Massachusetts Institute of Technology
Department of Electrical Engineering and Computer Science

6.453 Quantum Optical Communication

Problem Set 5 Solutions
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Problem 5.1
Here we shall derive the signal-to-noise ratio (SNR) optimality of squeezed states for quadarature measurements.

(a) We know that $\hat{a} = \hat{a}_1 + j\hat{a}_2$, with $\hat{a}_1$ and $\hat{a}_2$ Hermitian, and that $[\hat{a}_1, \hat{a}_2] = j/2$. Thus we have that,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle (\hat{a}_1 - j\hat{a}_2)(\hat{a}_1 + j\hat{a}_2) \rangle.$$

Multiplying out and using the commutator we get,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \hat{a}_1^2 \rangle + \langle \hat{a}_2^2 \rangle - 1/2$$

Because mean-squared values equal variances plus squared-mean values we then have,

$$\langle \hat{a}^\dagger \hat{a} \rangle = \langle \Delta\hat{a}_1^2 \rangle + \langle \hat{a}_1 \rangle^2 + \langle \Delta\hat{a}_2^2 \rangle + \langle \hat{a}_2 \rangle^2 - 1/2.$$

Rearranging terms, and using the average photon number constraint, we find that,

$$\text{SNR} \leq \frac{N + 1/2 - \langle \Delta\hat{a}_2^2 \rangle - \langle \hat{a}_2 \rangle^2}{\langle \Delta\hat{a}_1^2 \rangle} - 1,$$

with equality if and only if $\langle \hat{a}^\dagger \hat{a} \rangle = N$. By making $\langle \hat{a}^\dagger \hat{a} \rangle = N$ and $\langle \hat{a}_2 \rangle = 0$, we can increase the SNR to,

$$\text{SNR} = \frac{N + 1/2 - \langle \Delta\hat{a}_2^2 \rangle}{\langle \Delta\hat{a}_1^2 \rangle} - 1.$$

(b) For fixed N and $\langle \Delta\hat{a}_1^2 \rangle$, the SNR expression we have just derived is maximized by a minimum uncertainty state, i.e., one which satisfies $\langle \Delta\hat{a}_1^2 \rangle \langle \Delta\hat{a}_2^2 \rangle = 1/16$, in which case

$$\text{SNR} = \frac{N + 1/2}{\langle \Delta\hat{a}_1^2 \rangle} - \frac{1}{(4\langle \Delta\hat{a}_1^2 \rangle)^2} - 1.$$

Defining $x = \langle \Delta\hat{a}_1^2 \rangle$, we can differentiate the preceding SNR expression to obtain,

$$\frac{d\text{SNR}}{dx} = -\frac{N + 1/2}{x^2} + \frac{1}{8x^3},$$

which has a unique root at $x = 1/8(N + 1/2)$. Differentiating a second time gives,

$$\frac{d^2\text{SNR}}{dx^2} = \frac{2(N + 1/2)}{x^3} - \frac{3}{8x^4},$$

which equals $-8^3(N + 1/2)^4 < 0$ at $x = 1/8(N + 1/2)$, so that the stationary point we have found is a maximum. The resulting optimal SNR value is then found, by substitution, to be $\text{SNR} = 4N(N + 1)$.

(c) For the squeezed state $|\beta;\mu,\nu\rangle$, we know that

$$\begin{aligned}\langle\hat{a}_1\rangle &= \mathrm{Re}(\mu^*\beta - \nu\beta^*),\\ \langle\Delta\hat{a}_1^2\rangle &= \frac{|\mu-\nu|^2}{4},\\ \langle\hat{a}^\dagger\hat{a}\rangle &= |\mu^*\beta-\nu\beta^*|^2+|\nu|^2.\end{aligned}$$

Substituting in $\beta=\sqrt{N(N+1)}$, $\mu=(N+1)/\sqrt{2N+1}$, and $\nu=N/\sqrt{2N+1}$, we get

$$\begin{aligned}\langle\hat{a}_1\rangle &= \sqrt{N(N+1)/(2N+1)},\\ \langle\Delta\hat{a}_1^2\rangle &= 1/4(2N+1),\\ \langle\hat{a}^\dagger\hat{a}\rangle &= N(N+1)/(2N+1)+N^2/(2N+1)=N.\end{aligned}$$

This state therefore has N photons on average, and its quadrature-measurement SNR equals the optimal value $4N(N+1)$.

(d) For the coherent state $|\sqrt{N}\rangle$ we have,

$$\begin{aligned}\langle\hat{a}_1\rangle &= \mathrm{Re}(\sqrt{N})=\sqrt{N},\\ \langle\Delta\hat{a}_1^2\rangle &= 1/4,\\ \langle\hat{a}^\dagger\hat{a}\rangle &= N.\end{aligned}$$

This state therefore has N photons on average, and its quadrature-measurement SNR equals $4N$. The optimal squeezed state has a larger quadrature-measurement SNR by a factor of $N+1$; for $N\gg1$, this is an enormous SNR advantage.

Problem 5.2

Here we shall introduce the notion of normally-ordered forms.

(a) This is a straightforward exercise. We have that

$$\hat{F}\equiv\hat{a}\hat{a}^\dagger\hat{a}=\left(\hat{a}^\dagger\hat{a}+\left[\hat{a},\hat{a}^\dagger\right]\right)\hat{a}=\hat{a}^\dagger\hat{a}^2+\hat{a},$$

and

$$\hat{F}\equiv\hat{a}\hat{a}^\dagger\hat{a}=\hat{a}\left(\hat{a}\hat{a}^\dagger-\left[\hat{a},\hat{a}^\dagger\right]\right)=\hat{a}^2\hat{a}^\dagger-\hat{a},$$

whence

$$F^{(n)}(\hat{a}^\dagger,\hat{a})=\hat{a}^\dagger\hat{a}^2+\hat{a},$$

and

$$F^{(a)}(\hat{a}, \hat{a}^\dagger) = \hat{a}^2\hat{a}^\dagger - \hat{a},$$

Because $\hat{F} = F^{(n)}(\hat{a}^\dagger, \hat{a})$ we have that,

$$\begin{aligned}
\langle\alpha|\hat{F}|\alpha\rangle &= \langle\alpha|F^{(n)}(\hat{a}^\dagger, \hat{a})|\alpha\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}^2 + \hat{a}|\alpha\rangle \\
&= \alpha^*\alpha^2 + \alpha,
\end{aligned}$$

where the last equality follows from (repeated) use of the eigen relations,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \text{and} \quad \langle\alpha|\hat{a}^\dagger = \langle\alpha|\alpha^*.$$

(b) Using the coherent-state identity resolution twice, we get

$$\hat{G} = \hat{I}\hat{G}\hat{I} = \iint \frac{d^2\alpha}{\pi}\,\frac{d^2\beta}{\pi}\,\langle\alpha|\hat{G}|\beta\rangle\,|\alpha\rangle\langle\beta|.$$

(c) We know that $\hat{F} = F^{(n)}(\hat{a}^\dagger, \hat{a})$, thus

$$\langle\alpha|\hat{F}|\beta\rangle = \langle\alpha|F^{(n)}(\hat{a}^\dagger, \hat{a})|\beta\rangle = \langle\alpha|\hat{a}^\dagger\hat{a}^2 + \hat{a}|\beta\rangle = (\alpha^*\beta^2 + \beta)\langle\alpha|\beta\rangle.$$

Using $F^{(n)}(\alpha^*, \alpha) = \alpha^*\alpha^2 + \alpha$, from (a), with α^* and α treated as independent variables we now get,

$$\langle\alpha|\hat{F}|\beta\rangle = F^{(n)}(\alpha^*, \beta)\langle\alpha|\beta\rangle = F^{(n)}(\alpha^*, \beta)e^{-(|\alpha|^2+|\beta|^2)/2+\alpha^*\beta},$$

where the last result uses the coherent-state inner product that we have derived in a previous problem set.

(d) The density operator is an Hermitian operator whose eigenvalues form a probability distribution. Moreover, $\langle\psi|\hat{\rho}|\psi\rangle$, for any unit-length ket $|\psi\rangle$, is the probability that the oscillator will be found in state $|\psi\rangle$. Because the coherent states are normalized to unit length, we have that $\langle\alpha|\hat{\rho}|\alpha\rangle \geq 0$. Because the coherent states resolve the identity and the trace of an operator can be computed by summing its matrix elements in this overcomplete basis, we have that,

$$\int \frac{d^2\alpha}{\pi}\,\langle\alpha|\hat{\rho}|\alpha\rangle = \mathrm{tr}(\hat{\rho}) = 1,$$

where the last equality was proven on a previous problem set. It follows that $p(\alpha_1, \alpha_2) \equiv \rho^{(n)}(\alpha^*, \alpha)/\pi = \langle\alpha|\hat{\rho}|\alpha\rangle/\pi$ is a proper joint probability density for two real-valued random variables. We shall see in class that this density characterizes the measurement statistics of heterodyne detection, viz., a joint measurement of *both* quadratures of the oscillator.

Problem 5.3
Here we will introduce the three most important characteristic functions for quantum statistical analyses.

(a) We have that $[\hat{a}, \hat{a}^\dagger] = 1$, so that $[-\zeta^* \hat{a}, \zeta \hat{a}^\dagger] = -|\zeta|^2$. It follows that,

$$[-\zeta^* \hat{a}, [-\zeta^* \hat{a}, \zeta \hat{a}^\dagger]] = [\zeta \hat{a}^\dagger, [-\zeta^* \hat{a}, \zeta \hat{a}^\dagger]] = 0,$$

and hence (from the identities given in the problem statement),

$$e^{-\zeta^* \hat{a} + \zeta \hat{a}^\dagger} = e^{-\zeta^* \hat{a}} e^{\zeta \hat{a}^\dagger} e^{|\zeta|^2/2} = e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} e^{-|\zeta|^2/2}.$$

Multiplying these equalities by $\hat{\rho}$ and taking the trace we obtain the relations we were seeking:

$$\chi_W^\rho(\zeta^*, \zeta) = \chi_A^\rho(\zeta^*, \zeta) e^{|\zeta|^2/2} = \chi_N^\rho(\zeta^*, \zeta) e^{-|\zeta|^2/2}.$$

It is now easy to use these relations to find all three characteristic functions for the coherent-state density operator, $\hat{\rho} = |\alpha\rangle\langle\alpha|$. We start with the normally-ordered characteristic function,

$$\begin{aligned} \chi_N^\rho(\zeta^*, \zeta) &\equiv \mathrm{tr}\Big(\hat{\rho} e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}\Big) = \mathrm{tr}\Big(|\alpha\rangle\langle\alpha| e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}\Big) \\ &= \langle\alpha| e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}} |\alpha\rangle = e^{\zeta \alpha^* - \zeta^* \alpha}. \end{aligned}$$

We then immediately obtain,

$$\chi_W^\rho(\zeta^*, \zeta) = \chi_N^\rho(\zeta^*, \zeta) e^{-|\zeta|^2/2} = e^{\zeta \alpha^* - \zeta^* \alpha - |\zeta|^2/2},$$

and

$$\chi_A^\rho(\zeta^*, \zeta) = \chi_W^\rho(\zeta^*, \zeta) e^{-|\zeta|^2/2} = e^{\zeta \alpha^* - \zeta^* \alpha - |\zeta|^2}.$$

(b) We have that,

$$\chi_A^\rho(\zeta^*, \zeta) \equiv \mathrm{tr}\Big(\hat{\rho} e^{-\zeta^* \hat{a}} e^{\zeta \hat{a}^\dagger}\Big).$$

Introducing,

$$\hat{I} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha|,$$

in between the exponentials in the χ_A^ρ definition yields,

$$\begin{aligned}
\chi_A^\rho(\zeta^*,\zeta) &= \int \frac{d^2\alpha}{\pi}\,\mathrm{tr}\Big(\hat\rho e^{-\zeta^*\hat a}|\alpha\rangle\langle\alpha|e^{\zeta\hat a^\dagger}\Big)\\
&= \int \frac{d^2\alpha}{\pi}\,\mathrm{tr}\big(\hat\rho|\alpha\rangle\langle\alpha|e^{-\zeta^*\alpha+\zeta\alpha^*}\big)\\
&= \int \frac{d^2\alpha}{\pi}\,\langle\alpha|\hat\rho|\alpha\rangle e^{-\zeta^*\alpha+\zeta\alpha^*}\\
&= \iint d\alpha_1\,d\alpha_2\,\rho^{(n)}(\alpha^*,\alpha)\frac{e^{2j\zeta_2\alpha_1-2j\zeta_1\alpha_2}}{\pi}\\
&= \left.\frac{\mathcal{F}[\rho^{(n)}(\alpha^*,\alpha)]}{\pi}\right|_{\substack{f_1=-\zeta_2/\pi\\ f_2=\zeta_1/\pi}},
\end{aligned}$$

where $\mathcal{F}[x(t_1,t_2)]$ denotes the 2-D Fourier transform,

$$X(f_1,f_2)=\mathcal{F}[x(t_1,t_2)]\equiv\iint dt_1\,dt_2\,x(t_1,t_2)e^{-j2\pi(f_1t_1+f_2t_2)}.$$

For future use, we note that the standard 2-D inverse Fourier transform,

$$x(t_1,t_2)=\mathcal{F}^{-1}[X(f_1,f_2)]\equiv\iint df_1\,df_2\,X(f_1,f_2)e^{j2\pi(f_1t_1+f_2t_2)},$$

can be used to show that

$$\rho^{(n)}(\alpha^*,\alpha)=\iint d\zeta_1\,d\zeta_2\,\chi_A^\rho(\zeta^*,\zeta)\frac{e^{-2j\zeta_2\alpha_1+2j\zeta_1\alpha_2}}{\pi}.$$

(c) All we need to do is to show that we can recover the diagonal elements in the coherent-state representation from

$$\hat\rho=\int\frac{d^2\zeta}{\pi}\,\chi_A^\rho(\zeta^*,\zeta)e^{-\zeta\hat a^\dagger}e^{\zeta^*\hat a}.$$

This calculation is simple:

$$\begin{aligned}
\langle\alpha|\hat\rho|\alpha\rangle &= \int\frac{d^2\zeta}{\pi}\,\chi_A^\rho(\zeta^*,\zeta)\langle\alpha|e^{-\zeta\hat a^\dagger}e^{\zeta^*\hat a}|\alpha\rangle\\
&= \int\frac{d^2\zeta}{\pi}\,\chi_A^\rho(\zeta^*,\zeta)e^{-\zeta\alpha^*+\zeta^*\alpha}\\
&= \iint d\zeta_1\,d\zeta_2\,\chi_A^\rho(\zeta^*,\zeta)\frac{e^{-2j\zeta_2\alpha_1+2j\zeta_1\alpha_2}}{\pi},
\end{aligned}$$

which equals $\rho^{(n)}(\alpha^*,\alpha)$, as was to be shown, from the result of (b).

(d) If $\hat{\rho}$ has a proper P-representation, then

$$\begin{aligned}
\chi_N^\rho(\zeta^*, \zeta) &\equiv \mathrm{tr}\Big(\hat{\rho} e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}\Big) \\
&= \int d^2\alpha\, P(\alpha, \alpha^*) \mathrm{tr}\Big(|\alpha\rangle\langle\alpha| e^{\zeta \hat{a}^\dagger} e^{-\zeta^* \hat{a}}\Big) \\
&= \int d^2\alpha\, P(\alpha, \alpha^*) e^{\zeta\alpha^* - \zeta^*\alpha} \\
&= \int d^2\alpha\, P(\alpha, \alpha^*) e^{2j\zeta_2\alpha_1 - 2j\zeta_1\alpha_2},
\end{aligned}$$

again a 2-D Fourier transform relationship. Keeping track of the normalization constant (factors of π in each Fourier dimension), we have that the inverse Fourier relationship is

$$P(\alpha, \alpha^*) = \int \frac{d^2\zeta}{\pi^2}\, \chi_N^\rho(\zeta^*, \zeta) e^{-\zeta\alpha^* + \zeta^*\alpha}.$$

(e) This part is trivial. We are told that the characteristic function for the classical outcome of the $\hat{a}_\theta$ measurement is

$$M_{\alpha_\theta}(jv) = \mathrm{tr}\Big(\hat{\rho} e^{jv\hat{a}_\theta}\Big).$$

Substituting in the definition $\hat{a}_\theta = [\hat{a}e^{-j\theta} + \hat{a}^\dagger e^{j\theta}]/2$, we see that

$$M_{\alpha_\theta}(jv) = \mathrm{tr}\Big(\hat{\rho} e^{-(-jve^{-j\theta}/2)\hat{a} + (jve^{j\theta}/2)\hat{a}^\dagger}\Big) = \chi_W^\rho(-jve^{-j\theta}/2, jve^{j\theta}/2).$$

Problem 5.4

Here we shall that it is easy to calculate number-operator and quadrature-operator measurement statistics when the oscillator has a proper P-representation

(a) Suppose that the quantum harmonic oscillator is in the coherent state $|\alpha\rangle$ with classical probability density $p(\alpha_1, \alpha_2)$. If $\hat{O}$ is any observable, then the conditional probability that the outcome of this measurement will be the eigenvalue o, given that the oscillator is in the state $|\alpha\rangle$, is $|\langle o|\alpha\rangle|^2$. (Without appreciable loss of generality, we have assumed that the eigenspace associated with the eigenvalue o is one-dimensional, and spanned by the unit-length eigenkets $\{|o\rangle\}$.) The unconditional probability that we get outcome o is therefore,

$$\iint d\alpha_1\, d\alpha_2\, p(\alpha_1, \alpha_2) |\langle o|\alpha\rangle|^2 = \langle o| \left(\int d^2\alpha\, p(\alpha_1, \alpha_2) |\alpha\rangle\langle\alpha| \right) |o\rangle.$$

But we know that the unconditional probability for the $\hat{O}$ measurement to yield outcome o is given by, $\langle o|\hat{\rho}|o\rangle$, where $\hat{\rho}$ is the oscillator's density operator. It follows, because $|o\rangle$ can be an arbitrary unit-length ket, that

$$\hat{\rho} = \int d^2\alpha\, p(\alpha_1, \alpha_2)|\alpha\rangle\langle\alpha|,$$

specifies the density operator in terms of the classical probability density for the state to be $|\alpha\rangle$. We are given that the density operator has a proper P-representation, i.e.,

$$\hat{\rho} = \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|,$$

so it is clear that $p(\alpha_1, \alpha_2) = P(\alpha, \alpha^*)$ is the probability density that the state is $|\alpha\rangle$.

(b) When we are in the coherent state $|\alpha\rangle$ the $\hat{N}$-measurement has Poisson statistics,

$$\Pr(\,\hat{N}\text{ outcome } = n \mid \text{state is } |\alpha\rangle\,) = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots$$

Averaging over the proper P-representation—which specifies the probability density that the state will be $|\alpha\rangle$—then gives us the unconditional probability distribution:

$$\Pr(\hat{N}\text{ outcome } = n) = \int d^2\alpha\, P(\alpha, \alpha^*)\frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}, \quad \text{for } n = 0, 1, 2, \ldots$$

Because the variance of the $\hat{N}$ measurement equals the mean of the conditional variance plus the variance of the conditional mean, and the variance of the conditional mean is non-negative, we have that

$$\begin{aligned}\langle\Delta\hat{N}^2\rangle &\geq \int d^2\alpha\, P(\alpha, \alpha^*)\text{var}(\,\hat{N}\text{ measurement } \mid \text{ state is } |\alpha\rangle\,) \\ &= \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha|^2,\end{aligned}$$

where the last equality uses the fact that the conditional distribution for the $\hat{N}$-measurement is Poisson with mean (and hence variance) $|\alpha|^2$. By a similar iterated expectation calculation we have that the mean of the $\hat{N}$-measurement equals the mean of its conditional mean, viz.,

$$\langle\hat{N}\rangle = \int d^2\alpha\, P(\alpha, \alpha^*)E(\,\hat{N}\text{ measurement } \mid \text{ state is } |\alpha\rangle\,) = \int d^2\alpha\, P(\alpha, \alpha^*)|\alpha|^2,$$

completing the proof that states with proper P-representations satisfy,

$$\langle\Delta\hat{N}^2\rangle \geq \langle\hat{N}\rangle.$$

Note that the number state $|n\rangle$ has $\langle\hat{N}\rangle = n$ and $\langle\Delta\hat{N}^2\rangle = 0$, and so the density operator $\hat{\rho} = |n\rangle\langle n|$ does *not* have a proper P-representation for $n > 0$.

(c) When we are in the coherent state $|\alpha\rangle$, the probability density for the outcome of the $\hat{a}_1$ quadrature measurement is Gaussian with mean $\alpha_1 = \mathrm{Re}(\alpha)$ and variance 1/4. Thus the unconditional density function for this measurement outcome to be a_1, when the density operator has a proper P-representation is,

$$p(\hat{a}_1 \text{ outcome } = a_1) = \int d^2\alpha \, P(\alpha, \alpha^*) \frac{\exp[-2(a_1 - \alpha_1)^2]}{\sqrt{\pi/2}}.$$

Via the same iterated expectation approach used in (b), we know that the variance of the $\hat{a}_1$ measurement equals or exceeds the mean of the conditional variance, i.e.,

$$\begin{aligned}\langle \Delta\hat{a}_1^2 \rangle \;&\geq\; \int d^2\alpha \, P(\alpha, \alpha^*) \mathrm{var}(\, \hat{a}_1 \text{ measurement } \mid \text{ state is } |\alpha\rangle\,) \\ &= \int d^2\alpha \, P(\alpha, \alpha^*) 1/4 = 1/4.\end{aligned}$$

Note that the squeezed state $|\beta; \mu, \nu\rangle$ with $\mu, \nu > 0$ has $\langle \Delta\hat{a}_1^2 \rangle = (\mu - \nu)^2/4 < 1/4$, and so the density operator $\hat{\rho} = |\beta; \mu, \nu\rangle\langle\beta; \mu, \nu|$ for $\mu, \nu > 0$ does *not* have a proper P-representation.