Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Mid-Term Examination
Fall 2016

This examination is closed book except for one 8 1/2 × 11 handwritten formula sheet (both sides) of your own devising.

Problem 1 (20 points)
For each statement below, indicate whether it is True or whether it is False, and provide a brief explanation of your reasoning.

(a) (10 points) Consider a pair of single-mode electromagnetic fields, with annihilation operators $\hat{a}_A$ and $\hat{a}_B$, whose joint state $|\psi\rangle_{AB}$ is a pure state. Suppose that the $\hat{N}_A = \hat{a}_A^\dagger \hat{a}_A$ and $\hat{N}_B = \hat{a}_B^\dagger \hat{a}_B$ measurements are made on these modes and that the resulting classical outcomes, N_A and N_B, have measurement statistics which satisfy

$$\text{Var}(N_A - N_B) < \text{Var}(N_A) + \text{Var}(N_B),$$

where $\text{Var}(\cdot)$ denotes variance.

True or False: The joint state of the $\hat{a}_A$ and $\hat{a}_B$ modes *must* be non-classical.

(b) (10 points) Consider a single-mode electromagnetic field with photon annihilation operator $\hat{a}$ whose Wigner distribution is $W(\alpha^*, \alpha)$.

True or False: The function $F(\alpha_1) \equiv \int_{-\infty}^{\infty} \mathrm{d}\alpha_2\, W(\alpha^*, \alpha)$, where α_1 and α_2 are the real and imaginary parts of α, is non-negative for *all* values of α_1.

In case you've forgotten:

$$W(\alpha^*, \alpha) = \int \frac{\mathrm{d}^2\zeta}{\pi^2}\, \chi_W(\zeta^*, \zeta) e^{\zeta^*\alpha - \zeta\alpha^*},$$

where $\chi_W(\zeta^*, \zeta) \equiv \langle e^{-\zeta^*\hat{a} + \zeta\hat{a}^\dagger} \rangle$, and

$$\int \frac{\mathrm{d}\alpha_2}{\pi}\, e^{2j\zeta_1\alpha_2} = \delta(\zeta_1)$$

where $\delta(\cdot)$ is the impulse function.

Problem 2 (40 points)
Consider the asymmetric beam-splitter setup shown in Fig. 1. In this setup, the beam spitter is illuminated by a signal mode (with annihilation operator $\hat{a}_S$) and a local-oscillator (LO) mode (with annihilation operator $\hat{a}_{\text{LO}}$). We will be interested in the output mode from that beam splitter whose annihilation operator is $\hat{a}_{\text{out}} = \sqrt{\epsilon}\,\hat{a}_S + \sqrt{1-\epsilon}\,\hat{a}_{\text{LO}}$, where $0 < \epsilon < 1$ and the $\hat{a}_{\text{LO}}$ mode is in the coherent state $|\beta\sqrt{\epsilon/(1-\epsilon)}\rangle_{\text{LO}}$.

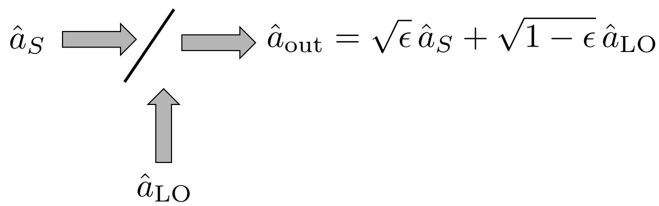

Figure 1: Asymmetric beam-splitter setup.

(a) (10 points) Suppose that the $\hat{a}_S$ mode is in the coherent state $|\gamma\rangle_S$.

(i) With only a *simple* statement of justification, find the state of the $\hat{a}_{\text{out}}$ mode.

(ii) Use your result from (i) to find $\rho^{(n)}_{a_{\text{out}}}(\alpha^*, \alpha) \equiv {}_{\text{out}}\langle\alpha|\hat{\rho}_{a_{\text{out}}}|\alpha\rangle_{\text{out}}$ in the limit $\epsilon \to 1$.

(b) (10 points) Figure 2 uses the beam-splitter setup in a photon-counting communication receiver with the following characteristics.

- The binary message b being communicated is equally likely to be 0 or 1.
- When $b = 0$, the $\hat{a}_S$ mode is in the coherent state $|-\sqrt{N_S}\rangle_S$. When $b = 1$, the $\hat{a}_S$ mode is in the coherent state $|\sqrt{N_S}\rangle_S$.
- The beam-splitter setup has $0 < \epsilon < 1$ and $\beta = \sqrt{N_S}$.
- The receiver's output is $\tilde{b} = 1$ when the $\hat{N}_{\text{out}} = \hat{a}^\dagger_{\text{out}}\hat{a}_{\text{out}}$ measurement's outcome N_{out} is non-zero. The receiver's output is $\tilde{b} = 0$ when $N_{\text{out}} = 0$.

(i) Use your result from (a) to find the states that the $\hat{a}_{\text{out}}$ mode is in when $b = 0$ and $b = 1$.

(ii) Use your results from (i) to find the receiver's error probability, $\Pr(\tilde{b} \neq b)$.

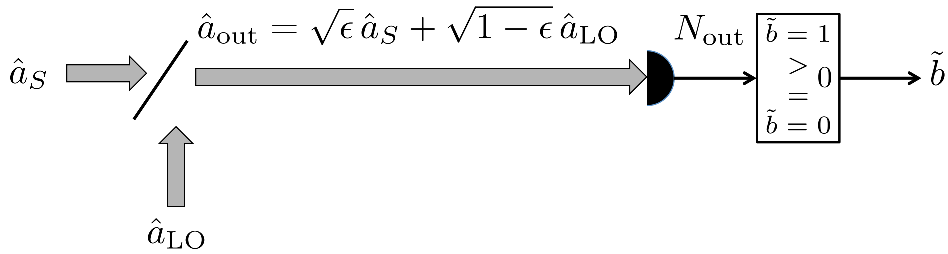

Figure 2: Photon-counting communication receiver.

(c) (10 points) Now, let the $\hat{a}_S$ mode be in an *arbitrary* state specified by the density operator $\hat{\rho}_S$.

(i) Find $\chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta)$, the anti-normally ordered characteristic function of the $\hat{a}_{\text{out}}$ mode. Your answer should be expressed in terms of the $\hat{a}_S$ mode's anti-normally ordered characteristic function, β, and ϵ.

(ii) Specialize your result from (i) to the limit $\epsilon \to 1$.

(d) (10 points) For your $\chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta)$ from (c), use the operator-valued inverse transform relation,

$$\hat{\rho}_{a_{\text{out}}} = \int \frac{\mathrm{d}^2\zeta}{\pi}\, \chi_A^{\rho_{a_{\text{out}}}}(\zeta^*, \zeta) e^{-\zeta \hat{a}_{\text{out}}^\dagger} e^{\zeta^* \hat{a}_{\text{out}}},$$

to obtain $\rho_{a_{\text{out}}}^{(n)}(\alpha^*, \alpha) \equiv {}_{\text{out}}\langle \alpha | \hat{\rho}_{a_{\text{out}}} | \alpha \rangle_{\text{out}}$ in the $\epsilon \to 1$ limit. Your answer should be expressed in terms of $\rho_S^{(n)}(\alpha^*, \alpha) \equiv {}_S\langle \alpha | \hat{\rho}_{a_S} | \alpha \rangle_S$, and β.

Problem 3 (40 points)
The system shown in Fig. 3 is a quantum non-demolition (QND) setup for measuring the photon number of an optical mode with annihilation operator $\hat{a}$. The cross-Kerr-effect box has the following input-output relation:

$$\hat{c} = e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}$$

$$\hat{d} = e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a},$$

where $\kappa > 0$ is a constant. The homodyne detector is set up to measure the $\hat{c}_2 = \mathrm{Im}(\hat{c})$ observable.

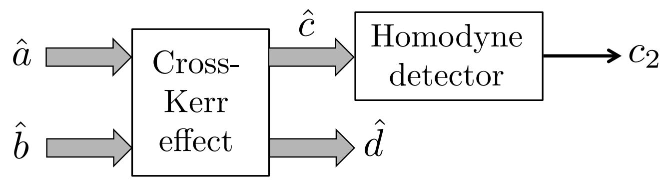

Figure 3: Quantum non-demolition detection setup.

(a) (10 points) Evaluate the number-ket matrix elements,

$${}_b\langle n_b|_a\langle n_a|e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}|m_a\rangle_a|m_b\rangle_b$$

and

$${}_b\langle n_b|_a\langle n_a|e^{j\kappa\hat{b}^\dagger\hat{b}}\hat{a}e^{j\kappa\hat{a}^\dagger\hat{a}}\hat{b}|m_a\rangle_a|m_b\rangle_b.$$

(b) (10 points) Assume that the $\hat{a}$ mode is in the number state $|m_a\rangle_a$. Let N_d be the outcome of the $\hat{N}_d = \hat{d}^\dagger\hat{d}$ measurement. Find the probability mass function $\Pr(N_d = n)$. **Hint**: You do *not* need to know the state of the $\hat{b}$ mode.

(c) (10 points) Assume that the $\hat{a}$ mode is in the number state $|m_a\rangle_a$ and the $\hat{b}$ mode is in the coherent state $|\sqrt{N_b}\,\rangle$. Find $\langle\hat{c}_2\rangle$ and $\langle\Delta\hat{c}_2^2\rangle$, the mean and variance of the $\hat{c}_2$ measurement.

(d) (10 points) Assume that the states of the $\hat{a}$ and $\hat{b}$ modes are as given in (c), and that $\kappa m_a \ll 1$. Let c_2 denote the outcome of the $\hat{c}_2$ measurement and define $\tilde{N}_a = c_2/\sqrt{N_b}\,\kappa$ to be the QND estimate of the $\hat{a}$ mode's photon number. Find the mean-squared error of this estimate, i.e., $\langle(\tilde{N}_a - m_a)^2\rangle$.

I'm sure you know these things:

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}, \qquad \sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\frac{1 - \cos(2x)}{2} = \sin^2(x), \qquad \sin(x) \approx x, \text{ for } |x| \ll 1$$