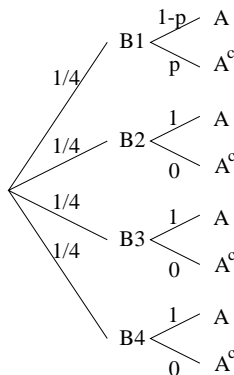


Problem Set 2: Solutions

Due: February 22, 2006

1. Let A be the event that your friend searches disk 1 and finds nothing, and let B_i be the event that your thesis is on disk i . The sample space is described below.

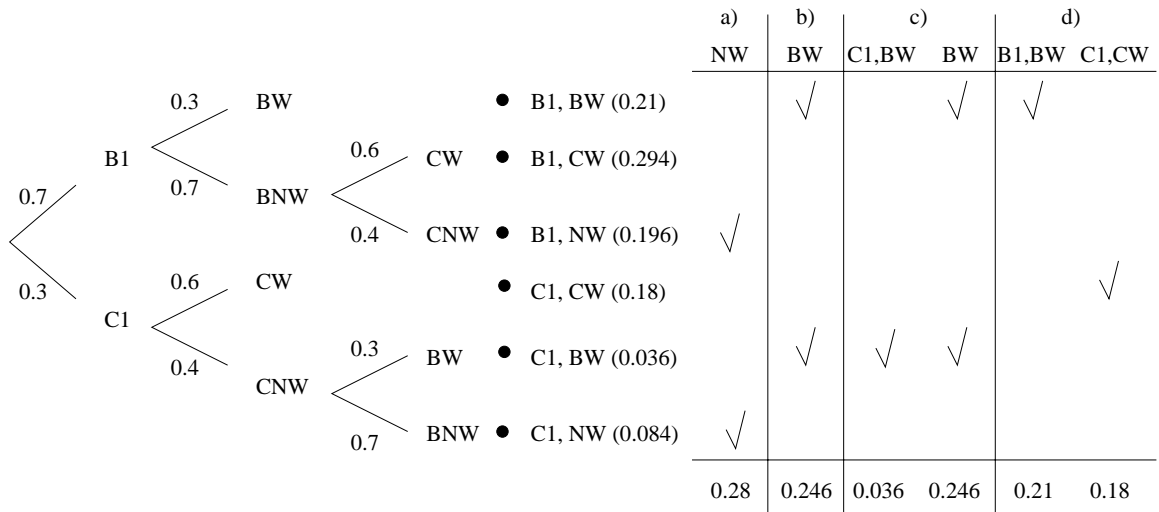


Note that $B_1, B_2, B_3,$ and B_4 partition the sample space, so applying Bayes' rule, we have

$$\begin{aligned}
 P(B_i | A) &= \frac{P(B_i)P(A | B_i)}{P(B_1)P(A | B_1) + P(B_2)P(A | B_2) + P(B_3)P(A | B_3) + P(B_4)P(A | B_4)} \\
 &= \frac{\frac{1}{4}P(A | B_i)}{\frac{1}{4}((1-p) + 1 + 1 + 1)} \\
 &= \frac{P(A | B_i)}{4-p} \\
 &= \begin{cases} (1-p)/(4-p) & \text{for } i = 1, \\ 1/(4-p) & \text{for } i = 2, 3, 4. \end{cases}
 \end{aligned}$$

2. Define the following events:
- $B1$: Bo submits his entry first.
 - $C1$: Ci submits his entry first.
 - BW : Bo's entry wins.
 - CW : Ci's entry wins.
 - BNW : Bo's entry does not win.
 - CNW : Ci's entry does not win.

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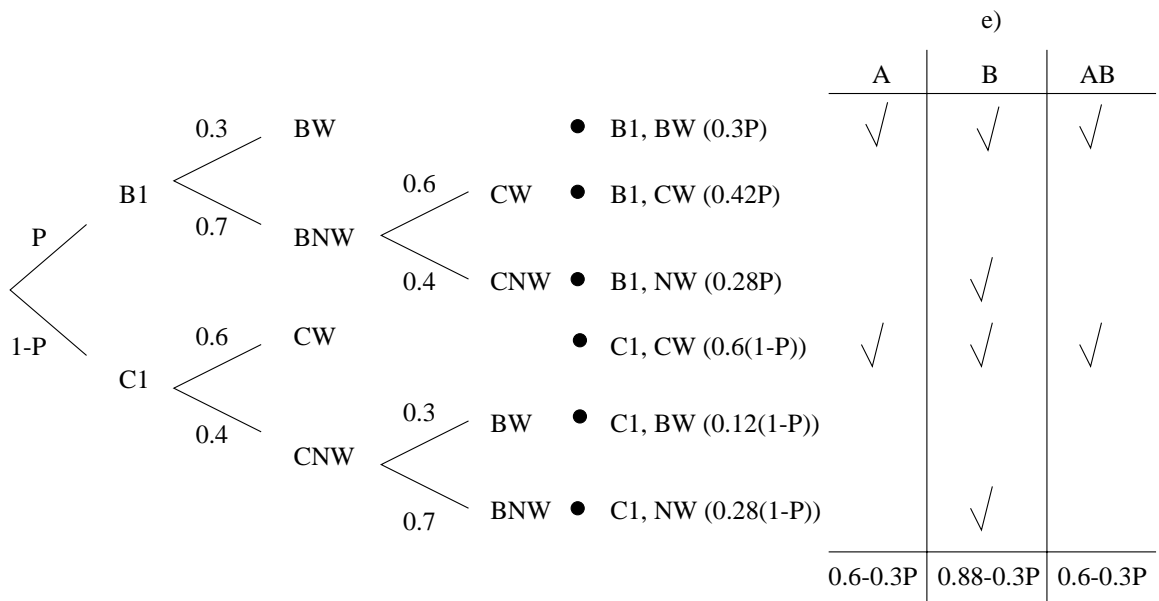
a) $P(\text{prize is not awarded}) = 0.196 + 0.084 = \boxed{0.28}$.

b) $P(\text{Bo will win}) = 0.21 + 0.036 = \boxed{0.246}$.

c) $P(\text{Ci's entry arrived first} | \text{Bo wins}) = P(C1, BW) / P(BW) = 0.036 / 0.246 = \boxed{0.1463}$.

d) $P(\text{First entry wins contest}) = P(B1, BW) + P(C1, CW) = 0.21 + 0.18 = \boxed{0.39}$.

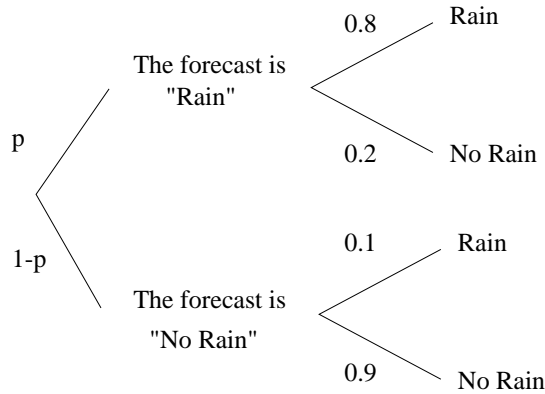
e) Defining A to be the event “First entry wins” and B to be the event “Second entry does not win” and modifying the above figure as follows:



We see that since $P(B)$ is not equal to 1 for any value of P , there is no value of P for which $P(A)P(B) = P(AB)$. Thus events A and B are never independent.

3. (a) The tree representation during the winter can be drawn as the following:

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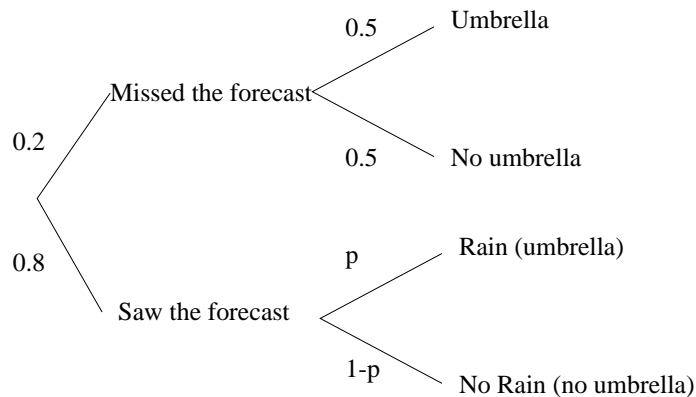
Let A be the event that the forecast was “Rain”,
 Let B be the event that it rained, Let p be the probability that the forecast says “Rain”,
 If it is in the winter, $p = 0.7$,

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)} = \frac{(0.8)(0.7)}{(0.8)(0.7) + (0.1)(0.3)} = \frac{56}{59}$$

Similarly, if it is in the summer, $p = 0.2$,

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)} = \frac{(0.8)(0.2)}{(0.8)(0.2) + (0.1)(0.8)} = \frac{2}{3}$$

- (b) Let C be the event that Victor is carrying an umbrella.
 Let D be the event that the forecast is no rain.
 The tree diagram in this case is:



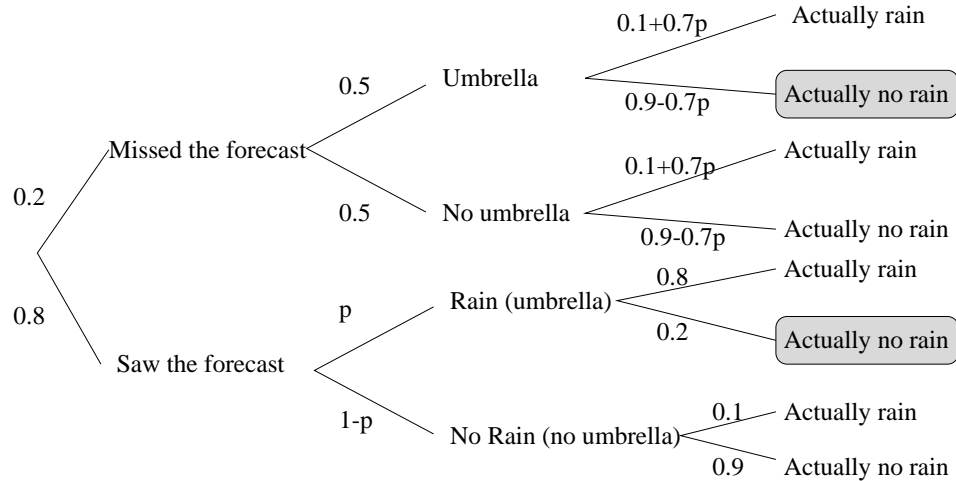
$$\begin{aligned} P(D) &= 1 - p \\ P(C) &= (0.8)p + (0.2)(0.5) = 0.8p + 0.1 \\ P(C | D) &= (0.8)(0) + (0.2)(0.5) = 0.1 \end{aligned}$$

Therefore, $P(C) = P(C | D)$ if and only if $p = 0$. However, p can only be 0.7 or 0.2, which implies the event C and D can never be independent, and this result does not depend on the season.

(c) Let us first find the probability of raining if Victor missed the forecast.

$$P(\text{actually rains} \mid \text{missed forecast}) = (0.8)p + (0.1)(1 - p) = 0.1 + 0.7p.$$

Then, we can extend the tree in part b) as follows,



Therefore, given that Victor is carrying an umbrella and it is not raining, we are looking at the two shaded cases.

$$\begin{aligned} & P(\text{saw forecast} \mid \text{umbrella and not raining}) \\ &= \frac{(0.8)p(0.2)}{(0.8)p(0.2) + (0.2)(0.5)(0.9 - 0.7p)} \end{aligned}$$

In fall and winter, $p = 0.7$, Probability = $\frac{112}{153}$
 In summer and spring, $p = 0.2$ Probability = $\frac{8}{27}$

4. Without prior bias on whether the exit of campus lies East or West, the exact answers of the passerby are not as important as whether a string of answers is similar or not. Let R_r denote the event that we receive r similar answers and T denote the event that these repeated answers are truthful. Let S denote the event that the questioned passerby is a student. Note that, because a professor always gives a false answer, $T \cap S^c = \emptyset$ and thus $\mathbf{P}(T \cap S^c) = 0$. Therefore,

$$\mathbf{P}(T|R_r) = \frac{\mathbf{P}(T \cap R_r)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r \cap S)}{\mathbf{P}(R_r)} = \frac{\mathbf{P}(T \cap R_r|S)\mathbf{P}(S)}{\mathbf{P}(R_r)}$$

where the stated independence of a passerby's successive answers implies $\mathbf{P}(T \cap R_r|S) = \left(\frac{3}{4}\right)^r$. Applying the Total Probability Theorem and again making use of independence, we also deduce

$$\mathbf{P}(R_r) = \mathbf{P}(R_r|S)\mathbf{P}(S) + \underbrace{\mathbf{P}(R_r|S^c)}_1 \mathbf{P}(S^c) = \left(\left(\frac{3}{4}\right)^r + \left(\frac{1}{4}\right)^r \right) \frac{2}{3} + \frac{1}{3} .$$

(a) Applying the above formulas for $r = 1$, we have $\mathbf{P}(R_1) = 1$ and thus

$$\mathbf{P}(T|R_1) = \frac{\frac{3}{4} \cdot \frac{2}{3}}{1} = \boxed{\frac{1}{2}} .$$

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(b) For $r = 2$, the formulas yield $\mathbf{P}(R_2) = \frac{3}{4}$ and thus

$$\mathbf{P}(T|R_2) = \frac{\left(\frac{3}{4}\right)^2 \frac{2}{3}}{\frac{3}{4}} = \boxed{\frac{1}{2}}.$$

(c) For $r = 3$, the formulas yield $\mathbf{P}(R_3) = \frac{15}{24}$ and thus

$$\mathbf{P}(T|R_3) = \frac{\left(\frac{3}{4}\right)^3 \frac{2}{3}}{\frac{15}{24}} = \boxed{\frac{9}{20}}.$$

(d) For $r = 4$, the formulas yield $\mathbf{P}(R_4) = \frac{35}{64}$ and thus

$$\mathbf{P}(T|R_4) = \frac{\left(\frac{3}{4}\right)^4 \frac{2}{3}}{\frac{35}{64}} = \boxed{\frac{27}{70}}.$$

(e) As soon as we receive a dissimilar answer from the same passerby, we know that this passerby is a student; a professor will always give the same (false) answer. Let D denote the event of receiving the first dissimilar answer. Given D on the fourth answer, either the student has provided three truthful answers followed by one untruthful answer, occurring with probability $\left(\frac{3}{4}\right)^3 \frac{1}{4}$, or the student has provided three untruthful answers followed by one truthful answer, occurring with probability $\left(\frac{1}{4}\right)^3 \frac{3}{4}$. Note that event T corresponds to the former; thus,

$$\mathbf{P}(T|R_3 \cap D) = \frac{\left(\frac{3}{4}\right)^3 \frac{1}{4}}{\left(\frac{3}{4}\right)^3 \frac{1}{4} + \left(\frac{1}{4}\right)^3 \frac{3}{4}} = \boxed{\frac{9}{10}}.$$

In parts (a) - (d), notice the decreasing trend in the probability of the passer-by being truthful as the number of similar answers grows. Intuitively, our confidence that the passerby is a professor grows as the sequence of similar answers gets longer, because we know a professor will always give the same (false) answer while a student has a chance to answer either way. However, as part (e) demonstrates, the first indication that the passerby is a student will boost our confidence that the previous string of similar answers are truthful, because any single answer by the student has a 3-to-1 chance of being a truthful one.

For the remainder of this problem, let E and W represent the events that a passerby provides East and West, respectively, as an answer and let T_E represent the event that East is the correct answer. We are told Ima's a-priori bias is $\mathbf{P}(T_E) = \epsilon$.

(f) Using Bayes's Rule and all the arguments used in parts (a) - (e), we have

$$\begin{aligned} \mathbf{P}(T_E|E) &= \frac{\mathbf{P}(E|T_E)\mathbf{P}(T_E)}{\mathbf{P}(E)} = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon} \quad \text{and} \\ \mathbf{P}(T_E|W) &= \frac{\mathbf{P}(W|T_E)\mathbf{P}(T_E)}{\mathbf{P}(W)} = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)\epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \frac{1}{3}\right)\epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)(1-\epsilon)} = \boxed{\epsilon}. \end{aligned}$$

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In particular, we have used that $\mathbf{P}(E) = \mathbf{P}(E|T_E)\mathbf{P}(T_E) + \mathbf{P}(E|T_E^c)\mathbf{P}(T_E^c)$ (and similarly for $\mathbf{P}(W)$)

(g) Likewise, given two consecutive and similar answers from the same passerby, we have

$$\mathbf{P}(T_E|EE) = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2 \epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2 \epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\epsilon} \quad \text{and}$$

$$\mathbf{P}(T_E|WW) = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right) \epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^2 + \frac{1}{3}\right) \epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^2 (1-\epsilon)} = \boxed{\epsilon} .$$

(h) Finally, given three consecutive and similar answers from the same passerby,

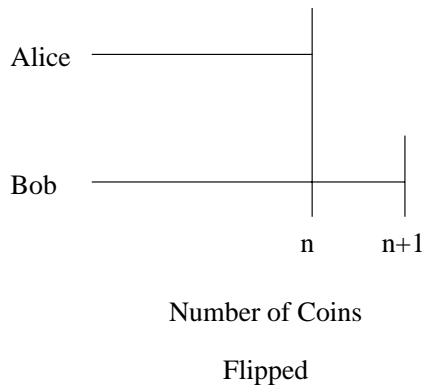
$$\mathbf{P}(T_E|EEE) = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3 \epsilon}{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3 \epsilon + \left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right)(1-\epsilon)} = \boxed{\frac{9\epsilon}{11-2\epsilon}} \quad \text{and}$$

$$\mathbf{P}(T_E|WWW) = \frac{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right) \epsilon}{\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^3 + \frac{1}{3}\right) \epsilon + \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)^3 (1-\epsilon)} = \boxed{\frac{11\epsilon}{9+2\epsilon}} .$$

For $\epsilon = \frac{9}{20}$, we calculate $\mathbf{P}(T_E|EEE) = \frac{81}{202}$ and $\mathbf{P}(T_E|WWW) = \frac{1}{2}$.

Notice that the E , EE and EEE answers to parts (f) - (h) match the answers to parts (a)-(c) when $\epsilon = \frac{1}{2}$, or when Ima's prior bias does not favor either possibility.

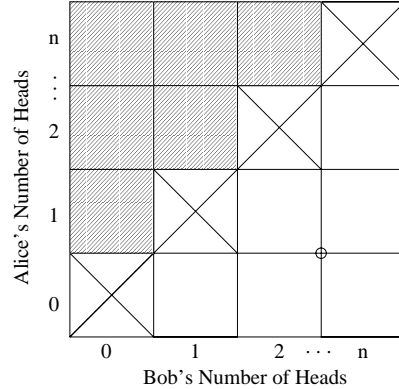
G1[†]. Through the first n coins, Alice and Bob are equally likely to have flipped the same number of heads as the other (since they are using fair coins and each flip is independent of the other flips). Given this, when Bob flips his last coin but Alice doesn't flip a coin, Bob has a $1/2$ chance of getting a head and thus having more heads than Alice.



Let's look at it another way. First define the following events:

1) A = the number of heads Alice tossed 2) B = the number of heads Bob tossed

Now suppose both Bob and Alice toss n coins. A sample space of interest is shown below where the shaded area represents Alice having more heads than Bob and the unshaded and uncrossed area represents Bob having more heads than Alice.



Since each coin is fair and each toss is independent of all the other tosses, each box in the diagram has an equal amount of probabilistic weight. We wish to show that

$$P(B > A) = P(A > B)$$

Well, each box has the same weight and there are the same number of boxes on each side of the diagonal. Consequently the above equation holds, meaning Alice and Bob are equally likely to have flipped more heads than the other. Furthermore, Alice and Bob are equally likely to have flipped the same number of heads, again since each box with an x in it has the same probabilistic weight.

Now Bob picks up the last coin. Given that both Alice and Bob are equally likely to have the same number of heads, the event Bob having more heads than Alice boils down to Bob getting a head on the last coin flip. Since this coin is fair and the flip is independent of past flips, this probability is simply $1/2$.

An alternative solution is shown as follows,

Let B be the event that Bob tossed more heads,

let X be the event that after each has tossed n of their coins, Bob has more heads than Alice,

let Y be the event that under the same conditions, Alice has more heads than Bob,

and let Z be the event that they have the same number of heads.

Since the coins are fair, we have $P(X) = P(Y)$, and also $P(Z) = 1 - P(X) - P(Y)$. Furthermore, we see that

$$P(B | X) = 1, \quad P(B | Y) = 0, \quad P(B | Z) = \frac{1}{2}.$$

Now we have, using the theorem of total probability,

$$P(B) = P(X) \cdot P(B | X) + P(Y) \cdot P(B | Y) + P(Z) \cdot P(B | Z)$$

$$\begin{aligned} &= P(X) + \frac{1}{2} \cdot P(Z) \\ &= \frac{1}{2} \cdot (P(X) + P(Y) + P(Z)) \\ &= \frac{1}{2} \end{aligned}$$

as required.