Massachusetts Institute of Technology
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6.453 Quantum Optical Communication

Problem Set 3
Fall 2016

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Problem 3.1
Here we shall extend the results of Problem 2.2 to include classically-random polarizations. Suppose we have a $+z$-propagating, frequency-ω photon whose polarization vector (in Problem 2.1 notation) is,

$$\mathbf{i} \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

where α_x and α_y are a pair of complex-valued classical random variables that satisfy

$$|\alpha_x|^2 + |\alpha_y|^2 = 1,$$

with probability one. (Two joint complex-valued random variables, α_x and α_y, are really four joint real-valued random variables, viz., the real and imaginary parts of α_x and α_y.)

The Poincaré sphere representation for the *average* behavior of this random polarization vector is,

$$\mathbf{r} \equiv \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} 2\text{Re}[\langle \alpha_x^* \alpha_y \rangle] \\ 2\text{Im}[\langle \alpha_x^* \alpha_y \rangle] \\ \langle |\alpha_x|^2 \rangle - \langle |\alpha_y|^2 \rangle \end{bmatrix},$$

where—in keeping with the quantum notation for averages—$\langle\cdot\rangle$ denotes ensemble average.

(a) Use the Schwarz inequality to prove that $\mathbf{r}^T\mathbf{r} \equiv r_1^2 + r_2^2 + r_3^2 \leq 1$, i.e., the $\mathbf{r}$ vector lies on or inside the unit sphere.

(b) Let $\mathbf{i}_a$ and $\mathbf{i}_b$ be a pair of deterministic, orthogonal, complex-valued unit vectors, viz.,

$$\mathbf{i}_k^\dagger \mathbf{i}_l = \delta_{kl} \equiv \begin{cases} 1, & k = l \\ 0, & k \neq l \end{cases},$$

where k and l can each be either a or b. By means of wave plates, a polarizing beam splitter, and a pair of ideal photon counters, it is possible to measure whether the photon is polarized along $\mathbf{i}_a$ or along $\mathbf{i}_b$, by which we mean whether the $\mathbf{i}_a$-polarization or the $\mathbf{i}_b$-polarization detector is the one that registers a photon detection. The statistics of this measurement satisfy,

$$\begin{aligned}
\Pr(\text{polarized along } \mathbf{i}_a) &= \frac{1+\mathbf{r}_a^T\mathbf{r}}{2}, && (1)\\
\Pr(\text{polarized along } \mathbf{i}_b) &= \frac{1+\mathbf{r}_b^T\mathbf{r}}{2}, && (2)
\end{aligned}$$

where $\mathbf{r}_a$ and $\mathbf{r}_b$ are the Poincaré sphere representations of $\mathbf{i}_a$ and $\mathbf{i}_b$, respectively. Use the orthogonality of $\mathbf{i}_a$ and $\mathbf{i}_b$ to show that $\mathbf{r}_a = -\mathbf{r}_b$, so that Eqs. (1) and (2) constitute a proper probability distribution.

(c) Suppose that the photon's random polarization leads to $\mathbf{r} = \mathbf{0}$, i.e., $r_1 = r_2 = r_3 = 0$. Show that

$$\Pr(\text{polarized along } \mathbf{i}_a) = \Pr(\text{polarized along } \mathbf{i}_b) = \frac{1}{2},$$

for all pairs of deterministic, orthogonal complex-valued unit vectors $\{\mathbf{i}_a, \mathbf{i}_b\}$, and thus that $\mathbf{r} = \mathbf{0}$ represents a state of completely random polarization. Contrast the preceding measurement statistics with what will be obtained when

$$\mathbf{r} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}_a = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{r}_b = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix},$$

and when

$$\mathbf{r} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}_a = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{r}_b = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix},$$

are the Poincaré sphere representations of the photon and the pair of orthogonal polarizations being measured.

Problem 3.2

Here we introduce the notion of a density operator, i.e., a way to account for classical randomness limiting our knowledge of a quantum system's state. Consider a quantum mechanical system whose state is not known. Instead, there is a classical probability distribution for this state. In particular, suppose that there are M distinct unit-length kets, $\{\, |\psi_m\rangle : 1 \le m \le M \,\}$, and that the system is known to be in one of these states. Moreover the probability that it is in state $|\psi_m\rangle$ is p_m, for $1 \le m \le M$, where $\{\, p_m : 1 \le m \le M \,\}$ is a classical probability distribution: $p_m \ge 0$ and $\sum_{m=1}^{M} p_m = 1$.

(a) Suppose that we measure the observable $\hat{O}$, where $\hat{O}$ has distinct eigenvalues, $\{ o_n : 1 \leq n < \infty \}$, and a complete orthonormal set of associated eigenkets, $\{ |o_n\rangle : 1 \leq n < \infty \}$. GIVEN that the state of the system is $|\psi_m\rangle$, we know that the $\hat{O}$ measurement will yield outcome o_n with conditional probability $\Pr(o_n \mid |\psi_m\rangle) \equiv |\langle o_n|\psi_m\rangle|^2$, for $1 \leq n < \infty$ and $1 \leq m \leq M$. Use this conditional probability distribution to obtain the unconditional probability, $\Pr(o_n)$, of getting the outcome o_n when we make the $\hat{O}$ measurement.

(b) Define a density operator for the system by,

$$\hat{\rho} \equiv \sum_{m=1}^{M} p_m |\psi_m\rangle\langle\psi_m|.$$

Show that $\hat{\rho}$ is an Hermitian operator, and verify that your answer to (a) can be reduced to

$$\Pr(o_n) = \langle o_n|\hat{\rho}|o_n\rangle, \quad \text{for } 1 \leq n < \infty.$$

(c) Show that the expected value of the $\hat{O}$ measurement, i.e.,

$$\langle \hat{O} \rangle \equiv \sum_{n=1}^{\infty} o_n \Pr(o_n),$$

satisfies

$$\langle \hat{O} \rangle = \text{tr}(\hat{\rho}\hat{O}),$$

where $\text{tr}(\hat{A})$ for any linear Hilbert-space operator, $\hat{A}$, is the trace of that operator, defined as follows. Let $\{ |k\rangle : 1 \leq k < \infty \}$ be an arbitrary complete set of orthonormal kets on the quantum system's state space, so that

$$\hat{I} = \sum_{k=1}^{\infty} |k\rangle\langle k|.$$

Then

$$\text{tr}(\hat{A}) \equiv \sum_{k=1}^{\infty} \langle k|\hat{A}|k\rangle,$$

i.e., it is the sum of the operator's diagonal matrix-elements in the $\{|k\rangle\}$ representation. **Comment:** The trace operation is invariant to the choice of the CON basis used for its calculation. Hence a propitious choice of the basis can be a great aid in simplifying the calculation of averages involving a density operator.

Problem 3.3

Here we will explore the difference between a pure state and a mixed state, i.e., the difference between knowing that a quantum system is in a definite state $|\psi\rangle$ as opposed to having a classically-random distribution over a set of such states, namely a density operator $\hat{\rho}$. Because the density operator is Hermitian, it has eigenvalues and eigenkets. Let us assume that these form a countable set, viz., $\hat{\rho}$ has eigenvalues, $\{\, \rho_n : 1 \leq n < \infty \,\}$, and associated eigenkets $\{\, |\rho_n\rangle : 1 \leq n < \infty \,\}$, that satisfy

$$\hat{\rho}|\rho_n\rangle = \rho_n|\rho_n\rangle, \quad \text{for } 1 \leq n < \infty.$$

Without loss of generality, we can assume that these eigenkets form a complete orthonormal set, i.e.,

$$\begin{aligned} \langle\rho_m|\rho_n\rangle &= \delta_{nm}, \\ \hat{I} &= \sum_{n=1}^{\infty} |\rho_n\rangle\langle\rho_n|. \end{aligned}$$

(a) Show that the eigenvalues $\{\rho_n\}$ satisfy

$$0 \leq \rho_n \leq 1, \quad \text{for } 1 \leq n < \infty,$$

and

$$\sum_{n=1}^{\infty} \rho_n = 1.$$

(b) Show that $\text{tr}(\hat{\rho}) = 1$ for any density operator

(c) Suppose that the quantum system is in a pure state, i.e., it is known to be in the state $|\psi\rangle$. Show that this situation can be represented in density-operator form by setting $\rho_1 = 1$ and $|\rho_1\rangle = |\psi\rangle$, viz., a pure state has a density operator with only one eigenket whose associated eigenvalue is non-zero. Show that $\text{tr}(\hat{\rho}^2) = 1$ for any pure-state density operator.

(d) When the density operator has two or more eigenkets with non-zero eigenvalues we say that the state is mixed, i.e., there are at least two different pure states that can occur with non-zero probabilities. Show that $\text{tr}(\hat{\rho}^2) < 1$ for any mixed-state density operator.

Problem 3.4

In this problem we shall explore the density operator for single-photon polarization. Suppose that we are interested in the polarization state of a frequency-ω, $+z$-propagating, single photon. We know that a pure state of such a photon can be written as the 2-D complex-valued ket vector,

$$|\mathbf{i}\rangle \equiv \begin{bmatrix} \alpha_x \\ \alpha_y \end{bmatrix},$$

in the x-y (horizontal-vertical) basis, with $|\alpha_x|^2 + |\alpha_y|^2 = 1$. If we measure the polarization state of this photon using the basis,

$$|\mathbf{i}'\rangle \equiv \begin{bmatrix} \alpha'_x \\ \alpha'_y \end{bmatrix},$$

and

$$|\mathbf{i}'_\perp\rangle \equiv \begin{bmatrix} \alpha'^*_y \\ -\alpha'^*_x \end{bmatrix},$$

where $|\alpha'_x|^2 + |\alpha'_y|^2 = 1$, then we will get outcome $\mathbf{i}'$ with probability

$$\Pr(\,\mathbf{i}' \mid |\mathbf{i}\rangle\,) = |\langle \mathbf{i}'|\mathbf{i}\rangle|^2,$$

and outcome $\mathbf{i}'_\perp$ with probability

$$\Pr(\,\mathbf{i}'_\perp \mid |\mathbf{i}\rangle\,) = |\langle \mathbf{i}'_\perp|\mathbf{i}\rangle|^2 = 1 - \Pr(\,\mathbf{i}' \mid |\mathbf{i}\rangle\,)$$

(a) Verify that the density operator for this pure state,

$$\hat{\rho} = |\mathbf{i}\rangle\langle\mathbf{i}|$$

gives these same probabilities via

$$\Pr(\,\mathbf{i}' \mid |\mathbf{i}\rangle\,) = \langle \mathbf{i}'|\hat{\rho}|\mathbf{i}'\rangle,$$

and

$$\Pr(\,\mathbf{i}'_\perp \mid |\mathbf{i}\rangle\,) = \langle \mathbf{i}'_\perp|\hat{\rho}|\mathbf{i}'_\perp\rangle = 1 - \Pr(\,\mathbf{i}' \mid |\mathbf{i}\rangle\,).$$

(b) Now suppose that the single photon is in a mixed state, i.e., that α_x and α_y are complex-valued random variables whose joint distribution ensures that $|\alpha_x|^2 + |\alpha_y|^2 = 1$ with probability one. Show that the density operator $\hat{\rho}$ can now be written in the form

$$\hat{\rho} = \begin{bmatrix} \langle|\alpha_x|^2\rangle & \langle\alpha_x\alpha_y^*\rangle \\ \langle\alpha_x^*\alpha_y\rangle & \langle|\alpha_y|^2\rangle \end{bmatrix},$$

by verifying that this expression yields the proper formulas for the unconditional measurement probabilities, $\Pr(\mathbf{i}')$ and $\Pr(\mathbf{i}'_\perp)$, i.e.,

$$\langle \mathbf{i}'|\hat{\rho}|\mathbf{i}'\rangle = \Pr(\mathbf{i}') = \int d\alpha_x \int d\alpha_y\, p(\alpha_x, \alpha_y)\, \Pr(\,\mathbf{i}' \mid |\mathbf{i}\rangle\,),$$

and

$$\langle \mathbf{i}'_\perp|\hat{\rho}|\mathbf{i}'_\perp\rangle = \Pr(\mathbf{i}'_\perp) = \int d\alpha_x \int d\alpha_y\, p(\alpha_x, \alpha_y)\, \Pr(\,\mathbf{i}'_\perp \mid |\mathbf{i}\rangle\,),$$

where $p(\alpha_x, \alpha_y)$ is the joint probability density for α_x and α_y. Note that you have just shown that the preceding form of the density operator is equivalent to the mixed-state Poincaré vector that you studied in Problem 3.1.

Problem 3.5
Let $\hat{A}$ and $\hat{B}$ be observables for some quantum system. In particular, let $\hat{A}$ and $\hat{B}$ each be Hermitian operators with complete orthonormal (CON) sets of eigenkets, $\{\, |a_n\rangle : 1 \leq n < \infty \,\}$ and $\{\, |b_n\rangle : 1 \leq n < \infty \,\}$, and associated eigenvalues, $\{\, a_n : 1 \leq n < \infty \,\}$ and $\{\, b_n : 1 \leq n < \infty \,\}$, respectively.

(a) The commutator of $\hat{A}$ and $\hat{B}$ is, by definition,

$$\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A}.$$

Show that $\frac{1}{j}\left[\hat{A}, \hat{B}\right]$ is an Hermitian operator.

(b) Assume that these observables commute, i.e.,

$$\left[\hat{A}, \hat{B}\right] \equiv \hat{A}\hat{B} - \hat{B}\hat{A} = 0,$$

and that the eigenvalues of $\hat{A}$ are distinct, as are the eigenvalues of $\hat{B}$. Show that every eigenket of $\hat{A}$ is also an eigenket of $\hat{B}$ and that every eigenket of $\hat{B}$ is also an eigenket of $\hat{A}$, i.e., $\hat{A}$ and $\hat{B}$ have a common, CON set of eigenkets.

Problem 3.6
Here we introduce the notation of tensor products, to permit us to deal with multiple quantum systems. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be the Hilbert spaces of possible states for two quantum systems, $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. If we are interested in making a joint measurement on these two systems, e.g., the sum of their "positions", etc., we need to have a way to describe states and observables for the joint system. Let $\{\, |\phi_n\rangle_1 : 1 \leq n < \infty \,\}$ and $\{\, |\theta_m\rangle_2 : 1 \leq m < \infty \,\}$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, where the subscripts 1 and 2 indicate to which Hilbert space the states belong. The Hilbert space of states for the *joint* quantum system—i.e., systems 1 and 2 together—is spanned by the tensor product states $\{\, |\phi_n\rangle_1 \otimes |\theta_m\rangle_2 : 1 \leq n, m < \infty \,\}$, i.e., an arbitrary state $|\psi\rangle \in \mathcal{H}$ can be expressed as a linear combination,

$$|\psi\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{nm}(|\phi_n\rangle_1 \otimes |\theta_m\rangle_2), \tag{3}$$

by appropriate choice of the coefficients $\{c_{nm}\}$. Thus, because the inner product between $|\phi_n\rangle_1 \otimes |\theta_m\rangle_2$ and $|\phi_k\rangle_1 \otimes |\theta_l\rangle_2$ is defined to be,

$$({}_2\langle\theta_l| \otimes {}_1\langle\phi_k|)(|\phi_n\rangle_1 \otimes |\theta_m\rangle_2) = ({}_2\langle\theta_l|\theta_m\rangle_2)({}_1\langle\phi_k|\phi_n\rangle_1),$$

the inner product between $|\psi\rangle$ from Eq. (3) and

$$|\psi'\rangle = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm}(|\phi_n\rangle_1 \otimes |\theta_m\rangle_2),$$

is

$$\langle\psi'|\psi\rangle = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} d_{nm}^* c_{nm}.$$

(a) Let $\hat{A}_1$ be an observable of system 1, i.e., an Hermitian operator on $\mathcal{H}_1$ with a complete set of eigenkets, and let $\hat{B}_2$ be an observable of system 2, i.e., an Hermitian operator on $\mathcal{H}_2$ with a complete set of eigenkets. The tensor product $\hat{C} = \hat{A}_1 \otimes \hat{B}_2$ is a linear operator that maps the state $|\phi_n\rangle_1 \otimes |\theta_m\rangle_2$ into the state $(\hat{A}_1|\phi_n\rangle_1) \otimes (\hat{B}_2|\theta_m\rangle_2)$.

Show that $\hat{C}$ is an Hermitian operator on $\mathcal{H}$ which has a complete set of eigenkets, so that $\hat{C}$ is an observable on the joint Hilbert space of systems 1 and 2.

(b) Let

$$\hat{A}_1 = \sum_{n=1}^{\infty} a_n |a_n\rangle_{11}\langle a_n| \quad \text{and} \quad \hat{B}_2 = \sum_{m=1}^{\infty} b_m |b_m\rangle_{22}\langle b_m|$$

be the diagonal (eigenvalue/eigenket) decompositions of $\hat{A}_1$ and $\hat{B}_2$, where the $\{a_n\}$ are assumed to be distinct, as are the $\{b_m\}$. When we measure $\hat{A}_1$ on system 1 *and* we measure $\hat{B}_2$ on system 2 with the joint system being in state $|\psi\rangle$, given by Eq. (3), the outcome will be an ordered pair $\{(a_n, b_m)\}$ of eigenvalues. The probability that (a_n, b_m) occurs is given by,

$$\Pr(a_n, b_m) = |\langle\psi|(|a_n\rangle_1 \otimes |b_m\rangle_2)|^2.$$

Show that this is a proper probability distribution. Express the marginal probabilities, $\Pr(a_n)$ and $\Pr(b_m)$, in terms of $|\psi\rangle$, the $\{|a_n\rangle_1\}$ and the $\{|b_m\rangle_2\}$.

(c) Specialize the results of (b) to the case of a product state, viz., a state that satisfies $|\psi\rangle = |\psi_1\rangle_1 \otimes |\psi_2\rangle_2$.

Problem 3.7

Here we prove that it is impossible to clone the unknown state of a quantum system by means of a unitary evolution. It is a proof by contradiction. Suppose that we have a quantum system whose Hilbert space of states is $\mathcal{H}_S$, where $_S$ indicates that this is the *source* system. Suppose too that we have a *target* system whose Hilbert space of states is $\mathcal{H}_T$. We will assume that these two Hilbert spaces have the same dimensionality, e.g., 2.

We wish to construct a perfect cloner, viz., a unitary operator, $\hat{U}$, on the tensor product space $\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_T$ such that

$$\hat{U}(|\psi\rangle_S \otimes |0\rangle_T) = |\psi\rangle_S \otimes |\psi\rangle_T, \tag{4}$$

where $|\psi\rangle_S$ is an *arbitrary* unit-length ket in $\mathcal{H}_S$, and $|0\rangle_T$ is a reference ("blank") unit-length ket in $\mathcal{H}_T$. Thus, the perfect cloner does not disturb the source state while it turns the target's "blank" state into a clone of the source state.

Let $|\psi_1\rangle_S$ and $|\psi_2\rangle_S$ be two distinct, unit-length kets in $\mathcal{H}_S$, let α and β be two non-zero complex numbers, and assume that we have found a perfect cloner operator $\hat{U}$ that satisfies Eq. (4) for all unit-length source kets.

(a) Define

$$|\psi'\rangle_S = \frac{\alpha|\psi_1\rangle_S + \beta|\psi_2\rangle_S}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta({}_S\langle\psi_1|\psi_2\rangle_S)]}}.$$

Use unitarity to evaluate the length of the ket $|\theta\rangle \equiv \hat{U}(|\psi'\rangle_S \otimes |0\rangle_T)$.

(b) Use the linearity of $\hat{U}$ to show that

$$|\theta\rangle = \alpha'(|\psi_1\rangle_S \otimes |\psi_1\rangle_T) + \beta'(|\psi_2\rangle_S \otimes |\psi_2\rangle_T). \tag{5}$$

where

$$\alpha' \equiv \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta({}_S\langle\psi_1|\psi_2\rangle_S)]}}$$

$$\beta' \equiv \frac{\beta}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta({}_S\langle\psi_1|\psi_2\rangle_S)]}}$$

(c) Use Eq. (5) to evaluate the length of $|\theta\rangle$. Show that this result contradicts what you found in (a), and thus conclude that there is no unitary $\hat{U}$ that can be a perfect cloner.

Problem 3.8

Here we prove that it is impossible to erase the unknown state of a quantum system by means of a unitary evolution. It is a proof by contradiction. Suppose that we have a quantum system whose Hilbert space of states is $\mathcal{H}_S$, where $_S$ indicates that this is the *source* system. Suppose too that we have an *ancilla* system whose Hilbert space of states is $\mathcal{H}_A$. We will assume that these two Hilbert spaces have the same dimensionality, e.g., 2.

We wish to construct a perfect eraser, viz., a unitary operator, $\hat{U}$, on the tensor product space $\mathcal{H} \equiv \mathcal{H}_S \otimes \mathcal{H}_A$ such that

$$\hat{U}(|\psi\rangle_S \otimes |0\rangle_A) = |0\rangle_S \otimes |0\rangle_A, \tag{6}$$

where $|\psi\rangle_S$ is an *arbitrary* unit-length ket in $\mathcal{H}_S$, and $|0\rangle_A$ is a reference ("blank") unit-length ket in $\mathcal{H}_A$. Thus, the perfect eraser does not disturb the ancilla state while it turns the source's input state into a "blank."

Let $|\psi_1\rangle_S$ and $|\psi_2\rangle_S$ be two distinct, unit-length kets in $\mathcal{H}_S$, let α and β be two non-zero complex numbers, and assume that we have found a perfect eraser operator $\hat{U}$ that satisfies Eq. (6) for all unit-length source kets.

(a) Define

$$|\psi'\rangle_S = \frac{\alpha|\psi_1\rangle_S + \beta|\psi_2\rangle_S}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta({}_S\langle\psi_1|\psi_2\rangle_S)]}}.$$

Use unitarity to evaluate the length of the ket $|\theta\rangle \equiv \hat{U}(|\psi'\rangle_S \otimes |0\rangle_A)$.

(b) Use the linearity of $\hat{U}$ to show that

$$|\theta\rangle = (\alpha' + \beta')(|0\rangle_S \otimes |0\rangle_A). \tag{7}$$

where

$$\alpha' \equiv \frac{\alpha}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta({}_S\langle\psi_1|\psi_2\rangle_S)]}}$$

$$\beta' \equiv \frac{\beta}{\sqrt{|\alpha|^2 + |\beta|^2 + 2\text{Re}[\alpha^*\beta({}_S\langle\psi_1|\psi_2\rangle_S)]}}$$

(c) Use Eq. (7) to evaluate the length of $|\theta\rangle$. Show that this result contradicts what you found in (a), and thus conclude that there is no unitary $\hat{U}$ that can be a perfect eraser.