6.641 Electromagnetic Fields, Forces, and Motion
Spring 2005

6.641 — Electromagnetic Fields, Forces, and Motion	Spring 2005
Quiz 2 - Solutions 2003	
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Problem 1

A

Question: Calculate the electric scalar potential inside and outside the cylinder at time $t = 0$.

Solution:

$$\Phi(r,\phi,t=0) = \begin{cases} A(t=0)r^3 \sin 3\phi & 0 \leq r \leq R \\ \frac{B(t=0)}{r^3} \sin 3\phi & r \geq R \end{cases}$$

$$E_r(r,\phi,t=0) = -\frac{\partial \Phi(r,\phi,t=0)}{\partial r} = \begin{cases} -3A(t=0)r^2 \sin 3\phi & 0 \leq r < R \\ \frac{3B(t=0)}{r^4} \sin 3\phi & r > R \end{cases}$$

$$\epsilon_2 E_r(r=R_+,\phi,t=0) - \epsilon_1 E_r(r=R_-,\phi,t=0) = \rho_s(t=0)\big|_{r=R} = \rho_{s_0} \sin 3\phi$$

$$\frac{3\epsilon_2 B(t=0)}{R^4} \cancel{\sin 3\phi} + 3\epsilon_1 A(t=0) R^2 \cancel{\sin 3\phi} = \rho_{s_0} \cancel{\sin 3\phi}$$

$$\Phi(r=R_+,\phi,t=0) = \Phi(r=R_-,\phi,t=0) \Rightarrow A(t=0)R^3 = \frac{B(t=0)}{R^3}$$

$$B(t=0) = A(t=0)R^6$$

$$3R^2 A(t=0)\left[\epsilon_1 + \epsilon_2\right] = \rho_{s_0}$$

$$A(t=0) = \frac{B(t=0)}{R^6} = \frac{\rho_{s_0}}{3R^2(\epsilon_1+\epsilon_2)}$$

$$\Phi(r,\phi,t=0) = \begin{cases} \frac{\rho_{s_0}}{3R^2(\epsilon_1+\epsilon_2)} r^3 \sin 3\phi & 0 \leq r \leq R \\ \frac{\rho_{s_0} R^4}{3(\epsilon_1+\epsilon_2)} \frac{\sin 3\phi}{r^3} & r \geq R \end{cases}$$

B

Question: Calculate the electric scalar potential inside and outside the cylinder for $t \geq 0$.

Solution:

$$\Phi(R,\phi,t) = \begin{cases} A(t)r^3 \sin 3\phi & 0 \leq r \leq R \\ \frac{B(t)}{r^3} \sin 3\phi & r \geq R \end{cases}$$

$$\Phi(r = R_+, \phi, t) = \Phi(r = R_-, \phi, t) \Rightarrow A(t)R^3 = \frac{B(t)}{R^3}$$
$$B(t) = A(t)R^6$$

$$\sigma_1 E_r(r = R_-, \phi, t) + \epsilon_1 \frac{\partial E_r(r = R_-, \phi, t)}{\partial t} = \sigma_2 E_r(r = R_+, \phi, t) + \epsilon_2 \frac{\partial E_r(r = R_+, \phi, t)}{\partial t}$$

$$E_r(r, \phi, t) = -\frac{\partial \Phi(r, \phi, t)}{\partial r} = \begin{cases} -3A(t)r^2 \sin 3\phi & 0 \le r < R \\ \frac{3B(t)}{r^4} \sin 3\phi & r > R \end{cases}$$

$$\begin{aligned} -3\sigma_1 R^2 A(t) - 3R^2 \epsilon_1 \frac{dA}{dt} &= \frac{3\sigma_2}{R^4} B(t) + \frac{3\epsilon_2}{R^4} \frac{dB}{dt} \\ &= 3R^2(\sigma_2 A(t) + \epsilon_2 \frac{dA}{dt}) \\ -\sigma_1 A(t) - \epsilon_1 \frac{dA}{dt} &= \sigma_2 A(t) + \epsilon_2 \frac{dA}{dt} \end{aligned}$$

$$\begin{aligned} (\epsilon_1 + \epsilon_2)\frac{dA}{dt} + (\sigma_1 + \sigma_2)A(t) = 0 \\ \frac{dA}{dt} + \frac{A(t)}{\tau} = 0; \tau = \frac{\epsilon_1 + \epsilon_2}{\sigma_1 + \sigma_2} \\ A(t) = A(t = 0)e^{-\frac{t}{\tau}} = \frac{\rho_{s_0}}{3R^2(\epsilon_1 + \epsilon_2)} e^{-\frac{t}{\tau}} \end{aligned}$$

$$\Phi(r, \phi, t) = \begin{cases} \frac{\rho_{s_0}}{3R^2(\epsilon_1+\epsilon_2)} r^3 \sin 3\phi e^{-\frac{t}{\tau}} & 0 \le r \le R \\ \frac{\rho_{s_0} R^4}{3(\epsilon_1+\epsilon_2)} \frac{\sin 3\phi}{r^3} e^{-\frac{t}{\tau}} & r \ge R \end{cases}$$

C

Question: What is the free surface charge density $\rho_s(r = R, t)$ for $t \ge 0$?

Solution:

$$\begin{aligned} \rho_s(r = R, t) &= \epsilon_2 E_r(R_+, \phi, t) - \epsilon_1 E_r(R_-, \phi, t) \\ &= \frac{3A(t)}{R^4} R^6 \epsilon_2 \sin 3\phi + 3A(t)R^2 \epsilon_1 \sin 3\phi \\ &= 3R^2 \sin 3\phi(\epsilon_1 + \epsilon_2)A(t) \\ &= \cancel{3R^2} \sin 3\phi \cancel{(\epsilon_1 + \epsilon_1)} \frac{\rho_{s_0}}{\cancel{3R^2}\cancel{(\epsilon_1 + \epsilon_2)}} e^{-\frac{t}{\tau}} \\ &= \rho_{s_0} \sin 3\phi e^{-\frac{t}{\tau}} \end{aligned}$$

Problem 2

A

Question: Determine the force of electric origin that acts on the upper capacitor plate in the direction of increasing x, as a function of the applied voltage V, the plate deflection x, and the parameters of the model.

Solution:

$$C(x) = \frac{\epsilon_0 A}{G - x}, f_x = \frac{1}{2}V^2\frac{dC}{dx} = \frac{1}{2}\frac{V^2\epsilon_0 A}{(G-x)^2}$$

B

Question: Determine a differential equation that describes the time response of the deflection x as the upper capacitor plate is deflected by the force of electric origin. Neglect gravity.

Solution:

$$M\frac{d^2x}{dt^2} = -Kx + \frac{\epsilon_0 AV^2}{2(G-x)^2}$$

C

Question: Assume that the deflection of the upper capacitor plate is in static equilibrium for a given voltage V. Determine a relation between the equilibrium deflection $x = X$, the applied voltage V, and the parameters of the model. You need not explicitly solve for X.

Solution: At equilibrium, $\frac{dx}{dt} = 0 \Rightarrow KX = \frac{\epsilon_0 AV^2}{2(G-X)^2}$.

D

Question: We wish to determine the voltage at which the equilibrium found in Part c becomes unstable. To do so, let $x = X + x'$ and linearize the differential equation found in Part b for small displacements x' from the equilibrium X, where $x' \ll X$.

Solution:

$$\begin{aligned} x &= X + x' \\ M\frac{d^2x'}{dt^2} &= -Kx' + \frac{\epsilon_0 AV^2}{\not{2}}\frac{\not{2}}{(G-X)^3}x' = \left[-K + \frac{\epsilon_0 AV^2}{(G-X)^3}\right]x' \\ M\frac{d^2x'}{dt^2} &= -K\left[1 - \frac{2X}{(G-X)}\right]x' \\ &= -K\frac{[G-3X]}{[G-X]}x' \end{aligned}$$

E

Question: Combine the answers to Parts c and d to show that the deflection of the upper plate becomes unstable (as determined by the dynamics of the linearized differential equation found in Part d) when the upper plate deflects a fraction of the gap G. Also, determine this fraction.

Solution: Unstable if $G - 3X < 0$. Incipience: $X = \frac{G}{3}$.

F

Question: Determine the voltage in terms of the parameters of the model at the onset of instability determined in Part e.

Solution:

$$-K + \frac{\epsilon_0 A V^2}{(G-X)^3} = 0 = -K + \frac{\epsilon_0 A V^2}{\left(\frac{2}{3}G\right)^3} = -K + \frac{27}{8}\frac{\epsilon_0 A V^2}{G^3}$$

$$V = \left[\frac{8KG^3}{27\epsilon_0 A}\right]^{\frac{1}{2}}$$

Problem 3

A

Question: Find the magnetic field $\bar{H}$ in each gap within the magnetic circuit. Neglect fringing field effects.

Solution:

$$H_a a = NI$$
$$H_b b = NI$$
$$H_a = \frac{NI}{a}, H_b = \frac{NI}{b}$$

B

Question: Find the self-inductance $L(x)$ of the coil as a function of block position x.

Solution:

$$\Phi = \mu_b H_b s_b d + H_a d\left(\mu_a x + \mu_0\left(s_a - x\right)\right)$$
$$= NId\left(\frac{\mu_b s_b}{b} + \frac{1}{a}\left(\mu_a x + \mu_0\left(s_a - x\right)\right)\right)$$
$$\lambda = N\Phi = N^2 Id\left[\frac{\mu_b s_b}{b} + \frac{1}{a}\left(\mu_a x + \mu_0\left(s_a - x\right)\right)\right]$$
$$L(x) = \frac{\lambda}{I} = N^2 d\left[\frac{\mu_b s_b}{b} + \frac{1}{a}\left(\mu_a x + \mu_0\left(s_a - x\right)\right)\right]$$

C

Question: Find the magnetic force on the movable block.

Solution:

$$f_x = \frac{1}{2} I^2 \frac{dL}{dx} = \frac{1}{2} \frac{N^2 I^2 d}{a} (\mu_a - \mu_0)$$

D

Question: What is the governing differential equation for the position of the movable block?

Solution:

$$M \frac{d^2 x}{dt^2} = -Kx + \frac{1}{2} N^2 \frac{I^2 d}{a} (\mu_a - \mu_0) = f_T(x)$$

E

Question: Find the equilibrium position $x = x_{eq}$ of the movable block assuming $0 < x < s_a$.

Solution:

$$f_T(x) = f_x - Kx_{eq} = 0 = \frac{1}{2} N^2 \frac{I^2 d}{a} (\mu_a - \mu_0) - Kx_{eq}$$

$$x_{eq} = \frac{1}{2} \frac{N^2 I^2 d}{Ka}$$

F

Question: Is this equilibrium stable or unstable?

Solution:

$$\left. \frac{df_T}{dx} \right|_{x = x_{eq}} = -K < 0 \text{ (stable)}$$

G

Question: The movable block of mass M is slightly displaced from its equilibrium at x_{eq} by an amount Δx and released with velocity $\left. \frac{dx}{dt} \right|_{t=0} = v_0$. To first order calculate $x'(t)$ where $x(t) = x_{eq} + x'(t)$ with $x'(t) \ll x_{eq}$. Neglect friction.

Solution:

$$M\frac{d^2x'}{dt^2} = \left.\frac{df_T}{dx}\right|_{x_{eq}} x' = -Kx' \Rightarrow \frac{d^2x'}{dt^2} + \omega_0^2 x' = 0, \omega_0^2 = \frac{K}{M}$$

$$\begin{aligned}
x' &= A\sin\omega_0 t + B\cos\omega_0 t \\
\frac{dx'}{dt} &= \omega_0\left[A\cos\omega_0 t - B\sin\omega_0 t\right] \\
x'(t=0) &= B = \Delta x \\
\left.\frac{dx'}{dt}\right|_{t=0} &= \omega_0 A = v_0 \Rightarrow A = \frac{v_0}{\omega_0} \\
x'(t) &= \frac{v_0}{\omega_0}\sin\omega_0 t + \Delta x\cos\omega_0 t
\end{aligned}$$