Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 7
Fall 2016

Issued: Thursday, October 20, 2016 **Due:** Thursday, October 27, 2016

Problem 7.1
Consider a single mode of a quantized electromagnetic field, viz., $\hat{a}e^{-j\omega t}/\sqrt{AT_o}$ for $(x,y) \in \mathcal{A}$ and $0 \le t \le T_o$ with $\mathcal{A}$ being a region in the $z = 0$ plane of area A. In class we have assumed that when this mode in unexcited it is in its vacuum state, $|0\rangle$. Strictly speaking this is not true if the field is in thermal equilibrium at absolute temperature T. Here we shall develop the quantum state that prevails in thermal equilibrium.

Let P_n be the probability that this field mode is in the number state $|n\rangle$. Statistical mechanics teaches that in thermal equilibrium this probability distribution, $\{ P_n : n = 0, 1, 2, \ldots \}$, maximizes the entropy of the system,

$$S(\{P_n\}) \equiv -\sum_{n=0}^{\infty} P_n \ln(P_n),$$

subject to a constraint on the system's average energy above the ground state, i.e., its average energy above the zero-point-fluctuation energy $\hbar\omega/2$, namely:

$$\hbar\omega\langle \hat{a}^\dagger \hat{a}\rangle = \sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$$

(a) Define an objective function,

$$F(\{P_n\}, \lambda_1, \lambda_2) \equiv -\sum_{n=0}^{\infty} P_n \ln(P_n) + \lambda_1\left(1 - \sum_{n=0}^{\infty} P_n\right) + \lambda_2\left(\mathcal{E} - \sum_{n=0}^{\infty} \hbar\omega n P_n\right),$$

where λ_1 and λ_2 are Lagrange multipliers, with the former being dimensionless and the latter having units $(\text{Joules})^{-1}$. Show that maximizing $S(\{P_n\})$ over the $\{P_n\}$ subject to the constraints that $\sum_{n=0}^{\infty} P_n = 1$ and $\sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$ is equivalent to maximizing $F(\{P_n\}, \lambda_1, \lambda_2)$ without constraints.

(b) Show that the maximum of $F(\{P_n\}, \lambda_1, \lambda_2)$ occurs at,

$$P_n = e^{-(1+\lambda_1+n\hbar\omega\lambda_2)}, \quad \text{for } n = 0, 1, 2, \ldots, \tag{1}$$

where λ_1 and λ_2 are used to ensure that $\sum_{n=0}^{\infty} P_n = 1$ and $\sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$ prevail.

(c) Use $\sum_{n=0}^{\infty} P_n = 1$ to eliminate λ_1 from Eq. (1).

(d) Use $\sum_{n=0}^{\infty} \hbar\omega n P_n = \mathcal{E}$ to find $\mathcal{E}$ as a function of $\hbar\omega$ and λ_2.

(e) Statistical mechanics tells us that $\lambda_2 = 1/kT$ where k is Boltzmann's constant ($k = 1.38 \times 10^{-23}$ Joules/K) and T is the absolute temperature (in degrees K). If you use this expression for λ_2, your result for $\mathcal{E}$ from (d) will become Planck's radiation law. Evaluate $N \equiv \mathcal{E}/\hbar\omega$, i.e., the average photon number of the thermal equilibrium state for wavelength $\lambda \equiv 2\pi c/\omega = 1.55\,\mu$m (the fiber-optic communication wavelength) and $T = 290\,$K (room temperature).

(f) Use the results of (c) and (d) to show that $\{P_n\}$ is the Bose-Einstein distribution with mean N, i.e.,

$$P_n = \frac{N^n}{(N+1)^{n+1}}, \quad \text{for } n = 0, 1, 2, \ldots$$

Problem 7.2

The density operator for a single-mode quantum field, $\hat{a}_{IN}$, that is in thermal equilibrium at temperature $T\,$K is

$$\hat{\rho} = \sum_{n=0}^{\infty} P_n |n\rangle\langle n|,$$

where $\{P_n\}$ is the Bose-Einstein distribution from Problem 7.1(f) with

$$N = \frac{1}{e^{\hbar\omega/kT} - 1}.$$

Suppose that this field mode is the input to a phase-sensitive amplifier whose output satisfies,

$$\hat{a}_{OUT} = \mu \hat{a}_{IN} + \nu \hat{a}_{IN}^{\dagger},$$

with μ, ν real, positive, and obeying $\mu^2 - \nu^2 = 1$.

(a) Let $\hat{a}_{OUT_1} \equiv \text{Re}(\hat{a}_{OUT})$ and $\hat{a}_{OUT_2} \equiv \text{Im}(\hat{a}_{OUT})$. Find $\langle \hat{a}_{OUT_1} \rangle$ and $\langle \hat{a}_{OUT_2} \rangle$.

(b) Find $\langle \Delta \hat{a}_{OUT_1}^2 \rangle$ and $\langle \Delta \hat{a}_{OUT_2}^2 \rangle$.

Problem 7.3

Consider the semiclassical photon-counting configuration shown in Fig. 1. Here, a single-mode classical signal field, $a_S e^{-j\omega t}/\sqrt{AT}$ for $(x, y) \in \mathcal{A}$ in the $z = 0$ plane and $0 \le t \le T$ is incident on a unity-quantum-efficiency ideal photodetector whose area-A photosensitive region is $\mathcal{A}$. Given knowledge of $|a_S|^2$, the output of this photon counter, N_S, is a Poisson random variable with mean $\langle N_S \rangle = |a_S|^2$. Suppose that $a_S = a_{S_1} + j a_{S_2}$, where a_{S_1} and a_{S_2} are statistically independent, identically distributed, zero-mean, real-valued Gaussian random variables each with variance $N/2$.

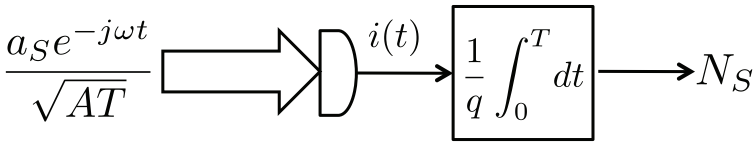

Figure 1: Semiclassical photon-counting configuration

(a) Use the results of Problems 1.4 and 1.3 (without rederiving them!) to find the probability density function of $|a_S|^2$.

(b) Use the result of Problem 1.5(a) (without rederiving it!) to find the unconditional probability distribution of the photon counter, viz., $\{ \Pr(N_S = n) : n = 0, 1, 2, \ldots \}$.

(c) Use the results of Problem 1.5(c) (without rederiving them!) to find $\langle N_S \rangle$ and $\langle \Delta N_S^2 \rangle$. Identify the shot noise and excess noise components of $\langle \Delta N_S^2 \rangle$.

Problem 7.4

Consider the semiclassical photon-counting configuration from Problem 7.3. Now we shall assume that $a_S = \alpha_S + n_S$, where α_S is a non-random positive-real number and $n_S = n_{S_1} + jn_{S_2}$ with n_{S_1} and n_{S_2} being statistically independent, identically distributed, zero-mean, real-valued Gaussian random variables each with variance $N/2$.

(a) Find $\langle N_S \rangle$, the unconditional mean of the photon count N_S.

(b) Find $\langle N_S^2 \rangle$, the unconditional mean-square of the photon count.

Hint: Complex-Gaussian moment factoring implies that $\langle |n_S|^4 \rangle = 2\langle |n_S|^2 \rangle^2$.

(c) Combine your answers to (a) and (b) to find $\langle \Delta N_S^2 \rangle$, and identify the shot noise and excess noise terms in your expression for this variance.

(d) Find the unconditional probability distribution of the photon counter.

Hint: Write the integral of the conditional probability distribution multiplied by the 2-D Gaussian distribution for a_S in polar coordinates, i.e., using $a_S = re^{j\phi}$ with $r \geq 0$. Integrate over ϕ and then use,

$$\int_0^\infty dr\, 2r^{2n+1} I_0(2|\alpha|r/N) e^{-r^2(N+1)/N} =$$

$$n! e^{\alpha^2/N(N+1)} \left(\frac{N}{N+1} \right)^{n+1} L_n\left(-\frac{\alpha^2}{N(N+1)} \right), \quad \text{for } \alpha \text{ real and } n = 0, 1, 2, \ldots,$$

where $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind, and

$$L_n(x) \equiv \sum_{m=0}^{n} (-1)^m \left(\begin{array}{c} n \\ n-m \end{array} \right) \frac{x^m}{m!},$$

is the nth Laguerre polynomial.

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