# Stirling's formula, Asymptotics



## Closed form for *n*!

Factorial defines a product:

$$n! \;::=\; 1\cdot 2\cdot 3\cdots(n-1)\cdot n \;=\prod_{i=1}^{n} i$$

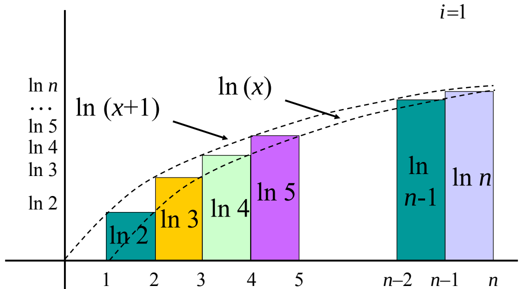
Turn product into a sum taking logs:

$$\begin{aligned}\ln(n!) &= \ln(1\cdot 2\cdot 3\cdots(n-1)\cdot n)\\ &= \ln 1+\ln 2+\cdots+\ln(n-1)+\ln(n)\\ &= \sum_{i=1}^{n}\ln(i)\end{aligned}$$



## Integral Method

Integral Method to bound $\sum_{i=1}^{n}\ln i$






## Integral Method

Bounds on ln(*n*!)

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n}\ln(i) \le \int_0^n \ln(x+1)\,dx$$



## Integral Method

*Reminder:*

$$\int \ln x\,dx = x\ln\left(\frac{x}{e}\right)$$



## Integral Method

Bounds on ln(*n*!)

$$\int_1^n \ln(x)\,dx \le \sum_{i=1}^{n}\ln(i) \le \int_0^n \ln(x+1)\,dx$$

$$n\ln(n/e)+1 \le \sum\ln(i) \le (n+1)\ln((n+1)/e)+1$$

So guess: $\sum_{i=1}^{n}\ln(i) \approx (n+\frac{1}{2})\ln\left(\frac{n}{e}\right)$

### Integral Method

$$\sum_{i=1}^{n} \ln(i) \approx (n+\frac{1}{2})\ln\left(\frac{n}{e}\right)$$

exponentiating:

$$n! \approx \sqrt{n/e}\left(\frac{n}{e}\right)^n$$




### Stirling's Formula

A precise approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$




### Asymptotic Equivalence

*Def.* $f(n) \sim g(n)$

iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$$




### Stirling's Formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$




### Asymptotic Equivalence

*Example:* $(n^2 + n) \sim n^2$

because

$$\lim_{n\to\infty} \frac{n^2+n}{n^2} = \lim[\frac{n^2}{n^2} + \frac{n}{n^2}]$$
$$= \lim[1 + \frac{1}{n}]$$
$$= 1 + \lim \frac{1}{n}$$
$$= 1 + 0 = 1$$




### Little Oh

*Asymptotically smaller*:

*Def.* $f(n) = o(g(n))$

iff

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

## Little Oh

$$n^2 = o(n^3)$$

because

$$\lim_{n\to\infty} \frac{n^2}{n^3} =$$

$$\lim_{n\to\infty} \frac{1}{n} = 0$$

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## Big Oh

*Asymptotic Order of Growth:*

$$f(n) = O(g(n))$$

$$\limsup_{n\to\infty} \left( \frac{f(n)}{g(n)} \right) < \infty$$

a technicality -- ignore now

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## Big Oh

$$3n^2 = O(n^2)$$

because

$$\lim_{n\to\infty} \frac{3n^2}{n^2}$$

$$= 3 < \infty$$

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## The Oh's

If $f = o(g)$ or $f \sim g$ then $f = O(g)$

$\lim = 0$ $\lim = 1$ $\lim < \infty$

converse is NOT true!

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## The Oh's

If $f = o(g)$, then $g \neq O(f)$

$\lim \frac{f}{g} = 0$ $\lim \frac{g}{f} = \infty$

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## Big Oh

Equivalent definition:

$$f(n) = O(g(n))$$

$$\exists c, n_0 \geq 0 \ \forall n \geq n_0 \ |f(n)| \leq c \cdot g(n)$$

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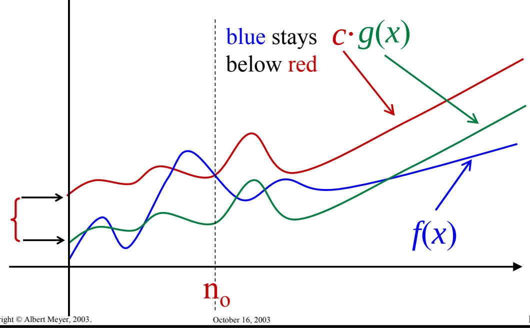
## Big Oh

$f(x) = O(g(x))$



# Problems 1, 2

## Little Oh

*Lemma:* $x^a = o(x^b)$ for $a < b$

Proof: $\dfrac{x^a}{x^b} = \dfrac{1}{x^{b-a}}$ and $b - a > 0$.

So as $x \to \infty$,

$x^{b-a} \to \infty$ and $\dfrac{1}{x^{b-a}} \to 0$.

## Little Oh

*Lemma:* $\ln x = o(x^\delta)$ for $\delta > 0$.

Proof: $\dfrac{1}{y} \le y$ for $y \ge 1$.

$$\int_1^z \frac{1}{y}\,dy \le \int_1^z y\,dy$$

$$\ln z \le \frac{z^2 - 1}{2}$$

## Little Oh

*Lemma:* $\ln x = o(x^\delta)$ for $\delta > 0$.

Proof: $\ln z \le \dfrac{z^2}{2}$ Let $z ::= \sqrt{x^\varepsilon}$

$$\frac{\varepsilon \ln x}{2} \le \frac{x^\varepsilon}{2}$$

$$\ln x \le \frac{x^\varepsilon}{\varepsilon} = o(x^\delta) \quad \text{for } \delta > \varepsilon.$$

## Theta

Same Order of Growth:

$f(n) = \Theta(g(n))$

$f(n) = O(g(n))$ and $g(n) = O(f(n))$

Not the same as " ~ " !

## Big Oh Mistakes

$f = O(g)$ defines a *relation* "$= O(\cdot)$"

Don't write $O(g) = f$.

Otherwise: $x = O(x)$, so $O(x) = x$.

But $2x = O(x)$, so

$2x = O(x) = x$,

therefore $2x = x$.

Nonsense!




## Big Oh Mistakes

Lower bound blunder:

"*f* is at least $O(n^2)$"




## Big Oh Mistakes

False Lemma: $\sum_{i=1}^{n} i = O(n)$

Of course really $\sum_{i=1}^{n} i = \theta(n^2)$




## Big Oh Mistakes

False Lemma: $\sum_{i=1}^{n} i = O(n)$

False Proof:

$0 = O(1),\ 1 = O(1),\ 2 = O(1), \ldots$

So each $i = O(1)$. So

$$\sum_{i=1}^{n} i = O(1) + O(1) + \cdots + O(1)$$
$$= n \cdot O(1) = O(n).$$




## Team Problems

# Problems 3 & 4