6.041SC Probabilistic Systems Analysis and Applied Probability, Fall 2013
Transcript – Recitation: The Monty Hall Problem

Hi. In the session, we'll be solving the Monty Hall problem. And this problem is based on an old game show that was called "Let's Make a Deal." And the host of this game show, his name was Monty Hall, which is why this problem is now known as the Monty Hall problem.

And this problem is actually pretty well-known, because there was some disagreement at the time over what the right answer to this problem should be. Even some really smart people didn't agree on what the right answer should be. And part of what might explain that disagreement is that they probably were considering slightly different variations of the problem, because as in all probability problems, the assumptions that you're working with are very important, because otherwise you may be solving an actually different problem. And so what we'll do first is really layout concretely what all the assumptions are, what the rules of the game are. And then we'll go through the methodology to solve for the actual answer.

So the game is actually relatively simple. So you're on a game show and you're presented with three doors. These doors are closed. And behind one of these doors is a prize, let's say, a car. And behind the other two doors, there's nothing. You don't know which one it is.

And the rules of the game are that, first, you get to choose any one of these three. So you pick one of the doors that you want. They don't show you what's behind that door, but your friend, who actually knows which door has the prize behind it, will look at the remaining doors.

So let's, just for example, let's say you chose door one. Your friend will look at the other two doors and open one of them. And you will make sure that the one that he opens is empty. That is the prize not behind that one.

And at this point, one of the doors is open and its empty, you have your original door plus another unopened door. And you're given an option-- you could either stay with your initial choice or you can switch to the other unopened door. And whichever one is your final choice, they will open that door. And if there's a price behind it, you win, and if there not, then you don't win.

So the question that we're trying to answer is what is the better strategy here? Is the better strategy to stay with your initial choice or is it better to switch to the other unopened door?

OK, so it turns out that the specific rules here actually are very important. Specifically, the rule about how your friend chooses to open doors. And the fact that he will always open one of the two other door that you haven't picked and he will make sure that that door doesn't have a prize behind it. And let's see how that actually plays out in this problem.

So the simplest way, I think, of thinking about this problem is just to think about under what circumstances does staying with your initial choice win? So if you think about it, the only way that you can win by staying with your initial choice is if your initial choice happened to be the door that has a prize behind it. And because you're sticking with the initial choice, you can actually kind of forget about the rest of the game, about opening of the other door and about switching.

It's as if you're playing a simpler game, which is just you have three doors, one of them has a prize behind it, and you choose one of them. And if you guessed right, then you win. If you didn't, then you don't win. And because the another important assumption is that the prize has an equal probability of being behind any one of three doors so one third, one third, one third.

Because of that, then if you stay with your first choice, you win only if your first choice happened to the right one. And that is the case with probably one third. So with that simple argument you can convince yourself that the probability of winning, given the strategy of staying with your first choice, is one third.

Now, let's think about the other strategy, which is to switch. So under what circumstances does switching win for you? Well, if your first choice happened to be the right door, then switching away from that door will always lose. But let's say, that happens with probably one third. But the rest of the time with probably 2/3, your first choice would be wrong.

So let's give an example here. Let's say, the prize, which I'll denote by happy face, is behind door two. And your first choice was door one. So your first choice was wrong.

Now, your friend can open door two, because door two has the prize behind it. He also doesn't open the door that you initially picked. So he has to open door three. So door three is open, and now you have an option of sticking with your first choice-- door one-- or switching to door two. So in this case, it's obvious to see that switching wins for you.

And now, if instead, you picked door one first, and the prize was behind door three, again, you are wrong. And again, your friend is forced to open door two. And switching, again, wins for you.

And so if you think about it, switching will win for you, as long as your initial pick was wrong. If your initial pick was wrong, then the prize is behind one of the doors. Your friend has to open one of the doors, but he can't open the door that has the prize behind it. So he has to open the other bad door, leaving the good door with the prize behind it, as the one that you can switch to.

And so by switching you will win in this scenario. And what is the probability of that happening? Well, that happens if your initial pick was wrong, which happens with probably 2/3.

So the final answer then, it's pretty simple, the probability of winning if you stay is one third, and the probability of winning if you switch is 2/3. And so maybe counterintuitively the result is that it's actually better for you, twice as good for you, to switch rather than stay. And so that was the argument, the kind of simple argument.

We can also be more methodical about this and actually list out all of the possible outcomes. Because it's relatively small problem-- there's only three doors-- we can actually just list out all the possible outcomes. So for example, if you chose door one first, and the prize was behind door one, your friend has a choice. He can open door two or door three, because they're both empty. And then in that case, if you stay, you win, you picked the door correctly. And if you switch to two or three, then you lose.

But if you chose door one, the prize is behind door two, then your friend has to open door three, he is forced to do that, then staying with lose but switching would win. And so on for the other cases. And so again, this is just an exhaustive list of all the possible outcomes, from which you can see that, in fact, staying wins, only if your first choice was correct. And switching wins in all the other cases. And so one third of the time, staying would win, 2/3 of the time switching would win.

OK, so now, we have the answer. Let's try to figure out and convince ourselves that it is actually right, because you might think before going through this process that maybe it doesn't matter whether you stay or you switch, they both have the same probably of winning, or maybe even staying is better. So why is staying worse and switching better?

Well, the first argument really is something that we've already talked about. By staying, you're essentially banking on your first choice being correct, which is a relatively poor bet, because you have only one in three chance of being right. But by switching, you're actually banking on your first choice being wrong, which is a relatively better bet, because you're more likely to be wrong than right in your first choice, because you're just picking blindly. OK, so that is one intuitive explanation for why switching is better.

Another slightly different way to think about it is that instead of picking single doors, you're actually picking groups of doors. So let's say that your first pick was door one. Then you're actually really deciding between door one or doors two and three combined. So why is that?

It's because by staying with door one, you're staying with door one. But by switching, you're actually getting two doors for the price of one, because you know that your friend will reveal one of these to be empty, and the other one will stay closed. But switching really kind of buys you both of these. And so because it buys you two opportunities to win, you get 2/3 chance of winning, versus a one third chance.

Another way of thinking about this is to increase the scale of the problem, and maybe that will help visualize the counterintuitive answer. So instead of having three doors, imagine that you have 1,000 doors that are closed. And again, one prize is behind one of the doors.

And the rules are similar-- you pick one door first, and then your friend will open 998 other doors. And these doors are guaranteed to be empty. And now you're left with your initial door plus one other door that is unopened.

So now the question is should you stay with your first choice or switch to your other choice? And it should be more intuitively obvious now that the better decision would be to switch, because you're overwhelmingly more likely to have picked incorrectly for your first pick. You have only 1 in 1,000 chance of getting it right. So that is kind of just taking this to a bigger extreme and really driving home the intuition.

OK, so what we've really discovered is that the fact that the rules of the game are that your friend has to open one of the other two doors and cannot reveal the prize plays a big role in this problem. And that is an important assumption.

OK, so now let's think about a slightly different variation now. So a different strategy. Instead of just always staying or always switching, we have a specific other strategy, which is that you will choose door one first and then, depending on what your friend does, you will act accordingly. So if your friend opens door two, you will not switch. And if your friend opens door three, you will switch. So let's draw out exactly what happens here.

So you have door one that you've chosen. And the prize can be behind doors one, two, or three. And again, it's equally likely. So the probabilities of these branches are one third, one third, and one third. And now given that, your friend in this scenario has a choice between opening doors two or three. And so because of doors, you chose one, the prize actually is behind one, and so two and three are both empty, so he can choose whichever one he wants to open.

And the problem actually hasn't specified how your friend actually decides between this. So we'll leave it in general. So we'll say that the probability p, your friend will open two, door two, in this case. And with the remaining probability 1 minus p, he will open door three.

What about in this case? Well, you chose door one. The prize is actually behind door two. So following the rules of the game, your friend is forced to open door three. So this happens with probability 1. And similarly, if the prize is behind door three, your friend is forced to open door two, which, again, happens with probably 1.

So now let's see how this strategy works. When do you win? You win when, according to the strategy, your final choice is the right door. So according to the strategy, in this case, your friend opened door two. And according to your strategy, if door two is open, you don't switch. So you stay with your first choice of one. And that happens to the right one, so you win in this case.

But what about here? Your friend opened door three, and by your strategy, you do switch, which is the wrong choice here, so you lose. Here, you switch, because you open door three, and you switch to the right door, so that wins. And this one, you don't switch, and you lose.

All right, so what is the final probability of winning? And the final probably of winning is the probability of getting to these two outcomes, which happens with probability one third times p plus one third times 1. So one third. So the final answer is one third p plus one third.

And notice now that the answer isn't just a number. Like in this case, the answer was one third and 2/3. And it didn't actually matter how your friend chose between these two doors when he had a choice. But in this case, it actually doesn't matter, because p stays in the answer.

But one thing that we can do is we can compare this with these strategies. So what we see is that, well p is a probability, so it has to be between 0 and 1. So this probability winning for this strategy is somewhere between one third times 0 plus one third, which is one third. And one third times 1 plus one third, which is 2/3. So the strategy is somewhere between 2/3 and one third.

So what we see is that no matter what, this strategy is at least as good as staying all the time, because that was only one third. And no matter what it can't be any better than switching, which was 2/3. So you can also come up with lots of other different strategies and see what the probabilities of winning are in that case.

OK, so what have we learned in this problem? What are the key takeaways? One important takeaway is that it's important to really understand a problem and arrive at a concrete and precise set of assumptions. So really have a precise problem that you're solving.

And another important takeaway is to think about your final answer, make sure that that actually makes sense to you, make sure that you can justify it somehow intuitively. In that case, you can actually convince yourself that your answer is actually correct, because sometimes go through a lot of formulas, and sometimes your formula may have an error in there somewhere. But you could take the final answer and ask yourself does this actually makes sense intuitively? That's often a very good check and sometimes you can catch errors in your calculations that way. OK so we'll see next time.