Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 8 Solutions
Fall 2016

Problem 8.1

Here we will derive the fidelity of a measure-and-prepare approach to qubit transmission. Suppose that Charlie has a single photon whose polarization state is

$$|\psi_C\rangle = \alpha|H\rangle + \beta|V\rangle,$$

where $|\alpha|^2 + |\beta|^2 = 1$ and $|H\rangle$ and $|V\rangle$ denote horizontally-polarized and vertically-polarized single photon states, respectively. Charlie wants to transmit this state to Bob, but Bob is too far away for reliable fiber-optic transmission of that single photon. Instead, Charlie gives his photon to Alice—who is located nearby—for her to measure in the H/V basis using a polarizing beam splitter and unity quantum efficiency photodetectors, as shown in Fig. 1. If Alice gets a click on her H detector, she sends Bob a classical message saying that he should prepare an H photon as his replica of $|\psi_C\rangle$. If Alice gets a click on her V detector, then she sends Bob a classical message saying that he should prepare a V photon as his replica of $|\psi_C\rangle$. Thus, Bob's state after this measure-and-prepare protocol is

$$|\psi_B\rangle = \begin{cases} |H\rangle, & \text{if Alice got an } H \text{ click} \\ |V\rangle, & \text{if Alice got a } V \text{ click.} \end{cases}$$

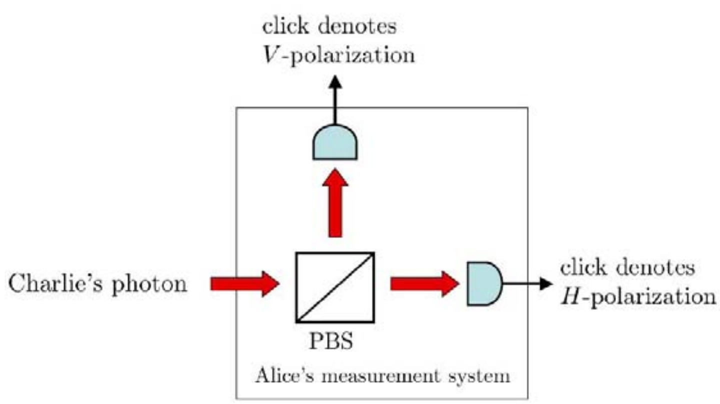

Figure 1: Alice's H/V polarization-measurement system. PBS denotes polarizing beam splitter.

(a) It is easy to find $\Pr(\,|\psi_B\rangle = |H\rangle \mid |\psi_C\rangle\,)$ and $\Pr(\,|\psi_B\rangle = |V\rangle \mid |\psi_C\rangle\,)$, i.e., the probabilities for Bob's two possible states conditioned on the value of Charlie's state. Bob will prepare a horizontally-polarized single photon when Alice's H-polarization detector has clicked. Given that Charlie's state was $|\psi_C\rangle = \alpha|H\rangle + \beta|V\rangle$, we have that

$$\Pr(\,|\psi_B\rangle = |H\rangle \mid |\psi_C\rangle\,) = |\langle H|\psi_C\rangle|^2 = |\alpha|^2.$$

A similar calculation—based on Bob's preparing a vertically-polarized single photon when Alice's V-polarization detector has clicked—gives

$$\Pr(\,|\psi_B\rangle = |V\rangle \mid |\psi_C\rangle\,) = |\langle V|\psi_C\rangle|^2 = |\beta|^2.$$

Because $|\alpha|^2 + |\beta|^2 = 1$, these two conditional probabilities—which represent the only states that Bob will prepare—sum to one. Indeed, we could have used that fact to find $\Pr(\,|\psi_B\rangle = |V\rangle \mid |\psi_C\rangle\,)$ as $1 - \Pr(\,|\psi_B\rangle = |H\rangle \mid |\psi_C\rangle\,)$.

(b) To use the results from (a) to find $\hat{\rho}_B(\alpha,\beta)$, Bob's density operator when Charlie's state is $|\psi_C\rangle$, we proceed as follows. When Alice's H-polarization detector clicks, Bob's state will be $|H\rangle$, and when her V-polarization detector clicks, his state will be $|V\rangle$. Thus, given that Charlie's state is $|\psi_C\rangle$, Bob's density operator will be

$$\begin{aligned}\hat{\rho}_B(\alpha,\beta) &= \Pr(\,|\psi_B\rangle = |H\rangle \mid |\psi_C\rangle\,)|H\rangle\langle H| + \Pr(\,|\psi_B\rangle = |V\rangle \mid |\psi_C\rangle\,)|V\rangle\langle V| \\ &= |\alpha|^2|H\rangle\langle H| + |\beta|^2|V\rangle\langle V|.\end{aligned}$$

(c) With the result from (b), we can easily evaluate the fidelity of the measure-and-prepare system conditioned on the value of Charlie's state, i.e.,

$$F(\alpha,\beta) \equiv \langle\psi_C|\hat{\rho}_B(\alpha,\beta)|\psi_C\rangle.$$

We have that

$$F(\alpha,\beta) = (\alpha^*\langle H| + \beta^*\langle V|)(|\alpha|^2|H\rangle\langle H| + |\beta|^2|V\rangle\langle V|)(\alpha|H\rangle + \beta|V\rangle) = |\alpha|^4 + |\beta|^4.$$

(d) When Charlie's state is random, and uniformly distributed over the Poincaré sphere, i.e., it has a 3-D unit vector representation,

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} \equiv \begin{bmatrix} 2\mathrm{Re}(\alpha^*\beta) \\ 2\mathrm{Im}(\alpha^*\beta) \\ |\alpha|^2 - |\beta|^2 \end{bmatrix},$$

that is uniformly distributed over the unit sphere, we must regard our result from (c) as the *conditional* fidelity given the value of Charlie's state. To find the numerical value of the measure-and-prepare system's average fidelity we must *average* over all possible states for Charlie's photon using the uniform distribution on the Poincaré sphere. We start with

$$\bar{F} \equiv \int_{\mathbf{r}\in\mathcal{P}} \mathrm{d}\mathbf{r}\, \frac{F(\alpha,\beta)}{4\pi}. = \int_{\mathbf{r}\in\mathcal{P}} \mathrm{d}\mathbf{r}\, \frac{|\alpha|^4+|\beta|^4}{4\pi}.$$

Then we use

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{bmatrix}, \quad \text{for } 0 \le \theta \le \pi \text{ and } 0 \le \phi \le 2\pi,$$

to obtain

$$|\alpha|^4 = \left(\frac{1+r_3}{2}\right)^2 = \left(\frac{1+\cos(\theta)}{2}\right)^2,$$

and

$$|\beta|^4 = \left(\frac{1-r_3}{2}\right)^2 = \left(\frac{1-\cos(\theta)}{2}\right)^2.$$

Substituting these results into our previous expression for $\bar{F}$ gives us

$$\bar{F} = \int_0^\pi \mathrm{d}\theta\, \sin(\theta) \int_0^{2\pi} \mathrm{d}\phi\, \frac{1+\cos^2(\theta)}{8\pi} = \int_0^\pi \mathrm{d}\theta\, \sin(\theta)\frac{1+\cos^2(\theta)}{4} = 2/3,$$

where the last equality used

$$\int_0^\pi \mathrm{d}\theta\, \sin(\theta) = -\cos(\theta)|_0^\pi = 2$$

and

$$\int_0^\pi \mathrm{d}\theta\, \sin(\theta)\cos^2(\theta) = -[\cos^3(\theta)/3]\big|_0^\pi = 2/3.$$

It can be shown that no *classical* communication scheme between Alice and Bob can lead to a higher average fidelity in Bob's reproducing Charlie's randomly-chosen qubit state. To get higher average fidelity, Alice and Bob must share entanglement and use qubit teleportation.

Problem 8.2

Here we will derive the fidelity of a measure-and-prepare approach to continuous-variable quantum communication. Suppose that Charlie has a single-mode field in the state $|\psi_C\rangle$ which he wishes to transmit to Bob. Because Bob is too far away for reliable quantum transmission, Charlie sends his field mode to Alice—who is located nearby—for her to measure via balanced heterodyne detection, i.e., by the positive operator-valued measurement (POVM) associated with the annihilation operator $\hat{a}_C$ of Charlie's field mode. Alice's outcome from this measurement is a complex-valued random variable α. She sends this classical α value to Bob over a classical communication channel and Bob uses this information to prepare a single-mode field in the coherent state $|\alpha\rangle$.

(a) Finding $p(\alpha_1, \alpha_2)$, the joint probability density function for α_1 and α_2, the real and imaginary parts of the random variable α, as a function of $|\psi_C\rangle$ is trivial. We know that the probability density for the outcome obtained from heterodyne detection is

$$p(\alpha_1, \alpha_2) = \frac{\langle\alpha|\hat{\rho}_C|\alpha\rangle}{\pi},$$

where $|\alpha\rangle$ is the coherent state, α_1 and α_2 are the real and imaginary parts of α, and $\hat{\rho}_C$ is the density operator for the $\hat{a}_C$ mode. We are given a pure state $|\psi_C\rangle$ for that mode, thus

$$p(\alpha_1, \alpha_2) = \frac{|\langle\alpha|\psi_C\rangle|^2}{\pi}.$$

(b) Expressing the density operator for Bob's state, $\hat{\rho}_B$, in P-representation form as a function of $|\psi_C\rangle$ is similarly trivial. Given the value of the heterodyne measurement's outcome, α, Bob's state is the coherent state $|\alpha\rangle$. Thus, because α is random—with joint probability density function $p(\alpha_1, \alpha_2)$ from (a) for its real and imaginary parts—we have that

$$\hat{\rho}_B = \int \mathrm{d}\alpha_1 \int \mathrm{d}\alpha_2\, p(\alpha_1, \alpha_2)|\alpha\rangle\langle\alpha|.$$

Identifying $P(\alpha, \alpha^*) = p(\alpha_1, \alpha_2)$ we get can rewrite this as

$$\hat{\rho}_B = \int \mathrm{d}^2\alpha\, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|,$$

and use (a) to get $P(\alpha, \alpha^*) = |\langle\alpha|\psi_C\rangle|^2/\pi$.

(c) The fidelity of this measure-and-prepare system is

$$F(|\psi_C\rangle) \equiv \langle\psi_C|\hat{\rho}_B|\psi_C\rangle.$$

Using the result of (b) this becomes

$$F(|\psi_C\rangle) = \int \mathrm{d}^2\alpha\, P(\alpha, \alpha^*)|\langle\alpha|\psi_C\rangle|^2 = \int \mathrm{d}^2\alpha\, \frac{|\langle\alpha|\psi_C\rangle|^4}{\pi}.$$

(d) When Charlie's state is the coherent state $|\alpha_C\rangle$, we have that

$$|\langle\alpha|\psi_C\rangle|^2 = |\langle\alpha|\alpha_C\rangle|^2 = e^{-|\alpha-\alpha_C|^2},$$

so that result from (c) becomes

$$F(|\alpha_C\rangle) = \int \mathrm{d}^2\alpha \, \frac{e^{-2|\alpha-\alpha_C|^2}}{\pi} = 1/2,$$

where the last equality follows from

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{-A(x-b)^2} = \sqrt{\pi/A}, \quad \text{for } A > 0 \text{ and arbitrary } b.$$

Problem 8.3

Here we shall begin a treatment of optimum binary hypothesis testing.

(a) The task here is straightforward. We will declare "state $= |\psi_{-1}\rangle$" when the $\hat{D}$ measurement yields outcome -1. Given that the state of the system is $|\psi_1\rangle$, this will occur with probability,

$$\Pr(\text{ say "state was } |\psi_{-1}\rangle\text{" } | \text{ state was } |\psi_1\rangle\,) \;=\; \Pr(\hat{D} = -1 \mid |\psi_1\rangle) = |\langle d_{-1}|\psi_1\rangle|^2.$$

Similarly, we find that,

$$\Pr(\text{ say "state was } |\psi_1\rangle\text{" } | \text{ state was } |\psi_{-1}\rangle\,) \;=\; \Pr(\hat{D} = 1 \mid |\psi_{-1}\rangle) = |\langle d_1|\psi_{-1}\rangle|^2.$$

The *unconditional* error probability now follows immediately:

$$\begin{aligned}
\Pr(e) &= \Pr(\text{state was } |\psi_1\rangle \text{ and say "state was } |\psi_{-1}\rangle\text{"}) \\
&+ \Pr(\text{state was } |\psi_{-1}\rangle \text{ and say "state was } |\psi_1\rangle\text{"}) \\
&= \Pr(\text{state was } |\psi_1\rangle)\,\Pr(\hat{D} = -1 \mid |\psi_1\rangle) \\
&+ \Pr(\text{state was } |\psi_{-1}\rangle)\,\Pr(\hat{D} = 1 \mid |\psi_{-1}\rangle) \\
&= \frac{1}{2}|\langle d_{-1}|\psi_1\rangle|^2 + \frac{1}{2}|\langle d_1|\psi_{-1}\rangle|^2.
\end{aligned}$$

(b) When $|\psi_{-1}\rangle$ and $|\psi_1\rangle$ are orthonormal we can make,

$$\Pr(\hat{D} = -1 \mid |\psi_1\rangle) = 0 \quad \text{and} \quad \Pr(\hat{D} = 1 \mid |\psi_{-1}\rangle) = 0,$$

by choosing

$$|d_{-1}\rangle = |\psi_{-1}\rangle \quad \text{and} \quad |d_1\rangle = |\psi_1\rangle.$$

We then have that $\{|d_{-1}\rangle, |d_1\rangle\}$ is an orthonormal set, so we have indeed found eigenkets for an observable, $\hat{D}$, on the reduced Hilbert space $\mathcal{H}$. From (a) we see that the unconditional error probability achieved by this decision operator is $\Pr(e) = 0$, i.e., we can perfectly distinguish between a pair of orthonormal $\{|\psi_{-1}\rangle, |\psi_1\rangle\}$ by making an appropriate observable measurement.

(c) Now we are given a pair of state $\{|\psi_{-1}\rangle, |\psi_1\rangle\}$ that are *not* orthogonal. Indeed their inner product is,

$$\langle\psi_{-1}|\psi_1\rangle = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta),$$

where $0 < 2\theta < \pi/2$. Using the expansions,

$$|d_{-1}\rangle = \cos(\phi)|x\rangle - \sin(\phi)|y\rangle \quad \text{and} \quad |d_1\rangle = \sin(\phi)|x\rangle + \cos(\phi)|y\rangle,$$

which specify an arbitrary orthonormal basis for $\mathcal{H}$ as ϕ ranges from 0 to 2π, and the results of (a) we have that,

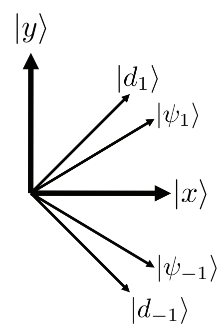

Figure 2: Geometry of the optimal detection problem for $\theta = \pi/6$

$$\begin{aligned}
\Pr(e) &= \frac{1}{2}[\cos(\phi)\cos(\theta) - \sin(\phi)\sin(\theta)]^2 + \frac{1}{2}[\sin(\phi)\cos(\theta) - \cos(\phi)\sin(\theta)]^2 \\
&= \frac{1}{2}[\cos^2(\phi)\cos^2(\theta) + \sin^2(\phi)\sin^2(\theta) - 2\sin(\phi)\cos(\phi)\sin(\theta)\cos(\theta)] \\
&+ \frac{1}{2}[\sin^2(\phi)\cos^2(\theta) + \cos^2(\phi)\sin^2(\theta) - 2\sin(\phi)\cos(\phi)\sin(\theta)\cos(\theta)] \\
&= \frac{1}{2}[\cos^2(\theta) + \sin^2(\theta) - 2\sin(2\phi)\sin(\theta)\cos(\theta)] \\
&= \frac{1}{2}[1 - \sin(2\phi)\sin(2\theta)].
\end{aligned}$$

It is now easy to optimize over ϕ, i.e., to optimize $\{|d_{-1}\rangle, |d_1\rangle\}$. Because $0 < \theta < \pi/4$, we have that $\sin(2\theta) > 0$. Thus, to minimize the error probability, we should make $\sin(2\phi) = 1$, i.e., we should choose $\phi = \pi/4$, whence

$$|d_{-1}\rangle = \frac{|x\rangle - |y\rangle}{\sqrt{2}} \quad \text{and} \quad |d_1\rangle = \frac{|x\rangle + |y\rangle}{\sqrt{2}},$$

gives the optimum (minimum) error probability,

$$\Pr(e)_{\text{opt}} = \frac{1}{2}[1 - \sin(2\theta)] = \frac{1}{2}\left[1 - \sqrt{1 - |\langle\psi_{-1}|\psi_1\rangle|^2}\right].$$

In Fig. 2, we have plotted the geometry of this optimal detection problem when $\theta = \pi/6$.

Problem 8.4

Here we shall continue our treatment of optimum binary hypothesis testing.

(a) When $|\psi_1\rangle = |n_{-1}\rangle$ and $|\psi_1\rangle = |n_1\rangle$, where $|n_{-1}\rangle$ and $|n_1\rangle$ are photon number states with $n_{-1} \neq n_1$, we have that $\langle\psi_{-1}|\psi_1\rangle = 0$, so that the optimum decision operator from Problem 8.3(b) achieves zero error probability. If we measure the number operator, $\hat{N}$, then the outcome of this measurement will be n_{-1} when the state is $|n_{-1}\rangle$ and the outcome will be n_1 when the state is $|n_1\rangle$. So zero-error-probability performance can be achieved by making the $\hat{N}$ measurement and saying "state was $|n_{-1}\rangle$" when n_{-1} occurs, and saying "state was $|n_1\rangle$" when n_1 occurs.

(b) When $|\psi_{-1}\rangle = |\alpha_{-1}\rangle$ and $|\psi_1\rangle = |\alpha_1\rangle$, where $|\alpha_{-1}\rangle$ and $|\alpha_1\rangle$ are coherent states with $\langle\alpha_{-1}|\alpha_1\rangle = \cos(2\theta)$ for a θ value satisfying $0 < \theta < \pi/4$, we can immediately apply the result of Problem 8.3(c) to show that,

$$\Pr(e) = \frac{1}{2}[1 - \sin(2\theta)] = \frac{1}{2}\left[1 - \sqrt{1 - \cos^2(2\theta)}\right] = \frac{1}{2}\left[1 - \sqrt{1 - |\langle\alpha_{-1}|\alpha_1\rangle|^2}\right],$$

gives the minimum achievable error probability.

(c) For OOK, we have that

$$\langle\alpha_{-1}|\alpha_1\rangle = \langle 0|\sqrt{N}\rangle = e^{-N/2},$$

which yields,

$$\Pr(e) = \frac{1}{2}\left[1 - \sqrt{1 - e^{-N}}\right] \approx \frac{1}{4}e^{-N}, \quad \text{for } N \gg 1,$$

for the error probability of the optimum decision rule.

Now, if we make the $\hat{N}$ measurement when the state of the mode is $|0\rangle$ then the outcome 0 will occur with probability one. Thus, for the given number-operator-based decision rule, we have that,

$$\Pr(\,\text{say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |0\rangle\,) = 0.$$

On the other hand, if we make the $\hat{N}$ measurement when the state of the mode is $|\sqrt{N}\rangle$ then the outcome will be zero with probability $|\langle 0|\sqrt{N}\rangle|^2 = e^{-N}$. Thus, for the given number-operator-based decision rule, we have that,

$$\Pr(\text{ say "state was } |0\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\,) = e^{-N}.$$

From these conditional probability results we get the unconditional error probability via,

$$\begin{aligned}
\Pr(e) &= \Pr(\text{state} = |0\rangle)\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |0\rangle\,) \\
&= \Pr(\text{state} = |\sqrt{N}\rangle)\Pr(\text{ say "state was } |0\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\,) \\
&= \frac{1}{2}0 + \frac{1}{2}e^{-N} = \frac{1}{2}e^{-N}.
\end{aligned}$$

So, for $N \gg 1$ this number-operator-based decision rule achieves an error probability only a factor of two larger than that of the optimum receiver.

(d) For BPSK, we have that

$$\langle \alpha_{-1}|\alpha_1\rangle = \langle -\sqrt{N}|\sqrt{N}\rangle = e^{-2N},$$

which yields,

$$\Pr(e) = \frac{1}{2}\Big[1 - \sqrt{1 - e^{-4N}}\Big] \approx \frac{1}{4}e^{-4N}, \quad \text{for } N \gg 1,$$

for the error probability of the optimum decision rule.

Now, if we make the $\hat{a}_1$ measurement when the state is $|-\sqrt{N}\rangle$, then the outcome is a Gaussian random variable with mean $-\sqrt{N}$ and variance $1/4$. Thus, for the quadrature-based decision rule, we find that,

$$\begin{aligned}
\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |-\sqrt{N}\rangle\,) &= \int_0^\infty d\alpha_1\,\frac{e^{-2(\alpha_1+\sqrt{N})^2}}{\sqrt{\pi/2}} \\
&= \int_{\sqrt{4N}}^\infty dt\,\frac{e^{-t^2/2}}{\sqrt{2\pi}} = Q(\sqrt{4N}),
\end{aligned}$$

where the second equality follows from the change of variables $t = 2(\alpha_1 + \sqrt{N})$. Likewise, if we make the $\hat{a}_1$ measurement when the state is $|\sqrt{N}\rangle$, then the outcome is a Gaussian random variable with mean $\sqrt{N}$ and variance $1/4$. Thus, for the quadrature-based decision rule, we find that,

$$\begin{aligned}
\Pr(\text{ say "state was } |-\sqrt{N}\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\,) &= \int_{-\infty}^0 d\alpha_1\,\frac{e^{-2(\alpha_1-\sqrt{N})^2}}{\sqrt{\pi/2}} \\
&= \int_{\sqrt{4N}}^\infty dt\,\frac{e^{-t^2/2}}{\sqrt{2\pi}} = Q(\sqrt{4N}),
\end{aligned}$$

where the second equality follows from the change of variables $t = -2(\alpha_1 - \sqrt{N})$. From these conditional probability results we get the unconditional error probability via,

$$
\begin{aligned}
\Pr(e) &= \Pr(\text{state} = |-\sqrt{N}\rangle)\Pr(\text{ say "state was } |\sqrt{N}\rangle\text{"} \mid \text{state was } |0\rangle\,) \\
&= \Pr(\text{state} = |\sqrt{N}\rangle)\Pr(\text{ say "state was } |-\sqrt{N}\rangle\text{"} \mid \text{state was } |\sqrt{N}\rangle\,) \\
&= \frac{1}{2}Q(\sqrt{4N}) + \frac{1}{2}Q(\sqrt{4N}) = Q(\sqrt{4N}).
\end{aligned}
$$

Because $Q(x) \leq \frac{1}{2}e^{-x^2/2}$, for $x \geq 0$, we find that,

$$\Pr(e) \leq \frac{1}{2}e^{-2N},$$

for this quadrature-based decision rule. Comparison with the performance of the optimum system reveals that the quadrature measurement needs about 3 dB more average photon number to achieve the same performance as the optimum system for BPSK.

Problem 8.5

Here we shall consider a different variant of the binary hypothesis testing problem, one involving a positive operator-valued measurement instead of an observable.

(a) Because a and b are positive and $(|\phi\rangle\langle\phi|)^\dagger = |\phi\rangle\langle\phi|$ for any ket vector $|\phi\rangle$, it is clear that $\hat{\Pi}_{-1}$, $\hat{\Pi}_1$, and $\hat{\Pi}_e$ are all Hermitian operators, and that $\langle\psi|\hat{\Pi}_j|\psi\rangle \geq 0$ for $j = -1, 1, e$ and all $|\psi\rangle$. So, all that remains to be done is to see if a and b can be chosen to make them a resolution of the identity. We have that,

$$
\begin{aligned}
\hat{\Pi}_{-1} + \hat{\Pi}_1 + \hat{\Pi}_e &= a|\xi_{-1}\rangle\langle\xi_{-1}| + a|\xi_1\rangle\langle\xi_1| + b|x\rangle\langle x| \\
&= 2a\sin^2(\theta)|x\rangle\langle x| + 2a\cos^2(\theta)|y\rangle\langle y| + b|x\rangle\langle x|.
\end{aligned}
$$

In the $\{|x\rangle, |y\rangle\}$ orthonormal basis, we have that,

$$\hat{I}_2 = |x\rangle\langle x| + |y\rangle\langle y|.$$

It follows that $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$ will resolve the identity—and hence be a POVM on $\mathcal{H}$—if and only if,

$$2a\sin^2(\theta) + b = 1 \quad \text{and} \quad 2a\cos^2(\theta) = 1.$$

These conditions are satisfied by,

$$a = 1/2\cos^2(\theta) \quad \text{and} \quad b = 1 - \tan^2(\theta).$$

Note that $0 < \theta < \pi/4$ implies that $\tan^2(\theta) < 1$, ensuring that $b > 0$.

(b) For the given decision rule we have that,

$$\Pr(\text{ say "state was } |\psi_1\rangle\text{" } | \text{ state was } |\psi_{-1}\rangle\ ) = \langle\psi_{-1}|\hat{\Pi}_1|\psi_{-1}\rangle$$

$$= a|\langle\psi_{-1}|\xi_1\rangle|^2 = a|-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)|^2 = 0.$$

Likewise, we have that,

$$\Pr(\text{ say "state was } |\psi_{-1}\rangle\text{" } | \text{ state was } |\psi_1\rangle\ ) = \langle\psi_1|\hat{\Pi}_{-1}|\psi_1\rangle$$

$$= a|\langle\psi_1|\xi_{-1}\rangle|^2 = a|-\cos(\theta)\sin(\theta) + \sin(\theta)\cos(\theta)|^2 = 0.$$

Thus the POVM decision rule will never be incorrect when it says "state was $|\psi_{-1}\rangle$" or when it says "state was $|\psi_1\rangle$."

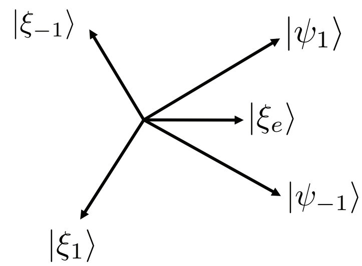

Figure 3: Geometry of the optimal POVM for $\theta = \pi/6$: $a = b = 2/3$ in this case.

These results come about because the ket vector $|\xi_{-1}\rangle$ associated with the POVM outcome -1 is orthogonal to $|\psi_1\rangle$, and the ket vector $|\xi_1\rangle$ associated with the POVM outcome 1 is orthogonal to $|\psi_{-1}\rangle$. Because $|\psi_{-1}\rangle$ and $|\psi_1\rangle$ are not orthogonal, it follows that $|\xi_{-1}\rangle$ and $|\xi_1\rangle$ will not be orthogonal either. Thus, to form them into a measurement, we need a POVM construction as opposed to an observable. In Fig. 3 we have sketched the geometry of this problem for the case $\theta = \pi/6$. Here we find that $a = 1/2\cos^2(\theta) = 2/3$ and $b = 1 - \tan^2(\theta) = 2/3$, so that the POVM has the geometry we explored in Problem 1.8.

(c) Our next task is to find the error probability of the POVM decision rule. We begin with the conditional error probabilities:

$$\Pr(\text{ say "error" } | \text{ state} = |\psi_{-1}\rangle\ ) = \langle\psi_{-1}|\hat{\Pi}_e|\psi_{-1}\rangle$$

$$= b|\langle\psi_{-1}|x\rangle|^2 = b\cos^2(\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta),$$

and

$$\Pr(\text{ say "error" } | \text{ state} = |\psi_1\rangle\ ) = \langle\psi_1|\hat{\Pi}_e|\psi_1\rangle$$

$$= b|\langle\psi_1|x\rangle|^2 = b\cos^2(\theta) = \cos^2(\theta) - \sin^2(\theta) = \cos(2\theta).$$

Because both conditional error probabilities are the same, they equal the unconditional error probability, i.e.,

$$\Pr(e) \equiv \Pr(\text{say “error”}) = \cos(2\theta).$$

(d) Our final task is very simple. For

$$|\psi_{-1}\rangle = \cos(\theta)|x\rangle - \sin(\theta)|y\rangle \quad \text{and} \quad |\psi_1\rangle = \cos(\theta)|x\rangle + \sin(\theta)|y\rangle,$$

we have that $\cos(2\theta) = \langle\psi_{-1}|\psi_1\rangle$, and for $|\psi_{-1}\rangle = |-\sqrt{N}\rangle$ and $|\psi_1\rangle = |\sqrt{N}\rangle$ we have that $\langle\psi_{-1}|\psi_1\rangle = e^{-2N}$. Thus, from (c), we have that $\Pr(e) = e^{-2N}$. This error probability is higher than what we found for the minimum error probability rule in Problem 8.2(d), but “error” in that problem meant confusing $|\psi_{-1}\rangle$ for $|\psi_1\rangle$ or vice versa, whereas “error” here means that our receiver says it cannot decide between the two hypotheses.