MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.011 Introduction to Communication, Control
and Signal Processing
Spring 2004

Second EVENING EXAM
Wednesday, April 21, 7:30 PM – 9:30 PM

- This is a closed book exam, but three $8\frac{1}{2}'' \times 11''$ sheets of notes (both sides) are allowed. They can be as big as $8\frac{1}{2}'' \times 11''$ or as small as you'd like and you can write on one side or two sides of each, but only three sheets are allowed.
- Everything on the notes must be in your original handwriting (i.e., material cannot be xeroxed from solutions, tables, books, etc).
- You have two hours for this exam.
- Calculators are NOT allowed.
- We will NOT provide a table of transforms.
- There are 4 problems on the exam with the percentage for each part and the total percentage for each problem as indicated. Note that the problems do not all have the same total percentage.
- The problems are not in order of difficulty. We recommend that you read through all the problems, then do the problems in whatever order suits you best.
- Please be neat—we cannot grade what we cannot decipher.
- We tried to provide ample space for you to write in. However, the amount of space provided is not an indication of the length of the explanation required. Short, to the point explanations are preferred to long ones that show little understanding.

All work and answers must be in the space provided on the exam booklet. You are welcome to use scratch pages that we provide but when you hand in the exam we will not accept any pages other than the exam booklet. There will be absolutely no exceptions.

Exam Grading

In grading all of the 6.011 exams we will be focusing on your level of understanding of the material associated with each problem. When we grade each part of a problem we will do our best to assess, from your work, your level of understanding. On each part of an exam question we will also indicate the percentage of the total exam grade represented by that part, and your numerical score on the exam will then be calculated accordingly.

Our assessment of your level of understanding will be based upon what is given in your solution. A correct final answer with no explanation will not receive full credit, and may even receive no credit at all. An incorrect final answer having a solution and explanation that shows excellent understanding quite likely will receive full (or close to full) credit.

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No work on this page will be evaluated.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.011 INTRODUCTION TO COMMUNICATION, CONTROL
AND SIGNAL PROCESSING
Spring 2004

SECOND EVENING EXAM - SOLUTIONS
Wednesday, April 21 , 2004

Full Name:

	Points	Grader
1(a)		
1(b)		
1(c)		
1(d)		
2(a)		
2(b)		
2(c)		
2(d)		
3(a)		
3(b)		
3(c)		
4		
Total		

Problem 1   (31%)

Consider the state space equations describing the position of a levitated ball in a magnetic suspension system as shown in Figure 1-1.
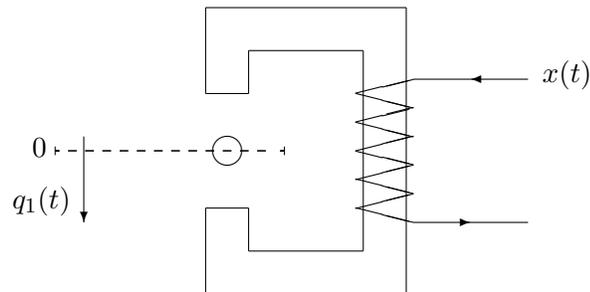

Figure 1-1:

The dynamical behavior of the ball is described by the following state equations where $q_1(t)$ is the vertical position and $q_2(t)$ is the vertical velocity of the ball

$$\frac{d\mathbf{q}(t)}{dt} = \mathbf{Aq}(t) + \mathbf{b}x(t)$$
$$y(t) = \mathbf{c}^T\mathbf{q(t)}$$

with

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}, \qquad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(a) (8%) Determine the eigenvalues and eigenvectors of the system.

$$\lambda_1 = 5 \qquad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 5 \end{bmatrix} \qquad\qquad \lambda_2 = -5 \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

Work to be looked at:

$$[\lambda I - A] = 0$$

$$\begin{bmatrix} \lambda & -1 \\ -25 & \lambda \end{bmatrix} = 0$$

$$\lambda^2 - 25 = 0$$

$$\lambda_1 = 5, \lambda_2 = -5$$

$For\,\lambda_1$,

$$\begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 5\begin{bmatrix} 1 \\ a \end{bmatrix}$$

a = 5

$For\,\lambda_2$,

$$\begin{bmatrix} 0 & 1 \\ 25 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = -5\begin{bmatrix} 1 \\ b \end{bmatrix}$$

b = -5

(b) (8%) Is it possible to arbitrarily place the modes of the system through the use of state feedback of the form

$$x(t) = \mathbf{g}^T \mathbf{q}(t)$$

$\boxed{\textbf{POSSIBLE}}$ **NOT POSSIBLE**

Explain:

The system is reachable

Work to be looked at:

$$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix}$$

$$V^{-1} = \frac{1}{-5-5}\begin{bmatrix} -5 & -1 \\ -5 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 5 & 1 \\ 5 & -1 \end{bmatrix}$$

$$\tilde{b} = V^{-1}b = \frac{1}{10}\begin{bmatrix} 5 & 1 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(c) (8%) Determine all possible values of $\mathbf{c}$ for which the system is bounded-input/bounded output (BIBO) stable.

$$\mathbf{c} = \mathbf{m} \begin{bmatrix} -5 & 1 \end{bmatrix}$$

Work to be looked at:

Let: $c^T = \begin{bmatrix} c_1 & c_2 \end{bmatrix}$

$$\tilde{c}^T = c^T V = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 5 & -5 \end{bmatrix} = \begin{bmatrix} c_1 + 5c_2 & c_1 - 5c_2 \end{bmatrix}$$

By choosing $c_1 + 5c_2 = 0$, the mode corresponding to λ_1 becomes unobservable.

(d) (7%) We assume that the state variables are not directly measureable but both the input $x(t)$ and the ouput $y(t)$ are with $c^T = (1,0)$. We want to estimate the state of the system based on the following equations

$$\frac{d\hat{\mathbf{q}}(t)}{dt} = \mathbf{A}\hat{\mathbf{q}}(t) + \mathbf{b}x(t) - \mathbf{l}(y(t) - \hat{y}(t))$$
$$\hat{y}(t) = \mathbf{c^T}\hat{\mathbf{q}}(t)$$

where $\hat{\mathbf{q}}$ is the estimate for $\mathbf{q}$. Can the gain vector l be chosen so that the error in the state estimate decays asymptotically. Clearly explain your answer.

<div align="center">

| YES | NO |

</div>

Explain:

There is no scalar m such that $c^T = \begin{bmatrix} 1 & 0 \end{bmatrix} = m \begin{bmatrix} -5 & 1 \end{bmatrix}$. Thus the unstable mode is observable.

Work to be looked at:

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Problem 2   (28%)

(a) (7%) A given system is asymptotically unstable but BIBO stable. Is it possible to implement observer-based feedback to make the system asymptotically stable?

<div align="center">

YES $\boxed{\textbf{NO}}$

</div>

Explain:

A system is asymptotically unstable but BIBO stable means that:

1. Unstable modes are hidden
2. Unstable modes are unreachable or unobservable
3. Observer based feedback cant stabilize the system asymptotically

(b) (7%) A reachable and observable 2nd order system is placed in series with a reachable and observable 3rd order system, see Figure 2-1. The overall system is found to have **one hidden mode**.

Figure 2-1:

If the hidden mode corresponds to a pole in the original 2nd order system (H), is the hidden mode of the overall system unreachable and/or unobservable?

<div align="center">

unreachable: YES $\boxed{\textbf{NO}}$ **unobservable:** $\boxed{\textbf{YES}}$ **NO**

</div>

Explain:
The original second order system H(z) is reachable and the hidden mode is a pole of H(z). The hidden mode must be reachable, but unobservable.

(c) (7%) Consider a pair of bivariate Gaussian random variables, X and Y. The MMSE estimator for Y in terms of X is $\hat{Y}(X) = 5$.

Are X and Y correlated?

correlated | **uncorrelated** |

Explain:

$\hat{Y}_{MMSE}(x) = \hat{Y}_{LMMSE}(x)$

$\hat{Y}_{MMSE}(x) = \frac{\sigma_{xy}}{\sigma_{x^2}}X + \mu_y - \frac{\sigma_{xy}}{\sigma_{x^2}}\mu_x$

$\hat{Y}_{MMSE}(x) = 5$

$\sigma_{xy} = 0$

Thus X, Y are uncorrelated.

(d) (7%) Consider a pair of random variables, X and Y, both with unity variance and zero mean. The LMMSE estimate $\hat{X}$ of X given Y is

$$\hat{X} = \frac{2}{3}Y$$

Can the LMMSE estimate $\hat{Y}$ of Y given X be

$$\hat{Y} = \frac{3}{2}X.$$

Clearly explain why or why not.

YES | **NO** |

Explain:

Solution 1:

$\hat{X}(Y) = \frac{\sigma_{xy}}{\sigma_{y^2}}Y = \sigma_{xy}Y = \frac{2}{3}Y$

$\sigma_{xy} = \frac{2}{3}$

$\hat{Y}(X) = \frac{\sigma_{xy}}{\sigma_{x^2}}Y = \sigma_{xy}X = \frac{2}{3}X$

But we are given that $\hat{Y}(X) = \frac{3}{2}X$. Thus there is a contradiction!!!

Solution 2:

$\hat{Y}(X) = \frac{3}{2}X \Rightarrow \rho\frac{\sigma_y}{\sigma_x} = \frac{3}{2}.$

This implies that $\rho = \frac{3}{2}$, but we know that $|\rho|$ must be less than or equal to 1. Thus there is a contradiction!!!

Problem 3   (26%)

Consider a digital communication system in which an i.i.d. bit stream $s[n]$ of ones and zeros is transmitted over a faulty, memoryless channel with ones and zeroes equally probable. The probability of a one being received as a zero is $1/8$ and the probability of a zero being received as a one is $1/4$. This type of channel is referred to as a binary memoryless channel and is depicted in Figure 3-1.
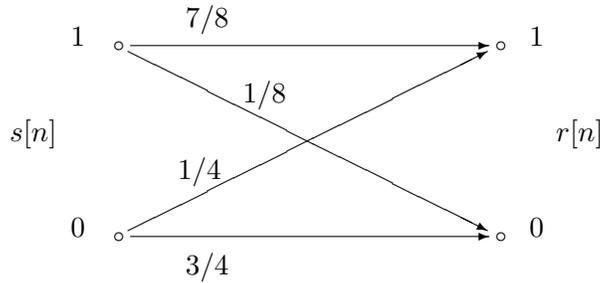
Figure 3-1:

(a) (10%) For any time index n, determine the joint probability mass function $P(r,s)$ and the marginal probability mass function $P(r)$

Work to be looked at:

$P(s=1) = P(s=0) = \frac{1}{2}$

$P(r=1, s=1) = P(r=1|s=1)P(s=1) = \frac{7}{8} \cdot \frac{1}{2} = \frac{7}{16}$

$P(r=1, s=0) = P(r=1|s=0)P(s=0) = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$

$P(r=1, s=1) = P(r=1|s=1)P(s=1) = \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{16}$

$P(r=0, s=0) = P(r=0|s=0)P(s=0) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8}$

$P(r=1) = P(r=1|s=1)P(s=1) + P(r=1|s=0)P(s=0) = \frac{7}{8} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{2} = \frac{9}{16}$

$$P(r = 0) = 1 - P(r = 1) = \tfrac{7}{16}$$

(b) (10%) We would like to process the received signal through a memoryless, possibly non-linear system F to obtain an estimate $\hat{s}[n]$ of $s[n]$ from $r[n]$. The memoryless system F is to be designed to minimize the mean-square error ε defined as:
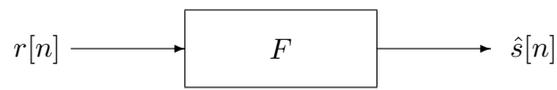
$$\varepsilon = E\left\{(s[n] - \hat{s}[n])^2\right\}$$

$$r[n] \longrightarrow \boxed{\quad F \quad} \longrightarrow \hat{s}[n]$$

Figure 3-2:

$$F = \begin{cases} \frac{1}{7} & r=0 \\ \frac{7}{9} & r=1 \end{cases}$$

Work to be looked at:

$$F = E[s|r] = \sum_s s \tfrac{P(s,r)}{P(r)}$$

$$r = 1 : 1 \cdot \tfrac{7/16}{9/16} + 0 \cdot \tfrac{1/8}{9/16} = \tfrac{7}{9}$$

$$r = 0 : 1 \cdot \tfrac{1/16}{7/16} + 0 \cdot \tfrac{3/8}{7/16} = \tfrac{1}{7}$$

(c) (6%) With your system from (b) what is the probability that at an arbitrary time index n_0 the estimate $\hat{s}[n_0]$ and the true value $s[n_0]$ will be equal?

Probability: 0

Work to be looked at:
Since our estimator gives out fractions, we can never get $\hat{s}[n_0] = s[n_0]$.

Problem 4   (15%)

A signal $s(t)$ is transmitted over a multipath channel as depicted in Figure 4-1 where $T = 10^{-3} = 1$ msec.
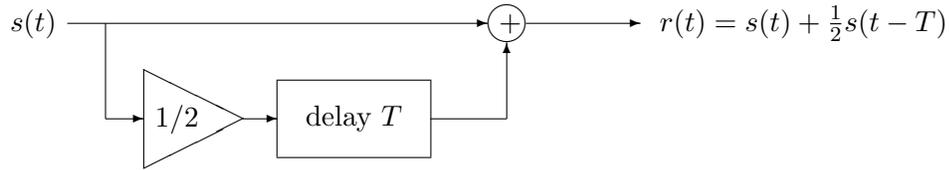

Figure 4-1:

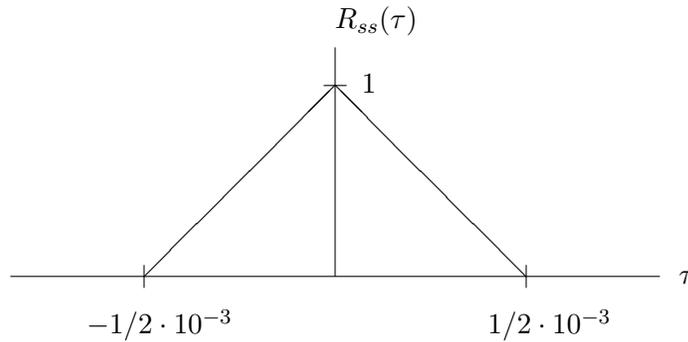$s(t)$ is a zero mean WSS random process with autocorrelation function shown in Figure 4-2.

Figure 4-2:

A receiver is to be designed to compensate for the multi-path propagation.
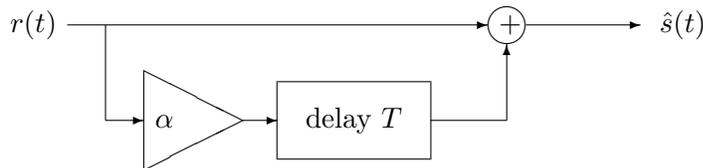Assume that the receiver has the structure shown in Figure 4-3.

Figure 4-3:

Determine the gain constant α in the receiver so that the mean square error between $\hat{s}(t)$ and $s(t)$ is minimized, i.e. determine α to minimize ε defined as:

$$\varepsilon = E\left\{(s(t) - \hat{s}(t))^2\right\}$$

Determine: $\alpha = -\frac{2}{5}$

Work to be looked at:

$r(t) = s(t) + \frac{1}{2}s(t - T)$

$\hat{s}(t) = r(t) + \alpha r(t - T) = s(t) + \frac{1}{2}s(t - T) + \alpha[s(t - T) + \frac{1}{2}s(t - 2T)]$

With $\varepsilon = E\left\{(s(t) - \hat{s}(t))^2\right\}$:

$\varepsilon = E\left[(s(t) - s(t) - \frac{1}{2}s(t - T) - \alpha\left[s(t - T) + \frac{1}{2}s(t - 2T)\right])^2\right]$

$\varepsilon = E\left[((\frac{1}{2} + \alpha)s(t - T) + \frac{1}{2}\alpha s(t - 2T))^2\right]$

$\varepsilon = (\frac{1}{2} + \alpha)^2 E\left[s^2(t - T)\right] + 2(\frac{1}{2} + \alpha)\frac{1}{2}\alpha E\left[s(t - T)s(t - 2T)\right] + \frac{1}{4}\alpha^2 E\left[s^2(t - 2T)\right]$

$\varepsilon = (\frac{1}{2} + \alpha)^2 R_{ss}(0) + (\frac{1}{2} + \alpha)\alpha R_{ss}(T) + \frac{1}{4}\alpha^2 R_{ss}(0)$

Thus, $\varepsilon = (\frac{1}{2} + \alpha)^2 + \frac{1}{4}\alpha^2$.

$R_{ss}(0) = 1, R_{ss}(T) = 0$

Now minimize ε with respect to α:

$\frac{\delta\varepsilon}{\delta\alpha} = 2(\frac{1}{2} + \alpha) + 2 \cdot \frac{1}{4}\alpha = 0$

$\alpha = -\frac{2}{5}$