Lecture Number 3
Fall 2016

Jeffrey H. Shapiro
©2006, 2008, 2012, 2015

Date: Thursday, September 15, 2016

Dirac-notation Quantum Mechanics.

Introduction

In Lecture 2 we established the following fundamentals of Dirac-notation quantum mechanics, as shown on Slides 2–6.

- A quantum-mechanical system $\mathcal{S}$ is a physical system governed by the laws of quantum mechanics.

- The state of a quantum mechanical system at a particular time t is the sum total of all information that can be known about the system at time t. It is a ket vector, $|\psi(t)\rangle$, in an appropriate Hilbert space $\mathcal{H}_\mathcal{S}$ of possible states. Finite energy states have unit length ket vectors, i.e., $\langle\psi(t)|\psi(t)\rangle = 1$.

- For $t \geq 0$, an isolated system with initial state $|\psi(0)\rangle$ evolves according to the Schrödinger equation

$$j\hbar\frac{\mathrm{d}|\psi(t)\rangle}{\mathrm{d}t} = \hat{H}|\psi(t)\rangle, \quad \text{for } t \geq 0, \tag{1}$$

where $\hat{H}$ is the Hamiltonian (energy) operator and $\hbar = h/2\pi$ is Planck's constant divided by 2π. This leads to

$$|\psi(t)\rangle = \hat{U}(t,0)|\psi(0)\rangle, \tag{2}$$

where $\hat{U}(t,0)$ is the unitary time-evolution operator associated with the Schrödinger equation.

- An observable is a measurable dynamical variable of the quantum system $\mathcal{S}$. It is represented by an Hermitian operator that has a complete set of eigenkets.

- For a quantum system $\mathcal{S}$ that is in state $|\psi(t)\rangle$ at time t, measurement of the observable

$$\hat{O} = \sum_n o_n |o_n\rangle\langle o_n|, \tag{3}$$

where $\{|o_n\rangle\}$ are the observable's orthonormal eigenkets and $\{o_n\}$ are its associated eigenvalues (assumed to be distinct), yields an outcome that is one of these eigenvalues according to the probability distribution

$$\Pr(o_n) = |\langle o_n|\psi(t)\rangle|^2. \tag{4}$$

If the observable that is measured has a continuum of non-degenerate eigenvalues, so that

$$\hat{O} = \int_{-\infty}^{\infty} \mathrm{d}o \, o |o\rangle\langle o|, \tag{5}$$

where the associated (infinite-length) orthonormal eigenkets are $\{|o\rangle\}$, then the probability density for obtaining the outcome o is

$$p(o) = |\langle o|\psi(t)\rangle|^2. \tag{6}$$

Today we will complete our foundational work on Dirac-notation quantum mechanics. We begin by continuing our treatment of quantum measurement statistics.

Moment Equations for Quantum Measurements

Probability mass functions and probability density functions provide complete statistical characterizations of discrete and continuous classical random variables, respectively. However, for many applications more limited—and hence incomplete—statistics will suffice. In particular, if x is a real-valued classical random variable, then its mean value

$$\langle x \rangle = \begin{cases} \sum_n X_n \Pr(X_n), & \text{for } x \text{ discrete valued} \\ \int \mathrm{d}X \, X p(X), & \text{for } x \text{ continuous valued}, \end{cases} \tag{7}$$

gives us useful information about the deterministic part of x, i.e., its *signal* component. The deviation of x from its mean,

$$\Delta x \equiv x - \langle x \rangle, \tag{8}$$

is then the random (noise) component of x. It is a zero-mean random variable whose mean-squared strength is the variance of x,

$$\mathrm{var}(x) = \langle \Delta x^2 \rangle = \begin{cases} \sum_n (X_n - \langle x \rangle)^2 \Pr(X_n), & \text{for } x \text{ discrete valued} \\ \int \mathrm{d}X \, (X - \langle x \rangle)^2 p(X), & \text{for } x \text{ continuous valued}. \end{cases} \tag{9}$$

2

By the linearity of expectation—which you are familiar with from your probability prerequisite—we have that

$$\text{var}(x) = \langle x^2 \rangle - \langle x \rangle^2, \tag{10}$$

a relation that we will use from time to time in performing variance calculations. Although knowledge of the mean and variance of x is far less information about its statistics than knowing its full characterization, it is nonetheless sufficient to evaluate the signal-to-noise ratio,

$$\text{SNR} \equiv \frac{\langle x \rangle^2}{\text{var}(x)}, \tag{11}$$

which gives us a quantitative measure of how noisy this random variable is. In particular, the Chebyschev inequality from probability theory can be cast in the following form:

$$\Pr\left(\left| \frac{x - \langle x \rangle}{\langle x \rangle} \right| \geq \delta \right) \leq 1/\delta^2 \text{SNR}, \quad \text{for } \langle x \rangle \neq 0 \text{ and any } \delta > 0. \tag{12}$$

Thus, for example, a random variable with a $60\,\text{dB}$ SNR (SNR $= 10^6$) has at most a 1% probability of being more than 1% away from its mean value.[1]

In light of the preceding remarks, you should not be surprised that much of our study of quantum measurement statistics will be limited to mean values and variances. To see how to simplify the calculations of these moments for observable measurements we turn to Slide 7. Suppose $\hat{O}$ has a discrete eigenvalue spectrum, with distinct eigenvalues. The mean value of the $\hat{O}$ measurement at time t is then

$$\langle \hat{O} \rangle \equiv \sum_n o_n \Pr(o_n) = \sum_n o_n |\langle o_n | \psi(t) \rangle|^2 = \sum_n o_n \langle \psi(t) | o_n \rangle \langle o_n | \psi(t) \rangle \tag{13}$$

$$= \langle \psi(t) | \left(\sum_n o_n | o_n \rangle \langle o_n | \right) | \psi(t) \rangle = \langle \psi(t) | \hat{O} | \psi(t) \rangle, \tag{14}$$

so that it can be calculated *without* explicitly evaluating $\Pr(o_n)$. You should verify that higher-order moments for this observable can be found via,[2]

$$\langle \hat{O}^k \rangle \equiv \sum_n o_n^k \Pr(o_n) = \langle \psi(t) | \hat{O}^k | \psi(t) \rangle, \quad \text{for } k = 2, 3, \ldots, \tag{15}$$

and thus

$$\text{var}(\hat{O}) = \langle \Delta \hat{O}^2 \rangle = \langle \hat{O}^2 \rangle - \langle \hat{O} \rangle^2 = \langle \psi(t) | \hat{O}^2 | \psi(t) \rangle - \langle \psi(t) | \hat{O} | \psi(t) \rangle^2. \tag{16}$$

---

[1]The Chebyschev inequality is very general, and so it is very weak. If x is a Gaussian random variable, then we have that the probability in (12) is no more than $e^{-\delta^2 \text{SNR}/2}$, which is far smaller than $1/\delta^2 \text{SNR}$ when δ^2 SNR $\gg 1$.

[2]To do so you will want to show that $\hat{O}^k = \sum_n o_n^k | o_n \rangle \langle o_n |$, which follows from the eigenket-eigenvalue relation for $\hat{O}$.

The corresponding results for an observable whose eigenvalue spectrum is the continuum $-\infty < o < \infty$ and non-degenerate are found in a similar manner,[3]

$$
\begin{aligned}
\langle \hat{O}^k \rangle &= \int_{-\infty}^{\infty} \mathrm{d}o \, o^k p(o) = \int_{-\infty}^{\infty} \mathrm{d}o \, o^k |\langle o|\psi(t)\rangle|^2 \tag{17} \\[2mm]
&= \int_{-\infty}^{\infty} \mathrm{d}o \, o^k \langle \psi(t)|o\rangle\langle o|\psi(t)\rangle = \langle \psi(t)| \left(\int_{-\infty}^{\infty} \mathrm{d}o \, o^k |o\rangle\langle o| \right) |\psi(t)\rangle \tag{18} \\[2mm]
&= \langle \psi(t)|\hat{O}^k|\psi(t)\rangle, \quad \text{for } k = 1, 2, 3, \ldots \tag{19}
\end{aligned}
$$

From this result it follows that Eq. (16) can also be used for observables whose eigenvalue spectra are continuous.

Schrödinger versus Heisenberg Pictures

First treatments of quantum mechanics—including what we have done so far—almost invariably take the Schrödinger equation route to characterizing the time evolution of a quantum system. Here, for an isolated system, the state vector evolves in time but the observables are time-independent operators. Ultimately, we'd like to have appropriate quantum versions for Maxwell's equations. Here, the electric and magnetic fields should be (3-D vector) observables that evolve in space and time. Then, for a quantized electromagnetic wave propagating in a source-free region of empty space, we would expect that the state vector should be constant if no measurements are made. We can convert our Dirac-notation quantum mechanics to an approach that has observables evolving in time and state vectors that are constants by going to the Heisenberg picture. We'll make that change today, after which we will stick with the Heisenberg picture for the rest of the semester.

Consider the descriptions given on Slide 8 for the Schrödinger and Heisenberg pictures of an isolated quantum system $\mathcal{S}$ for $t \geq 0$. The Schrödinger picture has time-independent observables $\{\hat{O}_S\}$, including its Hamiltonian $\hat{H}_S$, and, between measurements, a time-dependent state vector, $|\psi(t)\rangle_S$, that evolves according to the Schrödinger equation. Here, we are using the S subscript to emphasize that this is Schrödinger picture. Using the H subscript to denote Heisenberg picture, we have that observables, $\{\hat{O}_H(t)\}$, including the Hamiltonian $\hat{H}_H(t)$, now, in general, evolve in time—according to some appropriate equations of motion that we will soon determine—and, between measurements, the state vector, $|\psi\rangle_H$ is constant. These two pictures must be equivalent, i.e., a measurement made on the quantum system $\mathcal{S}$ at some time $t \geq 0$ must have the *same* statistics predicted from both of these pictures. At the initial time, $t = 0$, this will automatically occur because we will take

$$
\hat{O}_H(0) = \hat{O}_S, \quad \hat{H}_H(0) = \hat{H}_S, \quad |\psi\rangle_H = |\psi(0)\rangle_S. \tag{20}
$$

---

[3]Here, the eigenket-eigenvalue relation for $\hat{O}$ can be used to obtain the necessary intermediate result $\hat{O}^k = \int_{-\infty}^{\infty} \mathrm{d}o \, o^k |o\rangle\langle o|$.

For $t > 0$ we need to evolve the Heisenberg-picture observables so that the measurement statistics come out in agreement with what we already know from the Schrödinger picture description of quantum measurement.

Suppose that $\hat{O}_S$ is a Schrödinger-picture observable with discrete, non-degenerate eigenvalues $\{o_n\}$ and orthonormal eigenkets $\{|o_n\rangle_S\}$, so that

$$\hat{O}_S = \sum_n o_n |o_n\rangle_{SS}\langle o_n|. \tag{21}$$

Because these eigenvalues are the possible outcomes of a measurement of this observable, they must also be eigenvalues of the associated Heisenberg-picture observable $\hat{O}_H(t)$, which is why we did *not* add an S subscript to the eigenvalues in (21). From Axiom 3, we have that

$$\Pr(o_n) = |_S\langle o_n|\psi(t)\rangle_S|^2 = |_H\langle o_n(t)|\psi\rangle_H|^2, \tag{22}$$

where the first equality is for the Schrödinger picture and the second is for the Heisenberg picture, and we have used

$$\hat{O}_H(t) = \sum_n o_n |o_n(t)\rangle_{HH}\langle o_n(t)|, \tag{23}$$

with $\{|o_n(t)\rangle_H\}$ being the orthonormal eigenkets of $\hat{O}_H(t)$. It is now simple to see how to properly evolve $\hat{O}_H(t)$. From Eq. (22) we have that

$$\Pr(o_n) = {}_S\langle\psi(t)|o_n\rangle_{SS}\langle o_n|\psi(t)\rangle_S = (\hat{U}(t,0)|\psi(0)\rangle_S)^\dagger|o_n\rangle_{SS}\langle o_n|(\hat{U}(t,0)|\psi(0)\rangle_S), \tag{24}$$

where $\hat{U}(t,0)$ is the time-evolution operator for $\mathcal{S}$. Using $|\psi\rangle_H = |\psi(0)\rangle_S$ and $(\hat{U}(t,0)|\psi(0)\rangle_S)^\dagger = {}_S\langle\psi(0)|U^\dagger(t,0)$ we get

$$\Pr(o_n) = {}_H\langle\psi|[\hat{U}^\dagger(t,0)|o_n\rangle_S][{}_S\langle o_n|\hat{U}(t,0)]|\psi\rangle_H, \tag{25}$$

from which it follows that

$$|o_n(t)\rangle_H = \hat{U}^\dagger(t,0)|o_n\rangle_S, \tag{26}$$

is how the eigenkets of an observable are converted from the Schrödinger picture to the Heisenberg picture. You should use this result prove that

$$\hat{O}_H(t) = \hat{U}^\dagger(t,0)\hat{O}_S\hat{U}(t,0), \tag{27}$$

as stated on Slide 10.

In practice, we seldom calculate the time evolution of a Heisenberg-picture observable by first obtaining the unitary time-evolution operator from the Schrödinger

picture. Instead, we solve a differential equation that directly specifies the Heisenberg-picture observable's time evolution. Differentiating Eq. (27) with respect to time, we find that

$$j\hbar\frac{\mathrm{d}\hat{O}_H(t)}{\mathrm{d}t} = j\hbar\frac{\mathrm{d}\hat{U}^\dagger(t,0)}{\mathrm{d}t}\hat{O}_S\hat{U}(t,0) + j\hbar\hat{U}^\dagger(t,0)\hat{O}_S\frac{\mathrm{d}\hat{U}(t,0)}{\mathrm{d}t} \quad (28)$$

$$= -\hat{U}^\dagger(t,0)\hat{H}_S\hat{O}_S\hat{U}(t,0) + \hat{U}^\dagger(t,0)\hat{O}_S\hat{H}_S\hat{U}(t,0), \quad \text{for } t \geq 0. \quad (29)$$

On Problem Set 3 you will prove that

$$\hat{U}^\dagger(t,0)\hat{H}_S = \hat{H}_S\hat{U}^\dagger(t,0) \quad \text{and} \quad \hat{H}_S\hat{U}(t,0) = \hat{U}(t,0)\hat{H}_S, \quad (30)$$

whence

$$j\hbar\frac{\mathrm{d}\hat{O}_H(t)}{\mathrm{d}t} = -\hat{H}_S\hat{U}^\dagger(t,0)\hat{O}_S\hat{U}(t,0) + \hat{U}^\dagger(t,0)\hat{O}_S\hat{U}(t,0)\hat{H}_S \quad (31)$$

$$= \hat{O}_H(t)\hat{H}_S - \hat{H}_S\hat{O}_H(t) = \left[\hat{O}_H(t), \hat{H}_S\right], \quad \text{for } t \geq 0, \quad (32)$$

where $[\hat{A}, \hat{B}] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}$ is the *commutator* of the operators $\hat{A}$ and $\hat{B}$. Equation (32), which is known as the Heisenberg equation of motion for $\hat{O}_H(t)$, is to be solved subject to the initial condition $\hat{O}_H(0) = \hat{O}_S$.

Commutators play an essential role in quantum mechanics, as we will see momentarily. Lest you think that all linear operators commute, i.e., satisfy $[\hat{A}, \hat{B}] = 0$, we remind you that for $N \times N$ matrices $\mathbf{A}$ and $\mathbf{B}$ we generally find $\mathbf{AB} \neq \mathbf{BA}$.[4] As an example of the Heisenberg equation of motion, consider its behavior when $\hat{O}_H(t) = \hat{H}_H(t)$, i.e., when we are interested in finding the Heisenberg picture form of the Hamiltonian. Because $[\hat{H}_H(0), \hat{H}_S] = [\hat{H}_S, \hat{H}_S] = 0$, we find that $\mathrm{d}\hat{H}_H(t)/\mathrm{d}t|_{t=0} = 0$, from which we can show that $\hat{H}_H(t) = \hat{H}_S = \hat{H}$, i.e., for an isolated system, the Hamiltonian is a constant in *both* the Schrödinger and Heisenberg pictures.[5] Going forward, we shall drop the H subscript from Heisenberg picture states and operators, because we will not be returning to the Schrödinger picture.

Simultaneous Measurements

Let $\hat{A}$ and $\hat{B}$ be two observables of a quantum system $\mathcal{S}$. Saying that these two observables commute is equivalent to saying that they have a common set of orthonormal

---

[4]As a simple example, let $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$. Do the multiplication and see that $\mathbf{AB} \neq \mathbf{BA}$.

[5]This is hardly surprising. In an isolated system, no energy can be lost or gained, so it's pretty obvious that the energy operator (Hamiltonian) must be a constant of the motion.

eigenkets. So, for the case of discrete distinct eigenvalues we have[6]

$$\hat{A} = \sum_n a_n |\phi_n\rangle\langle\phi_n| \quad \text{and} \quad \hat{B} = \sum_n b_n |\phi_n\rangle\langle\phi_n|, \tag{33}$$

where the $\{|\phi_n\rangle\}$ comprise an orthonormal basis for $\mathcal{H}_\mathcal{S}$. Commuting observables may be measured *simultaneously*. Specifically, if the system is in the state $|\psi\rangle$ and we measure $\hat{A}$ *and* $\hat{B}$ simultaneously, then our outcome will be an ordered pair, (a_n, b_n), consisting of an eigenvalue of $\hat{A}$ and an eigenvalue of $\hat{B}$ that are associated with the *same* eigenket $|\phi_n\rangle$. Assuming that the eigenvalues are distinct, the probability of this event's occurring is

$$\text{Pr}(a_n, b_n) = |\langle\phi_n|\psi\rangle|^2. \tag{34}$$

A similar situation prevails if the commuting operators each have continuous eigenvalue spectra with distinct eigenvalues. Then we get that the joint probability density for the outcome (a, b) is

$$p(a, b) = |\langle\phi|\psi\rangle|^2, \tag{35}$$

where $|\phi\rangle$ is the $\hat{A}$ eigenket with eigenvalue a *and* it is also the $\hat{B}$ eigenket with eigenvalue b.[7]

The Heisenberg Uncertainty Principle

There is nothing in classical physics that precludes our simultaneously (and precisely) measuring the position and momentum of a particle, or any other pair of observables. Such is not the case in quantum mechanics. Commuting observables can be measured simultaneously. Those that do not commute cannot be measured simultaneously. Indeed, if

$$[\hat{A}(t), \hat{B}(t)] = j\hat{C}(t), \tag{36}$$

where $\hat{C}(t)$ is an Hermitian operator, then

$$\langle\Delta\hat{A}^2(t)\rangle\langle\Delta\hat{B}^2(t)\rangle \geq |\langle\hat{C}(t)\rangle|^2/4. \tag{37}$$

This result is the Heisenberg uncertainty principle, written in Dirac notation. Before delving into its proof, let us comment on what it says.

- If $\hat{A}(t)$ and $\hat{B}(t)$ are non-commuting observables then

$$[\hat{A}(t), \hat{B}(t)]^\dagger = [\hat{A}(t)\hat{B}(t)]^\dagger - [\hat{B}(t)\hat{A}(t)]^\dagger = \hat{B}^\dagger(t)\hat{A}^\dagger(t) - \hat{A}^\dagger(t)\hat{B}^\dagger(t) \tag{38}$$

$$= \hat{B}(t)\hat{A}(t) - \hat{A}(t)\hat{B}(t) = -[\hat{A}(t), \hat{B}(t)], \tag{39}$$

---

[6]You should verify, using these expansions, that $\hat{A}$ and $\hat{B}$ commute.

[7]The projection postulate applies to simultaneous measurements of commuting observables, but, as in the single measurement case, it is not of great interest to us because photodetection measurements are usually annihilative.

where the third equality uses the fact that $\hat{A}(t)$ and $\hat{B}(t)$ are Hermitian (because they are observables). It should then be clear that $[\hat{A}(t), \hat{B}(t)] = j\hat{C}(t)$, where $\hat{C}(t)$ is Hermitian.

- Because $\hat{A}(t)$ and $\hat{B}(t)$ cannot be measured simultaneously, the terms appearing on the left in the Heisenberg uncertainty principle have the following interpretations: $\langle \Delta\hat{A}^2(t)\rangle = \langle\psi|(\hat{A}(t) - \langle\hat{A}(t)\rangle)^2|\psi\rangle$ is the variance that results if we choose to measure $\hat{A}(t)$ on the system $\mathcal{S}$ when that system is in the state $|\psi\rangle$; likewise, $\langle \Delta\hat{B}^2(t)\rangle = \langle\psi|(\hat{B}(t) - \langle\hat{B}(t)\rangle)^2|\psi\rangle$ is the variance that results if we choose to measure $\hat{B}(t)$ on the system $\mathcal{S}$ when that system is in the state $|\psi\rangle$. Always remember, in using the Heisenberg uncertainty principle, that this is an either or proposition, i.e., we *cannot* make both measurements on $\mathcal{S}$ at the same time.

- The right-hand side of the uncertainty principle is $|\langle\hat{C}(t)\rangle|^2/4 = |\langle\psi|\hat{C}(t)|\psi\rangle|^2/4$, so that if $\hat{C}(t) = c\hat{I}$, where c is a non-zero scalar and $\hat{I}$ is the identity operator, then *all* states $|\psi\rangle$ have the same non-zero lower bound on the product of their $\hat{A}(t)$ and $\hat{B}(t)$ measurement variances. This will turn out to be the case for the quadrature components of the quantum harmonic oscillator, as we shall see next week.

Proof of the Uncertainty Principle

The proof is straightforward. With $\Delta\hat{A}(t) \equiv \hat{A}(t) - \langle\hat{A}(t)\rangle$ and $\Delta\hat{B}(t) \equiv \hat{B}(t) - \langle\hat{B}(t)\rangle$, we can multiply out to verify that

$$[\Delta\hat{A}(t), \Delta\hat{B}(t)] = j\hat{C}(t). \tag{40}$$

Next, we use the Schwarz inequality to show that

$$\langle\psi|\Delta\hat{A}^2(t)|\psi\rangle\langle\psi|\Delta\hat{B}^2(t)|\psi\rangle \geq |\langle\psi|\Delta\hat{A}(t)\Delta\hat{B}(t)|\psi\rangle|^2, \tag{41}$$

with equality if and only if $\Delta\hat{A}(t)|\psi\rangle = j\lambda\Delta\hat{B}(t)|\psi\rangle$ for some complex number λ.[8] Now we encounter a little double-bookkeeping algebra:

$$|\langle\psi|\Delta\hat{A}(t)\Delta\hat{B}(t)|\psi\rangle|^2$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}(t)\Delta\hat{B}(t) + \Delta\hat{B}(t)\Delta\hat{A}(t) + [\Delta\hat{A}(t), \Delta\hat{B}(t)]}{2}\right)|\psi\rangle\right|^2 \tag{42}$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}(t)\Delta\hat{B}(t) + \Delta\hat{B}(t)\Delta\hat{A}(t)}{2}\right)|\psi\rangle + \frac{j}{2}\langle\psi|\hat{C}(t)|\psi\rangle\right|^2. \tag{43}$$

---

[8] We have inserted the j here for later convenience.

Because the operators in the $\langle\psi|\cdot|\psi\rangle$ brackets are Hermitian, these brackets evaluate to real numbers and so

$$|\langle\psi|\Delta\hat{A}(t)\Delta\hat{B}(t)|\psi\rangle|^2$$

$$= \left|\langle\psi|\left(\frac{\Delta\hat{A}(t)\Delta\hat{B}(t) + \Delta\hat{B}(t)\Delta\hat{A}(t)}{2}\right)|\psi\rangle\right|^2 + \left|\frac{\langle\psi|\hat{C}(t)|\psi\rangle}{2}\right|^2 \tag{44}$$

$$\geq |\langle\psi|\hat{C}(t)|\psi\rangle|^2/4, \tag{45}$$

with equality if and only if $\langle\psi|\Delta\hat{A}(t)\Delta\hat{B}(t)|\psi\rangle = -\langle\psi|\Delta\hat{B}(t)\Delta\hat{A}(t)|\psi\rangle$. Using (45) in (41) then shows that

$$\langle\Delta\hat{A}^2(t)\rangle\langle\Delta\hat{B}^2(t)\rangle \geq |\langle\hat{C}(t)\rangle|^2/4. \tag{46}$$

Combining the equality conditions that were identified in (45) and (41) we see that equality is achieved in the Heisenberg Uncertainty Principle if and only if

$$\Delta\hat{A}(t)|\psi\rangle = j\lambda\Delta\hat{B}(t)|\psi\rangle, \quad \text{for } \lambda \text{ a real-valued constant.} \tag{47}$$

States that satisfy this condition, i.e., states that meet the Heisenberg lower bound on the product of variances for a particular pair of non-commuting observables, are called minimum uncertainty-product states for those two observables. The minimum uncertainty-product states for the quadrature components of the quantized single-mode electromagnetic field are what underlie the waveguide tap that was described in Lecture 1.

The Road Ahead

Now we have completed the quantum mechanics foundations that we will need for the entire semester. Next week we begin our treatment of the quantum harmonic oscillator. We will introduce this topic by quantizing the behavior of an LC circuit. This will lead us to operators that annihilate and create discrete energy quanta for the oscillator. Because a single mode of the quantized electromagnetic field is a quantum harmonic oscillator, these quanta can be thought of as photons and their associated annihilation and creation operators will be of great significance throughout the semester.

MIT OpenCourseWare

6.453 Quantum Optical Communication
Fall 2016