

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering & Computer Science
6.041/6.431: Probabilistic Systems Analysis
(Spring 2006)

Problem Set 7
Due: April 12, 2006

1. (a) Imagine that you first roll a fair 6-sided die and then you flip a fair coin the number of times shown by the die. Letting X denote the number of these flips that come up heads, find $\mathbf{E}[X]$ and $\text{var}(X)$.
(b) Repeat part (1a) only now assume you first roll two dice.
2. Sam and Pat are dating. All of their dates are scheduled to start at 9pm at the Kendall/MIT T stop. Sam always arrives promptly at 9pm. Pat is highly unorganized, and arrives at a time that is uniformly distributed between 8pm and 10pm. Sam gets irritated when Pat is late. Let X be the time in hours between 8pm and the time when Pat arrives. When Pat arrives after 9pm, their date will last for a time that is uniformly distributed between 0 and $(3 - X)$ hours. When Pat arrives before 9pm, their date will last exactly 3 hours. The date starts at the time that they meet. Sam will end the relationship after the second date on which Pat is late by more than 45 minutes. All dates are independent of any other dates.
 - (a) What is the expected number of hours Pat waits for Sam to arrive?
(Remember that waiting time must be non-negative.)
 - (b) What is the expected duration of any particular date?
 - (c) What is the expected number of dates they will have before breaking up?
3. Imagine that the number of people that enter a bar in a period of 15 minutes has a Poisson distribution with rate λ . Each person who comes in buys a drink. If there are N types of drinks, and each person is equally likely to choose any type of drink, independently of what anyone else chooses, find the expected number of different types of drinks the bartender will have to make.
4. Let X , Y and Z denote any three discrete random variables. You may find the result of part (a) useful to show the result of part (b).
 - (a) **Pull-Through Property:** For any suitable function g , show that

$$\mathbf{E}[Yg(X) | X] = g(X)\mathbf{E}[Y | X] \quad .$$

- (b) **Tower Property:** Show that

$$\mathbf{E}[\mathbf{E}[Y | X, Z] | X] = \mathbf{E}[Y | X] = \mathbf{E}[\mathbf{E}[Y | X] | X, Z] \quad .$$

5. Consider two random variables X and Y . Assume for simplicity that they both have zero mean.
 - (a) Show that X and $E[X | Y]$ are positively correlated.
 - (b) Show that the correlation coefficient of Y and $E[X | Y]$ has the same sign as the correlation coefficient of X and Y .
6. The transform for J , the total number of living groups contacted about the MIT blood drive, is:

$$M_J(s) = \left(\frac{1}{3} + \frac{2}{3}e^s\right)^{10}$$

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- (a) Determine the PMF $p_J(j)$.

Let K , the number of people in any particular living group, be an independent random variable with transform:

$$M_K(s) = \frac{\frac{1}{5}e^{4s}}{1 - \frac{4}{5}e^s}$$

- (b) Without doing much work, determine the PMF $p_K(k)$, and evaluate $\mathbf{E}[K]$ and $\text{Var}(K)$.
(c) Let L be the total number of people whose living groups are contacted about the blood drive. Determine either the PMF or the transform for random variable L . Also evaluate $\mathbf{E}[L]$ and $\text{Var}(L)$.

Suppose the probability that any particular person, whose living group is contacted about the MIT blood drive, donates blood is $\frac{1}{4}$, and that all such individuals make their decisions independently.

- (d) Let M denote the total number of blood donors from the contacted living groups. Determine an explicit numerical expression for $p_M(0)$. Also evaluate $\mathbf{E}[M]$ and $\text{Var}(M)$. (Your expressions, albeit not gracious, should be in closed form.)

7. Widgets are packed into cartons which are packed into crates. The weight (in pounds) of a widget, X , is a continuous random variable with PDF

$$f_X(x) = \lambda e^{-\lambda x}, \quad x \geq 0 \quad .$$

The number of widgets in any carton, K , is a random variable with PMF

$$p_K(k) = \frac{\mu^k e^{-\mu}}{k!}, \quad k = 0, 1, 2, \dots \quad .$$

The number of cartons in a crate, N , is a random variable with PMF

$$p_N(n) = p^{n-1}(1-p), \quad n = 1, 2, 3, \dots \quad .$$

Random variables X , K , and N are mutually independent. Determine

- (a) The probability that a randomly selected crate contains exactly one widget.
(b) The expected value and variance of the number of widgets in a crate.
(c) The transform or the PDF for the total weight of the widgets in a crate.
(d) The expected value and variance of the total weight of the widgets in a crate.

Let X_1, X_2, \dots be independent normal random variables with mean 2 and variance 4. Let N be a geometric random variable which is independent of the X_i , with parameter $p = 2/3$. (In particular, $\mathbf{E}[N] = 3/2$ and $\mathbf{E}[N^2] = 3$.)

8. (a) If δ is a small positive number, we have $\mathbf{P}(|X_1| \leq \delta) \approx \alpha\delta$, for some constant α . Find the value of α . (Your answer may involve π , no need to evaluate numerically.)
(b) Find $\mathbf{E}[X_1 N]$.
(c) Find the variance of $X_1 N$.

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(d) Find $\mathbf{E}[X_1 + \cdots + X_N \mid N \geq 2]$.

(e) Write down the transform associated with $N + X_1 + \cdots + X_N$.

9. Iwana Passe is taking a quiz with 12 questions. The amount of time she spends answering question i is T_i , and is exponentially distributed with $E[T_i] = \frac{1}{3}$ hour. The amount of time she spends on any particular question is independent of the amount of time she spends on any other question. Once she finishes answering a question, she immediately begins answering the next question.

Let N be the total number of questions she answers **correctly**.

Let X be the total amount of time she spends on questions that she answers **correctly**.

For parts (a) and (b), suppose we know she has probability $\frac{2}{3}$ of getting any particular quiz question correct, independently of her performance on any other quiz question.

(a) Find the expectation and variance of X .

(b) Assuming we know she spent at least $\frac{1}{6}$ of an hour on each question, find the transform of X .

For parts (c) and (d), suppose she has a fixed probability P of getting any particular quiz question correct, independently of her performance on any other quiz question, and with P uniformly distributed between 0 and 1. Assume P is the same value for all questions.

(c) Find the expectation and variance of N .

(d) Assuming there is only one question on the quiz, find the transform of N .

10. Consider n independent tosses of a die. Each toss has probability p_i of resulting in i . Let X_i be the number of tosses that result in i . Show that X_1 and X_2 are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).
11. When using a multiple access communication channel, a certain number of users N try to transmit information to a single receiver. If the real-valued random variable X_i represents the signal transmitted by user i , the received signal Y is

$$Y = X_1 + X_2 + \cdots + X_N + Z,$$

where Z is an additive noise term that is independent of the transmitted signals and is assumed to be a zero-mean Gaussian random variable with variance σ_Z^2 . We assume that the signals transmitted by different users are mutually independent and, furthermore, we assume that they are identically distributed, each Gaussian with mean μ and variance σ_X^2 .

(a) If N is deterministically equal to 2, find the transform or the PDF of Y .

(b) In most practical schemes, the number of users N is a random variable. Assume now that N is equally likely to be equal to $0, 1, \dots, 10$.

i. Find the transform or the PDF of Y .

ii. Find the mean and variance of Y .

iii. Given that $N \geq 2$, find the transform or PDF of Y .

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- (c) Depending on the time of day, the number of users N has different PMF's. Between 7 and 10AM, and between 5 and 8PM, N can be assumed to be Poisson distributed with parameter $\lambda = 10$. During all other hours, N is equally likely to be $0, 1, \dots, 10$. On a random time during the day, based on the reception of Y , the receiver estimates the transmitted signals. Determine the linear least-squared estimator of X_k (where k is a positive integer) given reception Y .