15.081J/6.251J Introduction to Mathematical Programming

Lecture 3: Geometry of Linear Optimization II

1 Outline

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- BFS for standard form polyhedra
- Deeper understanding of degeneracy
- Existence of extreme points
- Optimality of Extreme Points
- Representation of Polyhedra

2 BFS for standard form polyhedra

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- $\boldsymbol{Ax} = \boldsymbol{b}$ and $\boldsymbol{x} \geq \boldsymbol{0}$
- $m \times n$ matrix $\boldsymbol{A}$ has linearly independent rows
- $\boldsymbol{x} \in \Re^n$ is a basic solution if and only if $\boldsymbol{Ax} = \boldsymbol{b}$, and there exist indices $B(1), \ldots, B(m)$ such that:
  - The columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$ are linearly independent
  - If $i \neq B(1), \ldots, B(m)$, then $x_i = 0$

2.1 Construction of BFS

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Procedure for constructing basic solutions

1. Choose m linearly independent columns $\boldsymbol{A}_{B(1)}, \ldots, \boldsymbol{A}_{B(m)}$
2. Let $x_i = 0$ for all $i \neq B(1), \ldots, B(m)$
3. Solve $\boldsymbol{Ax} = \boldsymbol{b}$ for $x_{B(1)}, \ldots, x_{B(m)}$

$$\boldsymbol{Ax} = \boldsymbol{b} \quad \rightarrow \quad \boldsymbol{B x}_B + \boldsymbol{N x}_N = \boldsymbol{b}$$

$$\boldsymbol{x}_N = 0, \quad \boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}$$

2.2 Example 1

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$$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 8 \\ 12 \\ 4 \\ 6 \end{bmatrix}$$

- $\boldsymbol{A}_4, \boldsymbol{A}_5, \boldsymbol{A}_6, \boldsymbol{A}_7$ basic columns

- Solution: $\boldsymbol{x} = (0, 0, 0, 8, 12, 4, 6)$, a BFS
- Another basis: $\boldsymbol{A}_3, \boldsymbol{A}_5, \boldsymbol{A}_6, \boldsymbol{A}_7$ basic columns.
- Solution: $\boldsymbol{x} = (0, 0, 4, 0, -12, 4, 6)$, not a BFS

2.3 Geometric intuition

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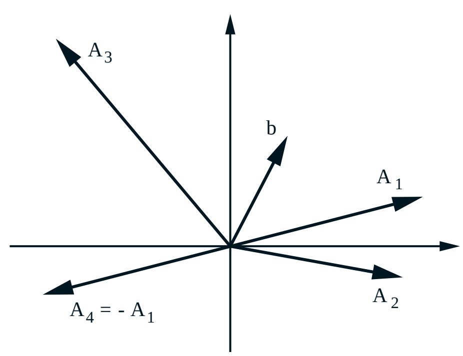

2.4 Example 2

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General form

$$\begin{array}{lcl} x_1 + x_2 + x_3 & \le & 4 \\ x_1 & \le & 2 \\ x_3 & \le & 3 \\ 3x_2 + x_3 & \le & 6 \\ x_1, x_2, x_3 & \ge & 0 \end{array}$$

Standard form

$$\begin{array}{lcl} x_1 + x_2 + x_3 + s_1 & = & 4 \\ x_1 + s_2 & = & 2 \\ x_3 + s_3 & = & 3 \\ 3x_2 + x_3 + s_4 & = & 6 \\ x_1, x_2, x_3, s_1, \ldots, s_4 & \geq & 0 \end{array}$$

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- Using the definition for BFS in polyhedra in general form :
- Choose tight constraints: $\left.\begin{array}{ll} x_1 + x_2 + x_3 & = 4 \\ x_3 & = 3 \\ x_2 & = 0 \end{array}\right\} \Rightarrow (1, 0, 3)$
- Check if $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ span $\Re^3$ (they do)

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- Using the definition for BFS in polyhedra in standard form :
- Pick the basic variables: $x_1, x_3, s_2, s_3 : \boldsymbol{x_B} = (x_1, x_3, s_2, s_3)$
- Pick the nonbasic variables: $x_2, s_1, s_4 : \boldsymbol{x_N} = (x_2, s_1, s_4)$

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- Partition A:

$$\boldsymbol{A} = \left[\begin{array}{ccccccc} x_1 & x_2 & x_3 & s_1 & s_2 & s_3 & s_4 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [\boldsymbol{B}, \boldsymbol{N}]$$

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$$\boldsymbol{B} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \boldsymbol{B} \text{ non-singular}$$

$$\boldsymbol{x_N} = \boldsymbol{0}, \boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} \Rightarrow \begin{pmatrix} x_1 \\ x_3 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 1 \\ 3 \end{pmatrix}$$

3 Degeneracy for standard form polyhedra

3.1 Definition

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- A BFS $\boldsymbol{x}$ of $P = \{\boldsymbol{x} \in \Re^n : \boldsymbol{Ax} = \boldsymbol{b}, \boldsymbol{A} : n \times n, \boldsymbol{x} \geq 0\}$ is called <u>degenerate</u> if it contains <u>more</u> than $n - m$ zeros.
- $\boldsymbol{x}$ is <u>non-degenerate</u> if it contains <u>exactly</u> $n - m$ zeros.

3.2 Example 2, revisited

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- In previous example:

$$(2,2,0,0,0,3,0)\text{degenerate}: \quad \begin{array}{l} n=7 \\ m=4 \end{array}$$

- More than $n - m = 7 - 4 = 3$ zeros.
- Ambiguity about which are basic variables.
- (x_1, x_2, x_3, x_6) one choice
  (x_1, x_2, x_6, x_7) another choice

3.3 Extreme points and BFS

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- Consider again the extreme point $(2,2,0,0,0,6,0)$
- How do we construct the basis?

$$\mathcal{B} = \left\{ \underset{\boldsymbol{A}_1}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}}, \underset{\boldsymbol{A}_2}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}}, \underset{\boldsymbol{A}_6}{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}} \right\}$$

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- Columns in $\mathcal{B}$ are linearly independent.
- Rank $(\boldsymbol{A}) = 4$
- $|\mathcal{B}| = 3 < 4$
- Can we augment $\mathcal{B}$?
- Choices:
  - $\mathcal{B}' = \mathcal{B} \cup \{\boldsymbol{A}_3\}$ basic variables x_1, x_2, x_3, x_6
  - $\mathcal{B}' = \mathcal{B} \cup \{\boldsymbol{A}_7\}$ basic variables x_1, x_2, x_6, x_7
  - How many choices do we have?

3.4 Degeneracy and geometry

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- Whether a BFS is degenerate may depend on the particular representation of a polyhedron.
- $P = \Big\{(x_1, x_2, x_3) \;\Big|\; x_1 - x_2 = 0,\; x_1 + x_2 + 2x_3 = 2,\; x_1, x_2, x_3 \geq 0\Big\}.$

- $n=3$, $m=2$ and $n-m=1$. $(1,1,0)$ is nondegenerate, while (0,0,1) is degenerate.
- Consider the representation $P = \Big\{(x_1,x_2,x_3) \mid x_1 - x_2 = 0,\ x_1 + x_2 + 2x_3 = 2,\ x_1 \geq 0,\ x_3 \geq 0\Big\}$. (0,0,1) is now nondegenerate.

3.5 Conclusions

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- An extreme point corresponds to possibly many bases in the presence of degeneracy.
- A basic feasible solution, however, corresponds to a unique extreme point.
- Degeneracy is not a purely geometric property.

4 Existence of extreme points

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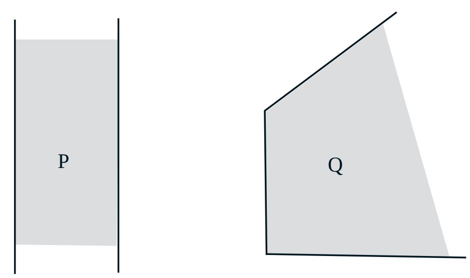

Note that $P = \{(x_1,x_2) : 0 \leq x_1 \leq 1\}$ does not have an extreme point, while $P' = \{(x_1,x_2) : x_1 \leq x_2, x_1 \geq 0, x_2 \geq 0\}$ has one. Why?

4.1 Definition

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A polyhedron $P \subset \Re^n$ **contains a line** if there exists a vector $\boldsymbol{x} \in P$ and a nonzero vector $\boldsymbol{d} \in \Re^n$ such that $\boldsymbol{x} + \lambda\boldsymbol{d} \in P$ for all scalars λ.

4.2 Theorem

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Suppose that the polyhedron $P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{a_i}'\boldsymbol{x} \geq b_i,\ i = 1,\ldots,m\}$ is nonempty. Then, the following are equivalent:

(a) The polyhedron P has at least one extreme point.

(b) The polyhedron P does not contain a line.

(c) There exist n vectors out of the family $\boldsymbol{a_1},\ldots,\boldsymbol{a_m}$, which are linearly independent.

4.3 Corollary

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- Polyhedra in standard form contain an extreme point.
- Bounded polyhedra contain an extreme point.

4.4 Proof

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Let $P = \{\boldsymbol{x} \mid \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} \neq \emptyset$, $\text{rank}(\boldsymbol{A}) = m$. If there exists a feasible solution in P, then there is an extreme point.
Proof

- Let $\boldsymbol{x} = (x_1, \ldots, x_t, 0, \ldots, 0)$, s.t. $\boldsymbol{x} \in P$. Consider $\boldsymbol{B} = \{\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_t\}$
- If $\{\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_t\}$ are linearly independent we can augment, to find a basis, and thus a BFS exists.
- If $\{\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_t\}$ are dependent

$$d_1\boldsymbol{A}_1 + \cdots + d_t\boldsymbol{A}_t = 0 \qquad (d_i \neq 0)$$

- But $x_1\boldsymbol{A}_1 + \cdots + x_t\boldsymbol{A}_t = \boldsymbol{b}$

$$\Rightarrow (x_1 + \theta d_1)\boldsymbol{A}_1 + \cdots + (x_t + \theta d_t)\boldsymbol{A}_t = b$$

- Consider $x_j(\theta) = \begin{cases} x_j + \theta d_j & j = 1 \ldots t \\ 0 & \text{otherwise.} \end{cases}$

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Clearly $\boldsymbol{A} \cdot \boldsymbol{x}(\theta) = \boldsymbol{b}$

$$\begin{array}{lll}
\text{Let:} & \theta_1 = \max\limits_{d_j > 0} \left\{-\frac{x_j}{d_j}\right\} & (\text{if all } d_j \leq 0 \\
& & \theta_1 = -\infty) \\
& \theta_2 = \min\limits_{d_j > 0} \left\{-\frac{x_j}{d_j}\right\} & (\text{if all } d_j \geq 0 \\
& & \theta_2 = +\infty) \\
\text{For} & \theta_1 \leq \theta \leq \theta_2 & (\text{sufficiently small}) \\
& \boldsymbol{x}(\theta) \geq 0 &
\end{array}$$

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Since at least one $(d_1, \ldots, d_t) \neq \boldsymbol{0} \Rightarrow$ at least one from θ_1, θ_2 is finite, say θ_1. But then $x(\theta_1) \geq 0$ and number of nonzeros decreased.

$$x_j + \theta \cdot d_j \geq 0 \quad \Rightarrow \quad x_j \geq -\theta d_j$$

4.5 Example 3

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- $P = \{\boldsymbol{x} \mid \begin{array}{lll} x_1 + x_2 + x_3 & & = 2 \\ x_1 & + x_4 & = 1, \quad x_1, \ldots, x_4 \geq 0\} \end{array}$
- $\boldsymbol{x} = \left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{1}{2}\right)$

- $$\boldsymbol{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

- $$1 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

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- Consider: $\boldsymbol{x}(\theta) = \left(\frac{1}{2} + \theta,\ \frac{1}{2} - \theta,\ 1,\ \frac{1}{2} - \theta \right)$ for $-\frac{1}{2} \leq \theta \leq \frac{1}{2}$.
- $\boldsymbol{x}(\theta) \in P$.
- Note $\boldsymbol{x}(-\frac{1}{2}) = (0, 1, 1, 1)$ and $\boldsymbol{x}(\frac{1}{2}) = (1, 0, 1, 0)$.

5 Optimality of Extreme Points

5.1 Theorem

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- Consider

$$\begin{aligned} \min \quad & \boldsymbol{c}'\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{x} \in P = \{\boldsymbol{x} \in \Re^n \mid \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}. \end{aligned}$$

- P has no line and it has an optimal solution.

Then, there exists an optimal solution which is an extreme point of P.

5.2 Proof

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- v optimal value of the cost $\boldsymbol{c}'\boldsymbol{x}$.
- $\boldsymbol{Q}$: set of optimal solutions, i.e.,

$$\boldsymbol{Q} = \{\boldsymbol{x} \mid \boldsymbol{c}'\boldsymbol{x} = v, \quad \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$$

- $Q \subset P$ and P contains no lines, Q does not contain any lines, hence is has an extreme point $\boldsymbol{x}^*$.

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- <u>Claim</u>: $\boldsymbol{x}^*$ is an extreme point of P.
- Suppose not; $\exists\, \boldsymbol{y}, \boldsymbol{w} \neq \boldsymbol{x}^*$: $\quad \boldsymbol{x}^* = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{w}, \quad \boldsymbol{y}, \boldsymbol{w} \in \boldsymbol{P},\ 0 < \lambda < 1$.
- $v = \boldsymbol{c}'\boldsymbol{x}^* = \lambda \boldsymbol{c}'\boldsymbol{y} + (1-\lambda)\boldsymbol{c}'\boldsymbol{w}$
- $\left.\begin{array}{l} \boldsymbol{c}'\boldsymbol{y} \geq v \\ \boldsymbol{c}'\boldsymbol{w} \geq v \end{array}\right\} \Rightarrow \boldsymbol{c}'\boldsymbol{y} = \boldsymbol{c}'\boldsymbol{w} = v \Rightarrow \boldsymbol{y}, \boldsymbol{w} \in \boldsymbol{Q}$
- $\Rightarrow \boldsymbol{x}^*$ is <u>NOT</u> an extreme point of Q, CONTRADICTION.

6 Representation of Polyhedra

6.1 Theorem

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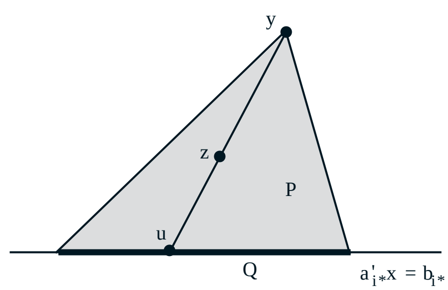
A nonempty and bounded polyhedron is the convex hull of its extreme points.



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