

In-Class Problems Week 9, Fri.

Problem 1. Solve the following problems using the Pigeonhole Principle. For each problem, try to identify the *pigeons*, the *pigeonholes*, and a *rule* assigning each pigeon to a pigeonhole.

- (a) In a room of 500 people, there exist two who share a birthday.
- (b) Every MIT ID number starts with a 9 (we think). Suppose that each of the 101 students in 6.042 sums the nine digits of his or her ID number and doubles the result. Explain why two students' results must be the same.
- (c) In every set of 100 integers, there exist two whose difference is a multiple of 37.
- (d) For any five points inside a unit square, there are two points at distance less than $\frac{1}{\sqrt{2}}$.
- (e) For any five points inside an equilateral triangle of side length 2, there are two points at distance less than 1.
- (f) Prove that every finite undirected graph with two or more vertices has two vertices of the same degree.

Problem 2. Your 6.001 tutorial has 12 students, who are supposed to break up into 4 groups of 3 students each. Your TA has observed that the students waste too much time trying to form balanced groups, so he decided to pre-assign students to groups and email the group assignments to his students.

(a) Your TA has a list of the 12 students in front of him, so he divides the list into consecutive groups of 3. For example, if the list is ABCDEFGHIJKL, the TA would define a sequence of four groups to be $(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\})$. This way of forming groups defines a mapping from a list of twelve students to a sequence of four groups. This is a k -to-1 mapping for what k ?

(b) A group assignment specifies which students are in the same group, but not any order in which the groups should be listed. If we map a sequence of 4 groups,

$$(\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}),$$

into a group assignment

$$\{\{A, B, C\}, \{D, E, F\}, \{G, H, I\}, \{J, K, L\}\},$$

this mapping is j -to-1 for what j ?

- (c) How many group assignments are possible?
- (d) In how many ways can $3n$ students be broken up into n groups of 3?

Problem 3. Answer the following questions using the Generalized Product Rule.

(a) Next week, I'm going to get really fit! On day 1, I'll exercise for 5 minutes. On each subsequent day, I'll exercise 0, 1, 2, or 3 minutes more than the previous day. For example, the number of minutes that I exercise on the seven days of next week might be 5, 6, 9, 9, 9, 11, 12. How many such sequences are possible?

(b) An r -*permutation* of a set is a sequence of r distinct elements of that set. For example, here are all the 2-permutations of $\{a, b, c, d\}$:

$$\begin{array}{lll} (a, b) & (a, c) & (a, d) \\ (b, a) & (b, c) & (b, d) \\ (c, a) & (c, b) & (c, d) \\ (d, a) & (d, b) & (d, c) \end{array}$$

How many r -permutations of an n -element set are there? Express your answer using factorial notation.

- (c) How many $n \times n$ matrices are there with *distinct* entries drawn from $\{1, \dots, p\}$, where $p \geq n^2$?

Problem 4. A certain company wants to have security for their computer systems. So they have given everyone a name and password. A length 10 word containing each of the characters:

a, d, e, f, i, l, o, p, r, s,

is called a *cword*. A password will be a cword which does not contain any of the subwords "fails", "failed", or "drop".

For example, the following two words are passwords:

adefiloprs, srpolifeda,

but the following three cwords are not:

adropeflrs, failedrops, dropefails.

- (a) How many cwords contain the subword "drop"?
- (b) How many cwords contain both "drop" and "fails"?
- (c) Use the Inclusion-exclusion Principle to find a simple formula for the number of passwords.

Counting Principles

Rule (Pigeonhole Principle). If $|A| > |B|$, then for every function $f : A \rightarrow B$ there exist two different elements of A that are mapped to the same element of B .

“If more than n pigeons are assigned to n holes, then there must exist two pigeons assigned to the same hole.”

A *k -to-1 function* maps exactly k elements of the domain to every element of the range. For example, the function mapping ears of 6.042 students to their owners is 2-to-1.

Rule (Division Rule). If $f : A \rightarrow B$ is k -to-1, then $|A| = k \cdot |B|$.

Rule (Generalized Product Rule). Let S be a set of length- k sequences. If there are:

- n_1 possible first entries,
- n_2 possible second entries for each first entry,
- n_3 possible third entries for each combination of first and second entries,;

then:

$$|S| = n_1 \cdot n_2 \cdot n_3 \cdots n_k$$

Rule (Inclusion-Exclusion for Three Sets).

$$\begin{aligned} |S_1 \cup S_2 \cup S_3| &= |S_1| + |S_2| + |S_3| \\ &\quad - |S_1 \cap S_2| - |S_1 \cap S_3| - |S_2 \cap S_3| \\ &\quad + |S_1 \cap S_2 \cap S_3| \end{aligned}$$