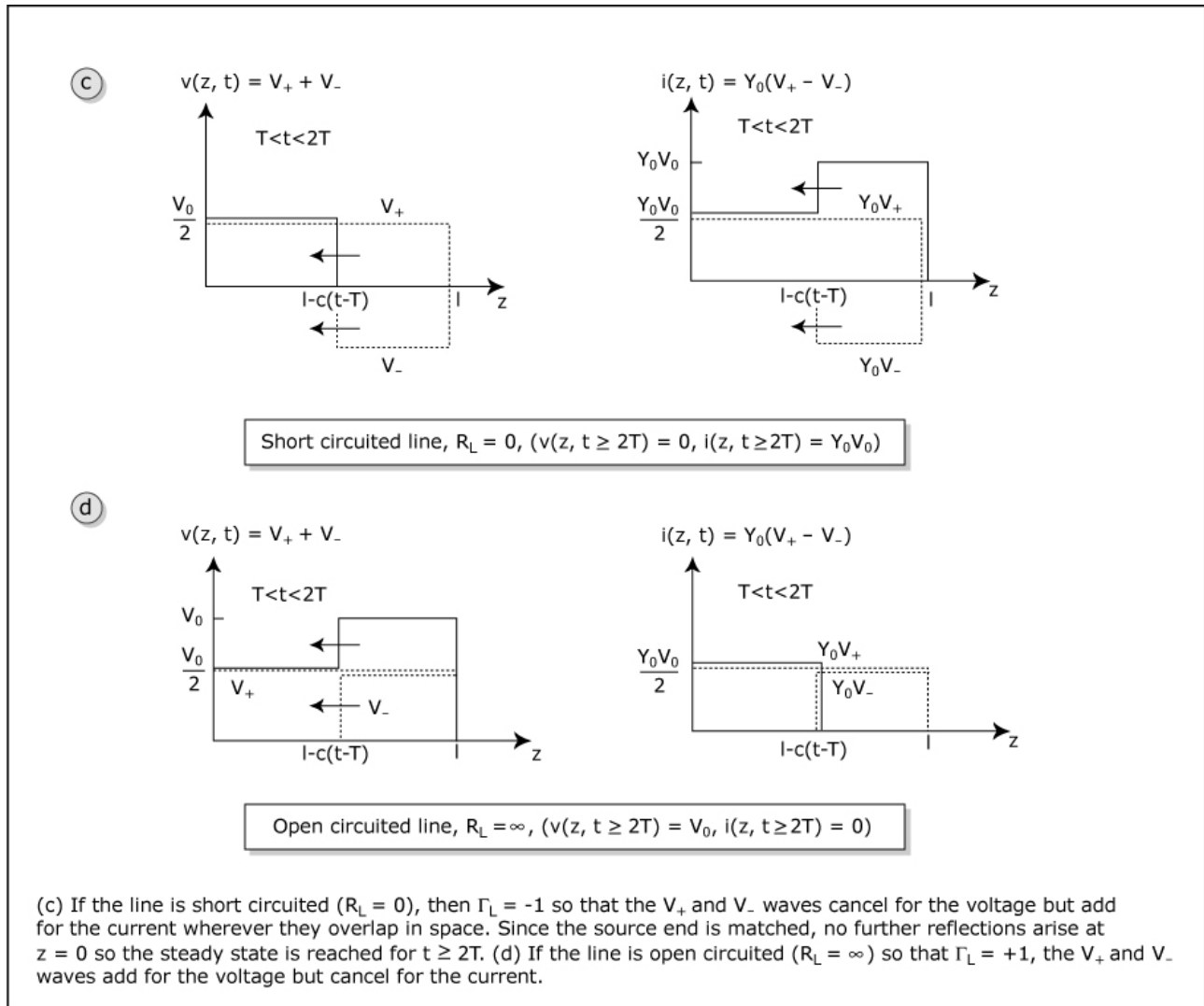


b. Short circuited line: $R_L = 0$, $\Gamma_L = -1$, $R_s = Z_0$, $\Gamma_s = 0$

$\Gamma_L = -1 \Rightarrow V_+ = -V_-$. When $V_+ \left(t - \frac{z}{c} \right)$ and $V_- \left(t + \frac{z}{c} \right)$ overlap

in space, $v(z, t) = 0$. For $t \geq 2T = 2l/c$, $v(z, t) = 0$,

$$i(z, t) = V_0/Z_0$$



c. Open circuited line: $R_L = \infty$, $\Gamma_L = +1$, $R_s = Z_0$, $\Gamma_s = 0$

$$\Gamma_L = +1 \Rightarrow V_+ = +V_-$$

For $t \geq 2T = 2l/c$, $v(z, t) = V_0$, $i(z, t) = 0$

3. Approach to the dc Steady State (neither end matched)

$$z = 0: \quad V_+(t) = \Gamma_0 V_0 + \Gamma_s V_-(t), \quad \Gamma_0 = \frac{Z_0}{R_s + Z_0}, \quad \Gamma_s = \frac{R_s - Z_0}{R_s + Z_0}$$

$$z = l: \quad V_-\left(t + \frac{l}{c}\right) = \Gamma_L V_+\left(t - \frac{l}{c}\right), \quad \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0}$$

at $z = l$:

$$V_{+n} = \Gamma_0 V_0 + \Gamma_s V_{-(n-1)}$$

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)}$$

$$V_{+n} = \Gamma_0 V_0 + \Gamma_s \Gamma_L V_{+(n-1)} \quad \Rightarrow \quad V_{+n} - \Gamma_s \Gamma_L V_{+(n-1)} = \Gamma_0 V_0$$

Particular Solution: $V_{+n} = \text{constant}$

$$\text{Constant} (1 - \Gamma_s \Gamma_L) = \Gamma_0 V_0 \Rightarrow \text{Constant} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L}$$

Homogeneous Solution: $V_{+n} - \Gamma_s \Gamma_L V_{+(n-1)} = 0$

Try a solution of the form: $V_{+n} = A \lambda^n$

$$A(\lambda^n - \Gamma_s \Gamma_L \lambda^{n-1}) \Rightarrow \lambda = \Gamma_s \Gamma_L$$

$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} + A (\Gamma_s \Gamma_L)^n$$

Initial Condition:

$$V_{+1} = \Gamma_0 V_0 = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} + A (\Gamma_s \Gamma_L) \Rightarrow A = -\frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L}$$

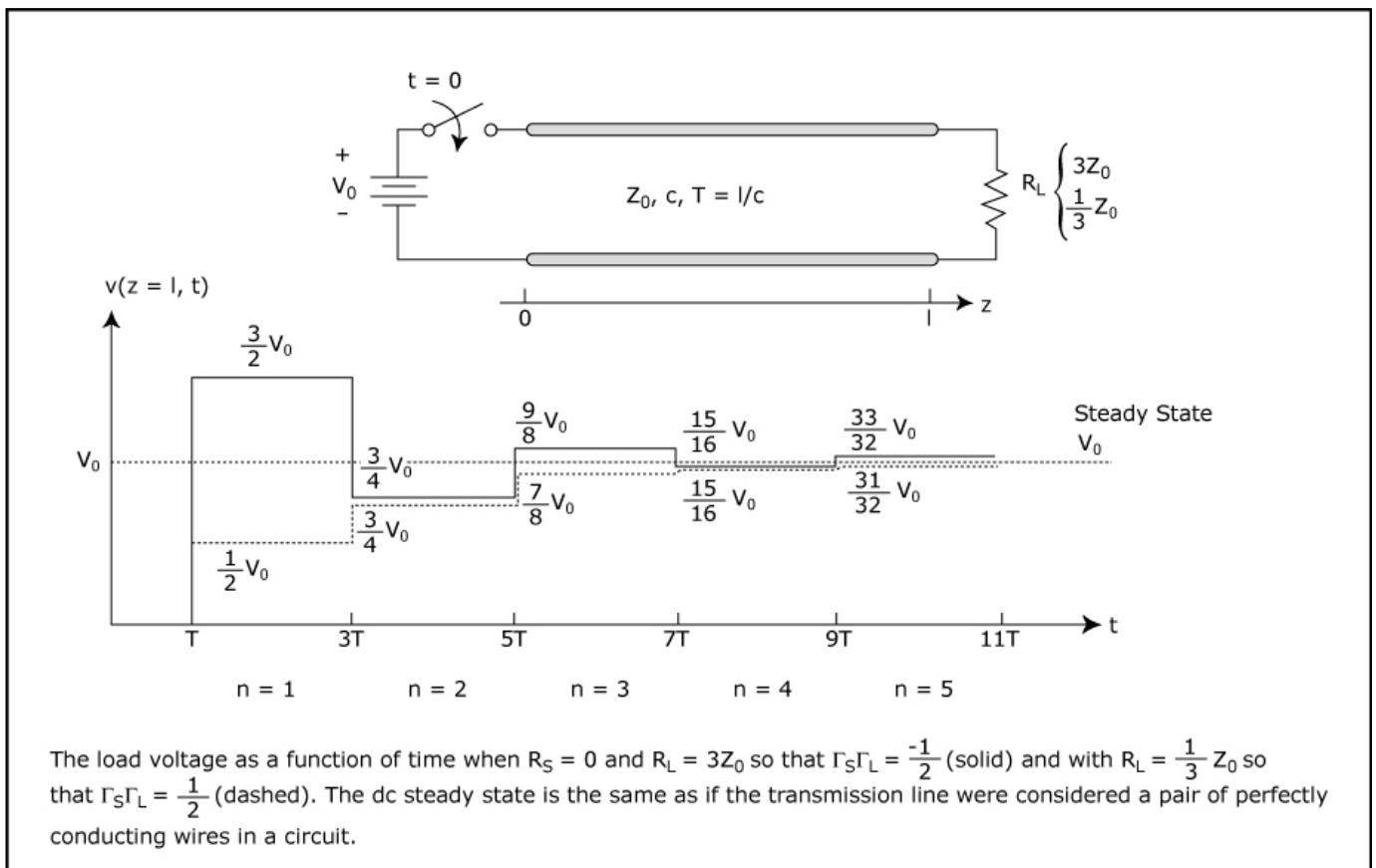
$$V_{+n} = \frac{\Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} \left[1 - (\Gamma_s \Gamma_L)^n \right]$$

$$V_{-(n-1)} = \Gamma_L V_{+(n-1)} \Rightarrow V_{-n} = \Gamma_L V_{+n}$$

$$V_n = V_{+n} + V_{-n} = V_{+n} (1 + \Gamma_L) = \frac{V_0 (1 + \Gamma_L) \Gamma_0 V_0}{1 - \Gamma_s \Gamma_L} [1 - (\Gamma_s \Gamma_L)^n]$$

$$= \frac{R_L}{R_L + R_s} V_0 [1 - (\Gamma_s \Gamma_L)^n]$$

$$\lim_{n \rightarrow \infty} V_n = \frac{R_L}{R_L + R_s} V_0$$



$$I_n = Y_0 [V_{+n} - V_{-n}] = Y_0 (1 - \Gamma_L) V_{+n} = \frac{Y_0 (1 - \Gamma_L) \Gamma_0 V_0 [1 - (\Gamma_s \Gamma_L)^n]}{1 - \Gamma_s \Gamma_L}$$

$$= V_0 [1 - (\Gamma_s \Gamma_L)^n] / (R_L + R_s)$$

a. Special Case: $R_s = 0, R_L = 3Z_0$

$$\Gamma_s = -1, \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{2}{4} = \frac{1}{2} \Rightarrow \Gamma_s \Gamma_L = -\frac{1}{2}$$

$$z = l \quad V_n = V_0 \left[1 - \left(-\frac{1}{2} \right)^n \right]$$

$$I_n = \frac{V_0}{3Z_0} \left[1 - \left(-\frac{1}{2} \right)^n \right]$$

b. Special Case: $R_s = 0, R_L = \frac{1}{3}Z_0$

$$\Gamma_s = -1, \Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} = \frac{-\frac{2}{3}}{\frac{4}{3}} = -\frac{1}{2} \Rightarrow \Gamma_s \Gamma_L = +\frac{1}{2}$$

$$z = l \quad V_n = V_0 \left[1 - \left(\frac{1}{2} \right)^n \right]$$

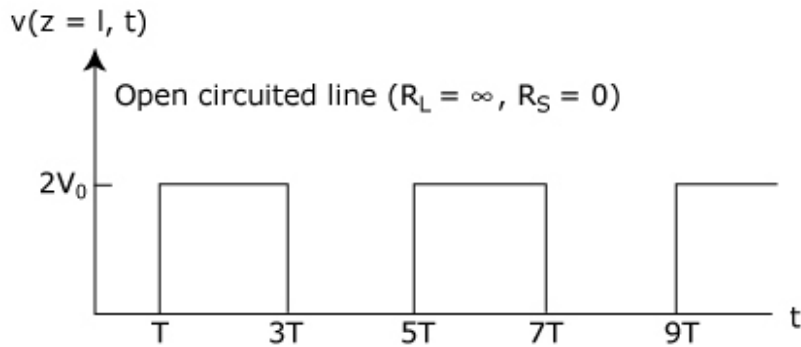
c. Special Case: $R_s = 0, R_L = \infty$ (open circuit)

$$\Gamma_s \Gamma_L = -1$$

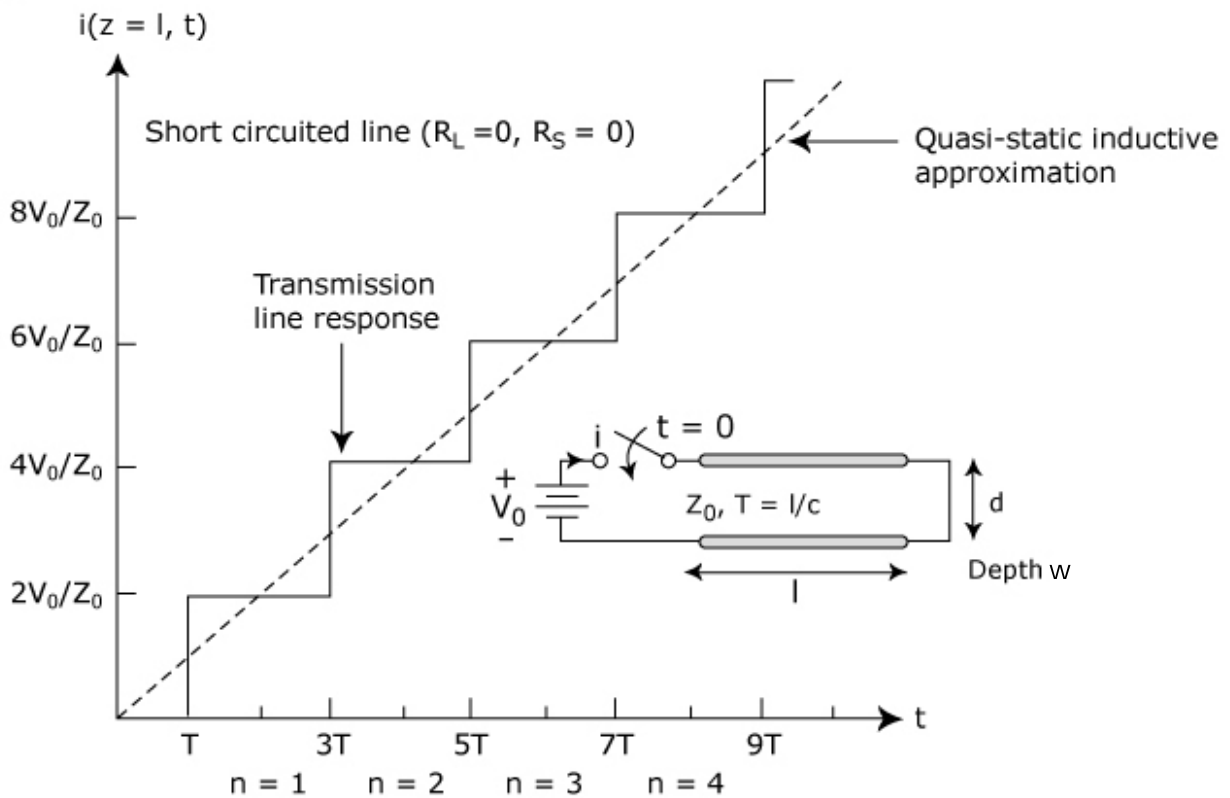
$$V_n = \frac{R_L}{R_s + R_L} V_0 \left[1 - (\Gamma_s \Gamma_L)^n \right] = V_0 (1 - (-1)^n)$$

$$= \begin{cases} 0 & n \text{ even} \\ 2V_0 & n \text{ odd} \end{cases}$$

(a)



(b)



The (a) open circuit voltage and (b) short circuit current at the $z = l$ end of the transmission line for $R_S = 0$. No dc steady state is reached because the system is lossless. If the short circuited transmission line is modeled as an inductor in the quasi-static limit, a step voltage input results in a linearly increasing current (shown dashed). The exact transmission line response is the solid staircase waveform.

d. Special Case: $R_S = 0$, $R_L = 0$ (Short circuit)

$$\Gamma_S \Gamma_L = +1$$

$$I_n = \frac{V_0}{R_L + R_S} \left[1 - (\Gamma_S \Gamma_L)^n \right] \text{ Indeterminate}$$

$$\Gamma_S = \frac{R_S - Z_0}{R_S + Z_0} = \frac{\frac{R_S}{Z_0} - 1}{\frac{R_S}{Z_0} + 1} \approx - \left(1 - \frac{R_S}{Z_0} \right)^2 \approx - \left(1 - \frac{2R_S}{Z_0} \right)$$

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \approx - \left(1 - \frac{2R_L}{Z_0} \right)$$

$$I_n = \frac{V_0}{R_L + R_S} \left[1 - \left[\left(1 - \frac{2R_L}{Z_0} \right) \left(1 - \frac{2R_S}{Z_0} \right) \right]^n \right]$$

$$\approx \frac{V_0}{R_L + R_S} \left[1 - \left(1 - \frac{2(R_L + R_S)}{Z_0} \right)^n \right]$$

$$\approx \frac{V_0}{R_L + R_S} \left[1 - 1 + 2n \frac{(R_L + R_S)}{Z_0} \right]$$

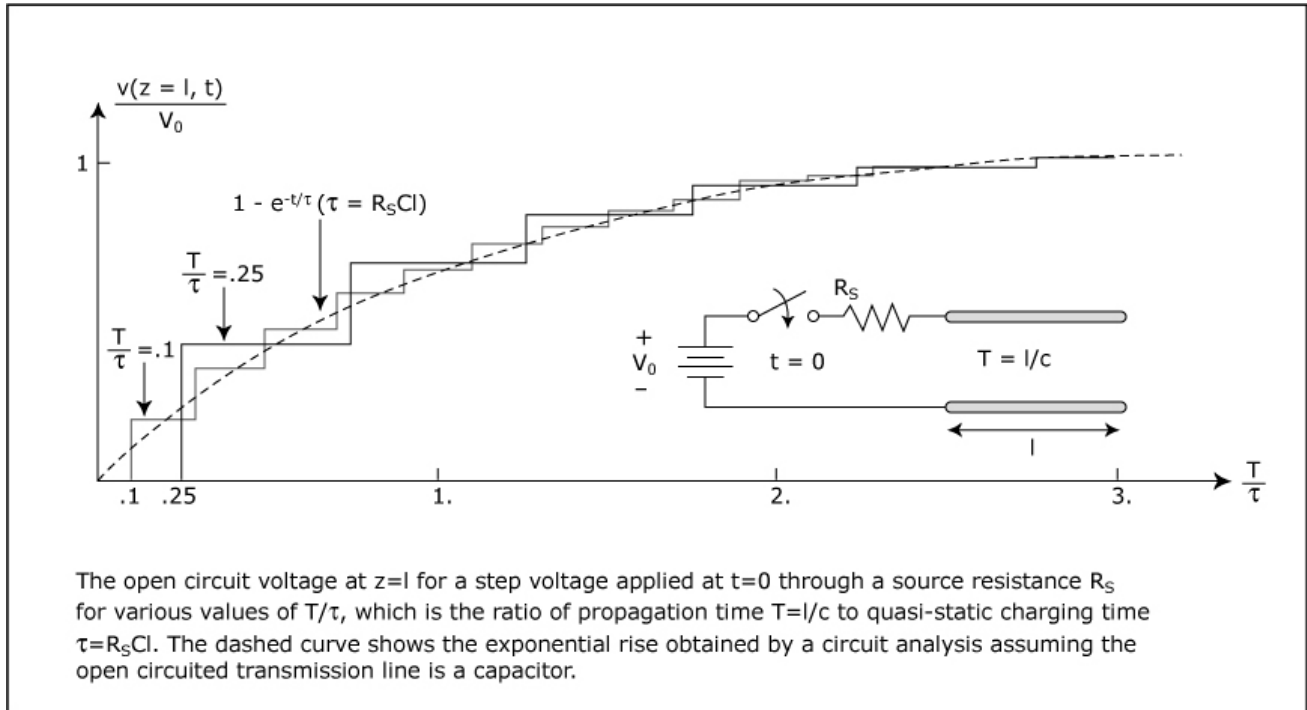
$$= \frac{V_0 2n}{Z_0}$$

Approximates Inductor

$$V_0 = (LI) \frac{di}{dt} \Rightarrow i = \frac{V_0}{LI} t$$

e. Special Case: $R_L = \infty$ (Open Circuit) , R_S finite

$$\Gamma_L = 1 \Rightarrow V_n = V_0 [1 - \Gamma_S^n]$$

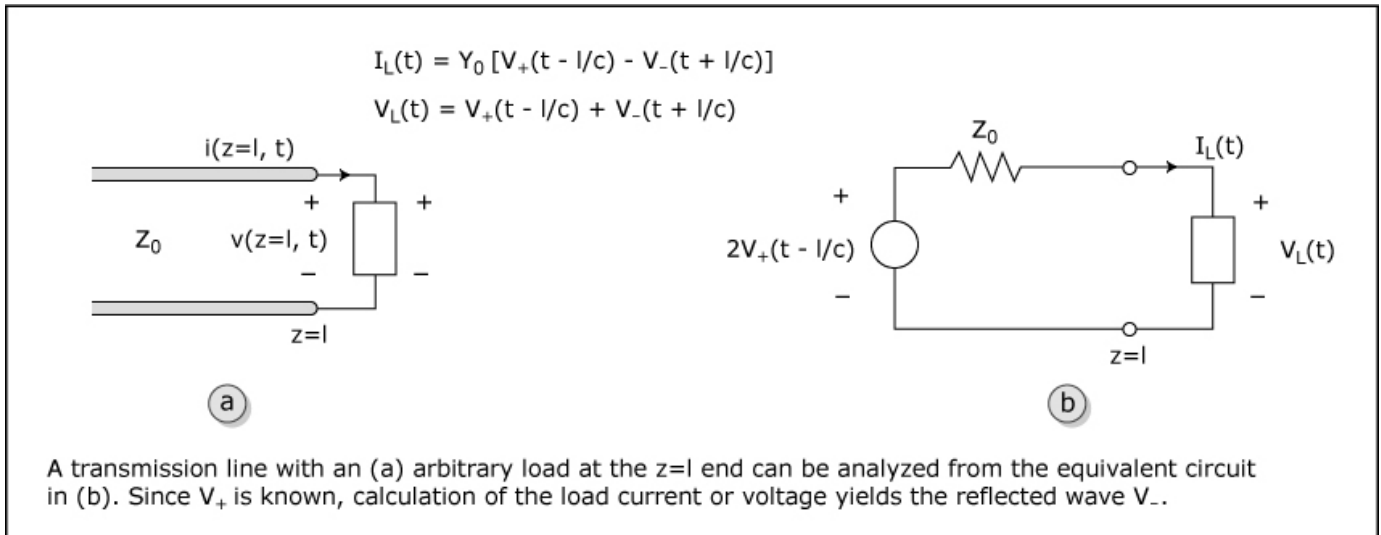


Approximate transmission line as capacitor being charged though resistor R_S

$$v(t) = V_0 (1 - e^{-t/\tau})$$

$$T = R_S C l$$

C. Reflections from Arbitrary Terminations



$$v(z=l, t) = V_L(t) = V_+ \left(t - \frac{l}{c} \right) + V_- \left(t + \frac{l}{c} \right)$$

$$i(z=l, t) = I_L(t) = Y_0 \left[V_+ \left(t - \frac{l}{c} \right) - V_- \left(t + \frac{l}{c} \right) \right]$$

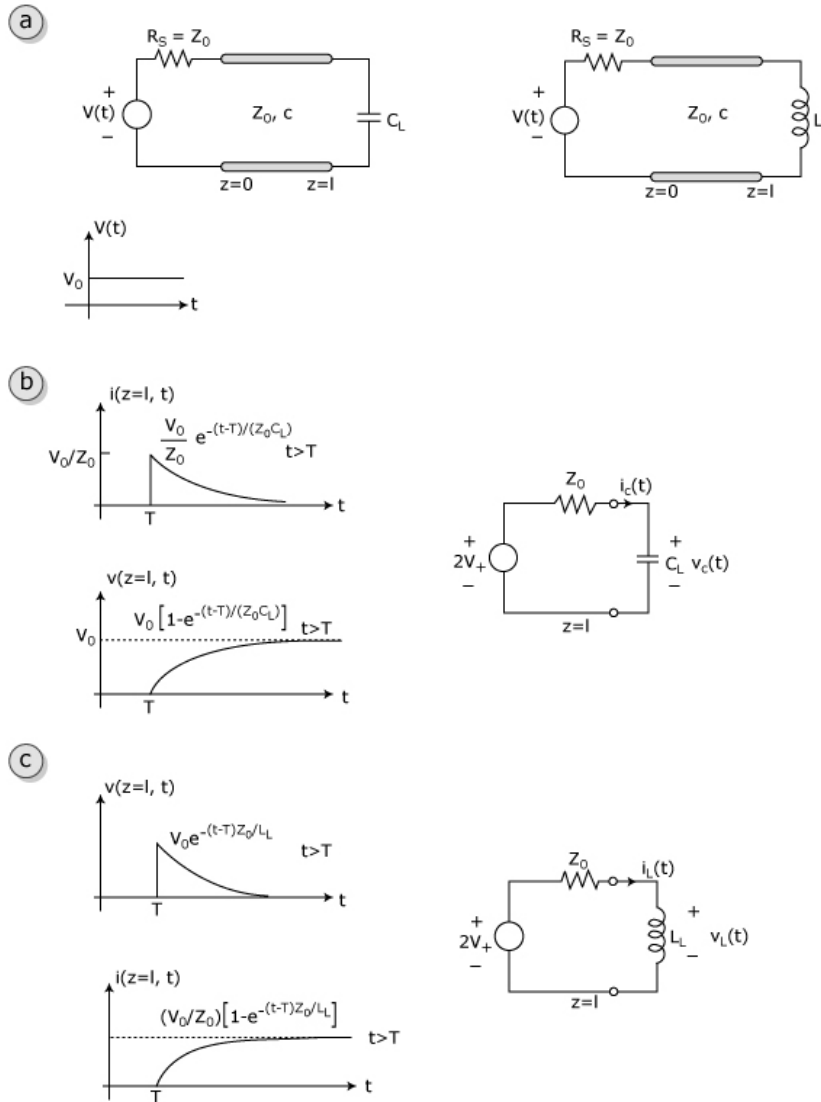
$$\text{Eliminate } V_- \left(t + \frac{l}{c} \right) \Rightarrow 2V_+ \left(t - \frac{l}{c} \right) = V_L(t) + I_L(t) Z_0$$

$$V_- \left(t + \frac{l}{c} \right) = V_L(t) - V_+ \left(t - \frac{l}{c} \right)$$

a. Capacitor C_L at $z=l$, $R_S = Z_0 \Rightarrow V_+ = \frac{V_0}{2}$

$$t > T \quad V_L(t) = v_c(t), \quad I_L(t) = C_L \frac{dv_c}{dt}$$

$$Z_0 C_L \frac{dv_c}{dt} + v_c = 2V_+ = V_0 \quad t > T$$



(a) A step voltage is applied to transmission lines loaded at $z=l$ with a capacitor C_L or inductor L_L . The load voltage and current are calculated from the (b) resistive-capacitive or (c) resistive-inductive equivalent circuits at $z=l$ to yield exponential waveforms with respective time constants $\tau = Z_0 C_L$ and $\tau = L_L / Z_0$ as the solutions approach the dc steady state. The waveforms begin after the initial V_+ wave arrives at $z=l$ after a time $T = l/c$. There are no further reflections as the source end is matched.

$$v_c(t) = V_0 \left[1 - e^{-(t-T)/(Z_0 C_L)} \right] \quad t > T$$

$$T = l/c$$

$$V_- = v_c(t) - V_+$$

$$= \frac{-V_0}{2} + V_0 \left[1 - e^{-(t-T)/(Z_0 C_L)} \right]$$

$$= \frac{V_0}{2} - V_0 e^{-(t-T)/(Z_0 C_L)}$$

$$i_c = C_L \frac{dv_c}{dt} = \frac{V_0}{Z_0} e^{-(t-T)/(Z_0 C_L)} \quad t > T$$

b. Inductor L_L at $z = l$

$$L_L \frac{di_L}{dt} + i_L Z_0 = 2V_+ = V_0 \quad t > T$$

$$i_L = \frac{V_0}{Z_0} \left(1 - e^{-(t-T)Z_0/L_L} \right) \quad t > T$$

$$v_L = L_L \frac{di_L}{dt} = V_0 e^{-(t-T)Z_0/L_L} \quad t > T$$

IV. Linear Constant Coefficient Difference Equations

$$D_n = \det \begin{bmatrix} a & c & 0 & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ c & a & c & 0 & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & c & a & c & 0 & 0 & 0 & 0 & - & - & - & - \\ 0 & 0 & c & a & c & 0 & 0 & 0 & - & - & - & - \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$\xleftarrow{\quad n \quad} \xrightarrow{\quad}$

\uparrow
 n
 \downarrow

$$= a \begin{bmatrix} \overbrace{a \ c \ 0 \ 0 \ 0 \ 0}^{n-1} \ - \ - \ - \ - \ - \ - \\ c \ a \ c \ 0 \ 0 \ 0 \ - \ - \ - \ - \ - \ - \\ 0 \ c \ a \ c \ 0 \ 0 \ - \ - \ - \ - \ - \ - \\ 0 \ 0 \ c \ a \ c \ 0 \ - \ - \ - \ - \ - \ - \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \end{bmatrix} \begin{matrix} \uparrow \\ n-1 \\ \downarrow \end{matrix}$$

$$-c^2 \begin{bmatrix} \overbrace{a \ c \ 0 \ 0 \ 0 \ 0}^{n-2} \ - \ - \ - \ - \ - \ - \\ c \ a \ c \ 0 \ 0 \ 0 \ - \ - \ - \ - \ - \ - \\ 0 \ c \ a \ c \ 0 \ 0 \ - \ - \ - \ - \ - \ - \\ 0 \ 0 \ c \ a \ c \ 0 \ - \ - \ - \ - \ - \ - \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \\ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \ \vdots \end{bmatrix} \begin{matrix} \uparrow \\ n-2 \\ \downarrow \end{matrix}$$

$$D_n = aD_{n-1} - c^2 D_{n-2}$$

$$D_n - a D_{n-1} + c^2 D_{n-2} = 0$$

$$D_n = A\lambda^n$$

$$A[\lambda^n - a\lambda^{n-1} + c^2 \lambda^{n-2}] = 0 \Rightarrow A\lambda^{n-2}[\lambda^2 - a\lambda + c^2] = 0$$

$$\lambda^2 - a\lambda + c^2 = 0 \Rightarrow \lambda = \frac{a}{2} \pm \left[\left(\frac{a}{2} \right)^2 - c^2 \right]^{1/2}$$

$$= c \left[\frac{a}{2c} \pm \left[\left(\frac{a}{2c} \right)^2 - 1 \right]^{1/2} \right]$$

$$\text{Let } \frac{a}{2c} = \cos \theta$$

$$\begin{aligned} \lambda &= c \left[\cos \theta \pm \left[\cos^2 \theta - 1 \right]^{1/2} \right] \\ &= c \left[\cos \theta \pm \left[-\sin^2 \theta \right]^{1/2} \right] \\ &= c \left[\cos \theta \pm j \sin \theta \right] \\ &= c e^{\pm j\theta} \end{aligned}$$

$$\begin{aligned} D_n &= A_1 (c e^{j\theta})^n + A_2 (c e^{-j\theta})^n \\ &= c^n \left[A_1 e^{jn\theta} + A_2 e^{-jn\theta} \right] \end{aligned}$$

$$D_1 = a = c \left[A_1 e^{j\theta} + A_2 e^{-j\theta} \right]$$

$$D_2 = a^2 - c^2 = c^2 \left[A_1 e^{j2\theta} + A_2 e^{-j2\theta} \right]$$

$$A_1 e^{j\theta} + A_2 e^{-j\theta} = \frac{a}{c}$$

$$\Rightarrow A_1 = \frac{e^{j\theta}}{2j \sin \theta}$$

$$A_2 = \frac{-e^{-j\theta}}{2j \sin \theta}$$

$$A_1 e^{j2\theta} + A_2 e^{-j2\theta} = \left(\frac{a}{c} \right)^2 - 1$$

$$D_n = \frac{c^n}{2j \sin \theta} \left[e^{j(n+1)\theta} - e^{-j(n+1)\theta} \right] = \frac{c^n \sin[(n+1)\theta]}{\sin \theta}$$

$$\text{Check: } D_1 = c \frac{\sin 2\theta}{\sin \theta} = 2 \cos \theta = 2c \left(\frac{a}{2c} \right) = a$$

$$D_2 = c^2 \frac{\sin 3\theta}{\sin \theta} = c^2 \left[4 \cos^2 \theta - 1 \right] = c^2 \left[4 \left(\frac{a}{2c} \right)^2 - 1 \right] = a^2 - c^2$$