Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

QUIZ 1, October 12, 2004

Your Full Name:	SOLUTIONS
Recitation Instructor & Time :	

- This quiz is **closed book**, but **one** sheet of notes is allowed. Calculators will not be necessary and are not allowed.
- Put your name in the space indicated above, and your recitation time next to the name of your instructor.
- Check that you have pages numbered up to 12. The booklet contains spaces for **all** answers.
- **Neat work and clear explanations count; show all relevant work and reasoning!** You may want to first work things through on scratch paper and then neatly transfer to this booklet the work you would like us to look at. Let us know if you need additional scratch paper. All scratch paper will be collected at the end of the quiz. However, **only** this booklet will be considered in the grading; **no additional answer or solution written elsewhere will be considered.** Absolutely no exceptions!
- There are **two problems**, **weighted as indicated on the question booklet**.
- **DO NOT DISCUSS THIS QUIZ WITH STUDENTS WHO HAVE NOT YET TAKEN IT TODAY!**

Problem	Your Score
1 (35 points)	
2 (15 points)	
Total (50 points)	

Problem 1 (35 points)

1(a) (10 points) The frequency response of a particular DT LTI system is

$$H(e^{j\Omega}) = \frac{e^{j2\Omega}}{1 - \frac{1}{2}e^{-j\Omega}} \ .$$

Determine its unit-sample response $h[n]$. (If you do this correctly, you will find that the system is neither causal nor anti-causal.) Also determine

$$\sum_{k=-\infty}^{\infty} h[k] \qquad \text{and} \qquad \int_0^{\pi} |H(e^{j\Omega})|^2 \, d\Omega \ .$$

Caution: You need to get the value of the integral exactly right to get credit for that part, so read carefully to see what is asked — and evaluate the integral without doing any integration! It may help you to recall that

$$\sum_{i=0}^{\infty} r^i = \frac{1}{1-r} \ , \qquad |r| < 1 \ .$$

Begin work for 1(a) here:

First, find $h[n]$:

$$H(z) = \frac{z^2}{1 - \frac{1}{2}z^{-1}} = z^2 \left(1 + \frac{1}{2}z^{-1} + (\frac{1}{2})^2 z^{-2} + ...\right)$$

$$h[n] = \left(\frac{1}{2}\right)^{n+2} u[n+2]$$

Next, compute:

$$\sum_{-\infty}^{\infty} h[k] = 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + ... = \frac{1}{1 - \frac{1}{2}} = 2$$

OR

$$\sum_{-\infty}^{\infty} h[k] = H(e^{j0}) = 2$$

Lastly, find:

$$\int_0^{\pi} |H(e^{j\Omega})|^2 \, d\Omega = \frac{2\pi}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\Omega})|^2 d\Omega\right)$$

By Parseval's theorem:

$$= \frac{2\pi}{2}\left(\sum_{k=-\infty}^{\infty} h^2[k]\right) = \pi\left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + ...\right) = \frac{\pi}{1-\frac{1}{4}} = \frac{4\pi}{3}$$

$$h[n] = \left(\frac{1}{2}\right)^{n+2} u[n+2]$$

$$\sum_{-\infty}^{\infty} h[k] = 2$$

$$\int_0^{\pi} |H(e^{j\Omega})|^2 \, d\Omega = \frac{4\pi}{3}$$

1(b) (5 points) If $x[n]$ denotes a wide-sense stationary process with mean value μ_x and autocovariance function $C_{xx}[m] = \sigma_x^2\,\delta[m]$, what is the linear minimum mean-square-error (LMMSE) estimate of $x[n+2]$ in terms of $x[n]$? In other words, find α and β in $\widehat{x}[n+2] = \alpha x[n] + \beta$ such that $E\{(x[n+2] - \widehat{x}[n+2])^2\}$ is minimized. Also find the associated (minimum) mean square error (MSE).

Work to be looked at:

$$\hat{x}[n+2] = \mu_x + \frac{C_{xx}[2]}{C_{xx}[0]}\left(x[n] - \mu_x\right)$$

But, we know that:

$$C_{xx}[2] = 0$$

Therefore,

$$\hat{x}[n+2] = \mu_x$$

Thus, $\alpha = 0$ and $\beta = \mu_x$.

The associated MMSE is:

$$MMSE = var\left(x[n+2]\right) = C_{xx}[0] = \sigma_x^2$$

Note that the general expression for MMSE is given as:

$$\sigma_x^2\left(1 - \rho^2\right)$$

but in this case, $\rho = 0$.

$\alpha = 0$ $\beta = \mu_x$

MSE $= \sigma_x^2$

1(c) (10 points) If the process $x[n]$ in 1(b) is applied to the input of the system in 1(a), what is the power spectral density $S_{yy}(e^{j\Omega})$ of the output process $y[n]$? Also evaluate $E\{y[n]\}$, $E\{y^2[n]\}$, and

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} y[k] \ .$$

Begin work for 1(c) here.

First find the PSD of $y[n]$:

$$R_{xx}[m] = C_{xx}[m] + \mu_x^2$$

$$S_{xx}\left(e^{j\Omega}\right) = \sigma_x^2 + 2\pi\mu_x^2\delta\left(\Omega\right)$$

$$S_{yy}\left(e^{j\Omega}\right) = \left|H\left(e^{j\Omega}\right)\right|^2 S_{xx} = H\left(e^{j\Omega}\right) H\left(e^{-j\Omega}\right) S_{xx} = \frac{e^{j2\Omega}}{1-\frac{1}{2}e^{-j\Omega}} \frac{e^{-j2\Omega}}{1-\frac{1}{2}e^{j\Omega}} S_{xx}$$

$$S_{yy}\left(e^{j\Omega}\right) = \frac{S_{xx}}{\frac{5}{4}+\cos\Omega} = \frac{\sigma_x^2 + 2\pi\mu_x^2\delta\left(\Omega\right)}{\frac{5}{4}+\cos\Omega}$$

Second, find $E(y[n])$:

$$E(y[n]) = \mu_x H\left(e^{j0}\right) = 2\mu_x$$

Third, find $E(y^2[n])$:

$$E(y^2[n]) = R_{yy}[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{yy}\left(e^{j\Omega}\right) d\Omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} S_{xx}\left(e^{j\Omega}\right)\left|H\left(e^{j\Omega}\right)\right|^2 d\Omega$$

$$E(y^2[n]) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sigma_x^2 + 2\pi\mu_x^2\delta\left(\Omega\right)\right)\left|H\left(e^{j\Omega}\right)\right|^2 d\Omega$$

$$E(y^2[n]) = \sigma_x^2\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|H\left(e^{j\Omega}\right)\right|^2 d\Omega\right) + \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\pi\mu_x^2\delta(\Omega)\left|H\left(e^{j\Omega}\right)\right|^2 d\Omega$$

$$E(y^2[n]) = \frac{4\sigma_x^2}{3} + \mu_x^2\left|H\left(e^{j0}\right)\right|^2 = \frac{4\sigma_x^2}{3} + 4\mu_x^2$$

Lastly, compute using ergodicity:

$$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} y[k] = \mu_y = 2\mu_x$$

.

$S_{yy}(e^{j\Omega}) = \frac{\sigma_x^2 + 2\pi\mu_x^2\delta(\Omega)}{\frac{5}{4} + \cos\Omega}$

$E\{y[n]\} = 2\mu_x$

$E\{y^2[n]\} = \frac{4\sigma_x^2}{3} + 4\mu_x^2$

$\lim_{N\to\infty} \frac{1}{2N+1} \sum_{k=-N}^{N} y[k] = 2\mu_x$

Caution: Make sure you move on to Problem 2 if you find yourself taking too long or getting stuck in part 1(d) below.

1(d) (10 points) With all quantities as previously defined, and assuming $\mu_x = 0$ for simplicity, suppose what you can measure is $q[n] = y[n] + v[n]$ for all n, where $v[n]$ is (zero-mean) white noise of intensity σ_v^2, and is uncorrelated with the process $x[k]$. Compute the frequency response $W(e^{j\Omega})$ of the non-causal Wiener filter that has $q[n]$ as input at time n and produces the LMMSE estimate $\widehat{x}[n+2]$ as output at time n. Explicitly check that your answer reduces to something that you expect in the case that $\sigma_v^2 = 0$.

Begin work for 1(d) here.

Let $g[n] = x[n+2]$

$$W\left(e^{j\Omega}\right) = \frac{S_{gq}\left(e^{j\Omega}\right)}{S_{qq}\left(e^{j\Omega}\right)}$$

Since y and v are uncorrelated (since x and v are uncorrelated), we have:

$$S_{qq} = S_{yy} + S_{vv} = HH^*\sigma_x^2 + \sigma_v^2$$

Since g and v are uncorrelated, we also have that:

$$S_{gq} = S_{gy}$$

Making use of the fact that $g[n] = x[n+2]$, we can say:

$$S_{gy} = e^{2j\Omega} S_{xy} = e^{2j\Omega} H^* \sigma_x^2$$

Putting it all together, we have:

$$W\left(e^{j\Omega}\right) = \frac{e^{j2\Omega} H\left(e^{-j\Omega}\right)\sigma_x^2}{H\left(e^{j\Omega}\right) H\left(e^{-j\Omega}\right)\sigma_x^2 + \sigma_v^2}$$

.

$$W\left(e^{j\Omega}\right) = \frac{e^{j2\Omega} H\left(e^{-j\Omega}\right) \sigma_x^2}{H\left(e^{j\Omega}\right) H\left(e^{-j\Omega}\right) \sigma_x^2 + \sigma_v^2}$$

When $\sigma_v^2 = 0$, this becomes:

$$\frac{e^{j2\Omega}}{H\left(e^{j\Omega}\right)}$$

Which is reasonable because: This is the inverse system to get $x[n]$ from $y[n]$, then advance by 2 steps, to get $x[n+2]$.

Problem 2 (15 points)

2(a) (5 points) Suppose $q_1(t)$ is obtained from $x_1(t)$ by filtering through a stable system with frequency response $\frac{1-j\omega}{1+j\omega}$, and $q_2(t)$ is obtained from $x_2(t)$ by filtering through another stable system with the same frequency response $\frac{1-j\omega}{1+j\omega}$. Express the cross-spectral density $S_{q_1q_2}(j\omega)$ in terms of $S_{x_1x_2}(j\omega)$. (Assume x_1 and x_2 are jointly wide-sense-stationary.)

$$S_{q_1q_2}(j\omega) = H\,(j\Omega)\,H^*\,(j\Omega)\,S_{x_1x_2}$$

$$S_{q_1q_2}(j\omega) = \left(\frac{1-j\omega}{1+j\omega}\right)\left(\frac{1+j\omega}{1-j\omega}\right)S_{x_1x_2}(j\omega) = S_{x_1x_2}(j\omega)$$

$S_{q_1q_2}(j\omega) = S_{x_1x_2}(j\omega)$

2(b) (10 points) For each of the following functions $R[m]$, state whether or not it can be the autocorrelation function of a DT WSS random process, where m denotes the lag. If it cannot be, explain why not. If it can be, explain in detail how you would obtain such a process by appropriately filtering a Bernoulli process that takes values at each time instant of $+1$ or -1, with equal probability.

(i) $R[m] = 1$ for $m = 0$, 0.7 for $|m| = 1$, and 0 elsewhere.

(ii) $R[m] = 2$ for $m = 0$, -1 for $|m| = 1$, and 0 elsewhere.

2(b)(i) $R[m] = 1$ for $m = 0$, 0.7 for $|m| = 1$, and 0 elsewhere:

Begin work here:

There are three easy criteria we can check for this candidate:

1. Is $R[m]$ an even function of m, i.e. is $R[m] = R[-m]$? YES
2. Is $|R[m]| \leq |R[0]|$? YES
3. Is the PSD always non-negative? NO, see work below.

$$\mathcal{F}(R[m]) = S(e^{j\Omega}) = 1 + 1.4\frac{(e^{j\Omega} + e^{-j\Omega})}{2} = 1 + 1.4\cos\Omega$$

In the above expression, $S(e^{j\Omega})$ can be negative for some Ω, and therefore it cannot be a valid PSD.

Thus, $R[m]$ CANNOT be a valid autocorrelation function.

2(b)(ii) $R[m] = 2$ for $m = 0$, -1 for $|m| = 1$, and 0 elsewhere:

Begin work here:

There are three easy criteria we can check for this candidate:

1. Is $R[m]$ an even function of m, i.e. is $R[m] = R[-m]$? YES
2. Is $|R[m]| \leq |R[0]|$? YES
3. Is the PSD always non-negative? YES, see work below.

$$\mathcal{F}(R[m]) = S(e^{j\Omega}) = 2 + 2\frac{\left(e^{j\Omega} + e^{-j\Omega}\right)}{2} = 2 + 2cos\Omega$$

In the above expression, $S(e^{j\Omega})$ is non-negative for all Ω, and therefore it CAN be a valid PSD. Thus, $R[m]$ CAN be a valid autocorrelation function.

To actually demonstrate that it is valid, we now show how to generate a process with this autocorrelation function, by appropriate LTI filtering of a Bernoulli process, with zero mean and autocorrelation $\delta[m]$. We need to find $h[m]$ such that $h[m] * h[-m] = R[m]$. The following filter will work for any integer value of l:

$$h[n] = \pm\left(\delta[n-l] - \delta[n-l-1]\right)$$

You can also obtain an expression for the filter $H(e^{j\Omega})$ in the frequency domain as:

$$H(j\Omega)H(-j\Omega) = 2 + 2\cos\Omega$$
$$H(j\Omega) = 1 - e^{-j\Omega}$$

Additional work: