Massachusetts Institute of Technology

Department of Electrical Engineering and Computer Science

6.011: Introduction to Communication, Control and Signal Processing

Fall 2004 Final Exam SOLUTIONS

YOUR NAME:

Recitation Hour:

- This is a closed book exam, but you may use FOUR $8\frac{1}{2}$" $\times$ 11" sheets of notes (both sides). Calculators are not allowed.
- The questions are in two parts. Part I, worth 60% of the exam, comprises several relatively short questions, which should require only somewhat brief calculations and explanations. We estimate that this will take you around 100 minutes. Part II comprises two longer problems, which are worth 20% each, and which we estimate will take you around 40 minutes each.
- We would rather see you do 80% of the exam quite well than 100% of the exam quite poorly!
- Be clear and precise in your reasoning and **show all relevant work.**
- **If we can't read it, we can't/won't grade it!** So please write neatly and legibly.
- You are to **hand in only this ANSWER booklet**. **No additional pages will be considered in the grading.** You may want to first work things through in the blank areas of the question booklet or on scratch paper, and then neatly transfer to thisr booklet the work you would like us to look at. Let us know if you need additional scratch paper.

Problem	**Your Score**
1 (8 points)	
2 (8 points)	
3 (8 points)	
4 (8 points)	
5 (10 points)	
6 (8 points)	
7 (10 points)	
8 (20 points)	
9 (20 points)	
Total (100 points)	

Problem 1 (8 points)

The voltage waveform $v(t)$ between the red terminal and ground of a power supply in your lab is equally likely to be either +5V for all time t, or −5V for all time t, because the power supply is equally likely to have been manufactured (in the distant past) by the Duraplus or Everminus companies. For this random process $v(t)$, determine: (i) the mean value of the process; (ii) its autocorrelation function; (iii) its autocovariance function; (iv) whether it is wide-sense stationary; (v) whether it is strict-sense stationary; and (vi) whether it is ergodic in mean value.

(i) $E[v(t)] = \mu_v = \frac{1}{2}(5) + \frac{1}{2}(-5) = 0$

(ii) $E[v(t+\tau)v(t)] = \frac{1}{2}(5)(5) + \frac{1}{2}(-5)(-5) = 25$

(iii) $E[(v(t+\tau) - \mu_v)(v(t) - \mu_v)] = \frac{1}{2}(5)(5) + \frac{1}{2}(-5)(-5) = 25$

(iv) YES, it is WSS: Since the mean and autocorrelation are constant, this process is WSS.

(v) YES, it is SSS: Since the time origin is irrelevant to the joint densities of samples, this process is SSS. In other words, the probabilistic descriptions are time invariant.

(vi) NO, it is not ergodic. The time average is either 5 for all time, with probability $\frac{1}{2}$, or −5 for all time, with probability $\frac{1}{2}$, while the ensemble average is 0.

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 1 (continued)

(i) Mean $= 0$

(ii) Autocorrelation function $= 25$

(iii) Autocovariance function $= 25$

(iv) Is the process wide-sense stationary? YES

(v) Is the process strict-sense stationary? YES

(vi) Is the process ergodic in mean value? NO

Problem 2 (8 points)

Which of the following are valid autocorrelation functions for a continuous-time wide-sense stationary random process $x(t)$? Briefly justify your answers. For each case that represents a valid autocorrelation function, determine the mean value of the process (explaining your reasoning!), and also compute $E[x^2(5)]$.

(a) $R_{xx}(\tau) = 2$, for $|\tau| < 2$, and is zero elsewhere:

The PSD $S_{xx}(j\omega)$ is a sinc function (the ampltude and frequency scaling is irrelevant), which goes negative, making it an invalid PSD for a WSS process.

Is this $R_{xx}(\tau)$ a valid autocorrelation function? (Justify your answer.)

NO - see explanation above

If it is, what is $E[x(t)]$? (Explain your reasoning.)

And if it is, then $E[x^2(5)] =$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 2 (continued)

(b) $R_{xx}(\tau) = 2 - |\tau|$, for $|\tau| < 2$, and is zero elsewhere:

The PSD $S_{xx}(j\omega) = sinc^2$, which is real, even, and non-negative for all ω, making it a valid PSD. This process can be generated by passing white noise through a frequency response of $\sqrt{sinc^2}$.

Is this $R_{xx}(\tau)$ a valid autocorrelation function? (Justify your answer.)

YES - see explanation above

If it is, what is $E[x(t)]$? (Explain your reasoning.)

The mean is 0. Proof by contradiction: Assume the mean is not equal to 0, then the transform of the autocovariance would be $D_{xx}(j\omega) = S_{xx}(j\omega) - \mu_x^2\delta(\omega)$. For this to be a valid autocorrelation, both the PSD and the transform of the autocovariance must be non-negative for all ω. Since S_{xx} does not have an impulse at DC to cancel with μ_x^2, D_{xx} would be negative at DC. This violates our criteria, hence there is a contradiction!

And if it is, then $E[x^2(5)] = R_{xx}(0) = 2$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 2 (continued further)

(c) $R_{xx}(\tau) = \frac{\sin(\pi\tau)}{\pi\tau}$.

The PSD S_{xx} is a box, which is real, even and non-negative for all ω. Thus this is also a valid PSD.

Is this $R_{xx}(\tau)$ a valid autocorrelation function? (Justify your answer.)

YES - see explanation above.

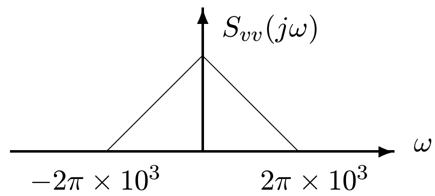
If it is, what is $E[x(t)]$? (Explain your reasoning.)

The mean is again 0. See the explanation for part b.

And if it is, then $E[x^2(5)] = R_{xx}(0) = 1$

Problem 3 (8 points)

Suppose $v(t)$ is a CT random process with power spectral density $S_{vv}(j\omega)$ having the triangular shape shown in the figure below (so the PSD is finite at $\omega = 0$). Is it possible to find a T such that the sequence $v[n] = v(nT)$ is a DT white noise process? If so, determine T and clearly explain your reasoning; if not, explain why not. (**Caution**: It is unlikely that your reasoning is complete if it doesn't deal with the relation between the autocorrelation function $R_{vv}(\tau)$ of a CT process and the autocorrelation function $R_{vv}[m]$ of the DT process obtained by sampling!)



YES, it is possible.

The PSD shown can also be expressed as the convolution of a box with itself, where the box is constant over the $-\pi \times 10^3 \leq \omega \leq \pi \times 10^3$ and 0 otherwise. Therefore the autocorrelation of the process is a $sinc^2$, with zero crossings at $T = \frac{\pi}{\pi \times 10^3} = 10^{-3}$. If we sample $v(t)$ every T seconds, the autocorrelation of $v[n]$ will be $R_{vv}[m] = R_{vv}(mT) = \alpha\delta[m]$, where α is some constant. The discrete time process is therefore white noise.

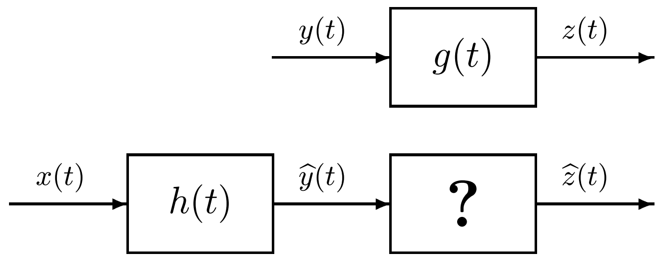
(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 3 (continued)

Is there a T such that $v[n] = v(nT)$ is a DT white noise process?

If so, $T =$

Explanation of whether or not there is a T:



Problem 4 (8 points)

Suppose that the random processes $x(\cdot)$ and $y(\cdot)$ are zero-mean and jointly wide-sense stationary, and that you have already designed a non-causal Wiener filter to use measurements of $x(\cdot)$ for all time in order to produce the linear minimum mean-square error (LMMSE) estimate $\widehat{y}(t)$ of $y(t)$ for each t. If we denote the impulse response of this Wiener filter by $h(\cdot)$, then $\widehat{y}(t) = h * x(t)$, as shown in the figure here.

Suppose you have computed and stored $\widehat{y}(t)$ for all t but have thrown away the original measurements $x(\cdot)$. Now you regret having done that, because you realize you also wanted to find the LMMSE estimate $\widehat{z}(t)$ of another process $z(t)$ using the (now discarded) measurements $x(\cdot)$. What saves you, however, is that the $z(t)$ you are interested in happens to be simply a filtered version of the process $y(\cdot)$ that you originally estimated: $z(t) = g * y(t)$, where $g(\cdot)$ is the known impulse response of a stable system.

Show that you can actually compute $\widehat{z}(t)$ by appropriate LTI filtering of $\widehat{y}(\cdot)$. In other words, specify the impulse response of an LTI filter that will take $\widehat{y}(\cdot)$ as input and produce $\widehat{z}(t)$ as output. Explicitly verify for this choice that the orthogonality principle is satisfied, i.e., that $R_{\widehat{z}x}(\tau) = R_{zx}(\tau)$ for all τ.

Guess $g(t)$ is the filter.

Now verify that this satisfies the orthogonality principle:

$$R_{\hat{z}x}(\tau) = g * R_{\hat{y}x}(\tau)$$

$$g * R_{\hat{y}x}(\tau) = g * R_{yx}(\tau) \text{ since } R_{\hat{y}x} = R_{yx} \text{ due to the fact that } h(t) \text{ is optimal}$$

$$g * R_{yx}(\tau) = R_{zx}(\tau)$$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

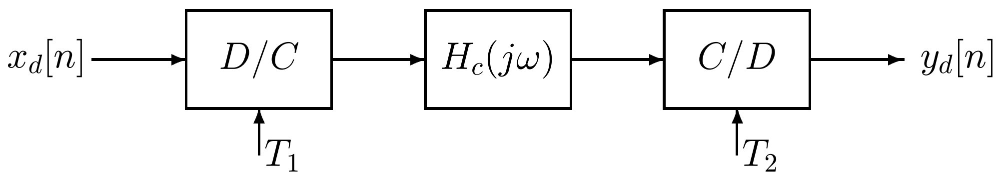
Problem 4 (continued)

Impulse response of an LTI filter that will take $\widehat{y}(\cdot)$ as input and produce $\widehat{z}(t)$ as output:

Explicit verification that the orthogonality principle is satisfied, i.e., that $R_{\widehat{z}x}(\tau) = R_{zx}(\tau)$ for all τ:

Problem 5 (10 points)

(i) (4 points) Suppose $T_1 = T_2 = T$ in the figure below. Determine all values of T for which the overall system from $x_d[n]$ to $y_d[n]$ is linear and time-invariant (LTI), and for these values of T write down the overall transfer function $H_d(e^{j\Omega})$ (specified for $|\Omega| < \pi$), expressing the transfer function in terms of $H_c(j\omega)$ and T.



We are assuming we have an ideal D/C converter. The output of the D/C converter, $x_c(t)$, is always bandlimited to $|\omega| = \frac{\pi}{T}$. As a result, the input to the C/D is always appropriately bandlimited, and no aliasing occurs.

$$Y_d(e^{j\Omega}) = \frac{1}{T} Y_c j\omega = \frac{1}{T} H_c(j\omega) X_c(j\omega)$$

$$Y_d(e^{j\Omega}) = \frac{1}{T} H_c(j\omega) T X_d(e^{j\Omega}) = H_c(j\omega) X_d(e^{j\Omega})|_{\omega=\frac{\Omega}{T}, |\Omega|<\pi}$$

Values of T for which the overall system is LTI: ALL T

Overall transfer function $H_d(e^{j\Omega})$ for these values of T (specify for $|\Omega| < \pi$):

$$H_c(j\frac{\Omega}{T}) \text{ for } |\Omega| < \pi$$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 5 (continued)

(ii) (6 points) Suppose $T_1 = 1$, $T_2 = 2$, $H_c(j\omega) = e^{-j3\omega}$ for all ω, and $x_d[n] = \cos(0.2n)$. What is $y_d[n]$?

$$x_d[n] \longrightarrow \boxed{D/C} \longrightarrow \boxed{H_c(j\omega)} \longrightarrow \boxed{C/D} \longrightarrow y_d[n]$$

$$\uparrow T_1 \qquad\qquad\qquad \uparrow T_2$$

$$x_c(nT) = x_d[n] = cos[0.2n]$$
$$x_c(t) = cos(0.2t)$$
$$y_c(t) = cos(0.2(t-3)) = cos(0.2t - 0.6)$$
$$y_d[n] = cos[0.2(2n) - 0.6] = cos[.4n - 0.6]$$

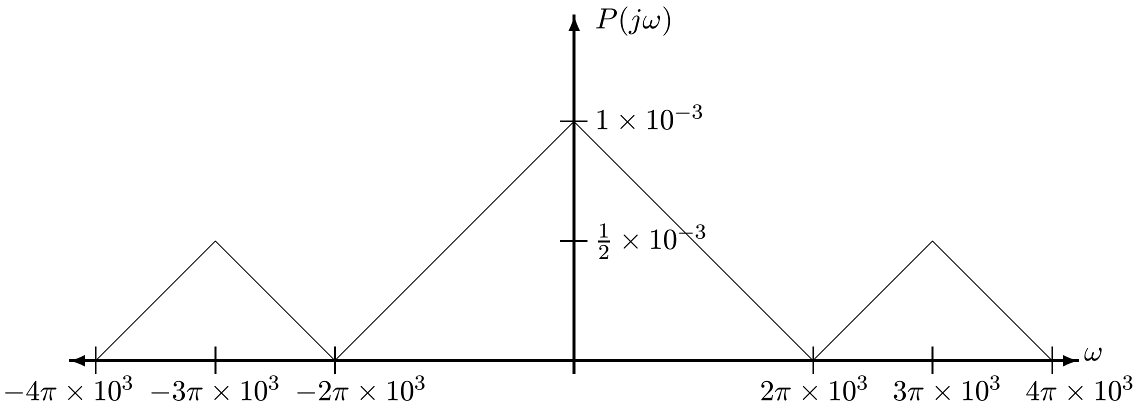
$y_d[n] = cos[.4n - 0.6]$

Problem 6 (8 points)

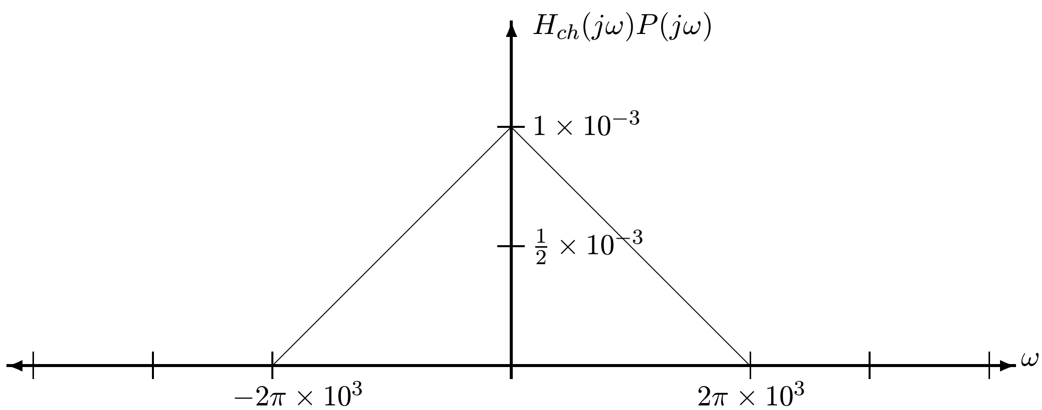
Consider the PAM communication system shown in the problem statement, where the LTI channel has frequency response

$$H_{ch}(j\omega) = \begin{cases} 1 & \text{for } |\omega| < 2\pi \times 10^3 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $P(j\omega)$ denote the CTFT of the basic pulse $p(t)$ used for the PAM system. Determine the smallest T for which the $P(j\omega)$ shown will yield $b[n] = c\,a[n]$, where c is a constant, and determine c.



First, we must examine how the channel affects the pulse.



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Problem 6 (continued)

This has to satisfy the Nyquist zero-ISI condition:

$$\frac{2\pi}{T} = 2\pi \times 10^3$$

Thus the smallest T is given by:

$$T = 10^{-3}$$

Integer multiples of T would also work, but the value given above is the smallest.

The Nyquist condition gives:

$$\sum_{-\infty}^{\infty} G\left(j\left(\omega - \frac{2\pi n}{T}\right)\right)$$

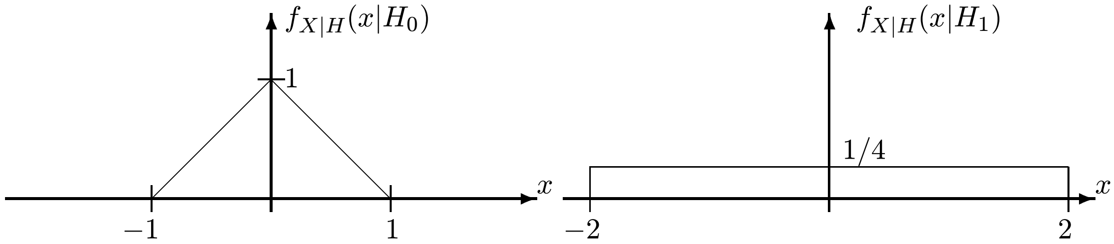
Evaluating this at $\omega = 0$ shows that $c = 1$.

Smallest T for which $b[n] = c\,a[n]$: 10^{-3}

$c = 1$

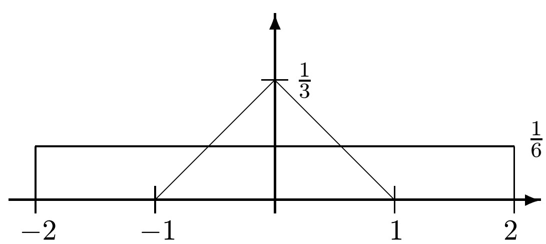
Problem 7 (10 points)

Assume we have to decide between hypotheses H_0 and H_1 based on a measured random variable X. The conditional densities for H_0 and H_1 are given below. Fully specify the decision rule that yields minimum probability of error, assuming the prior probabilities of the hypotheses satisfy $P(H_1) = 2P(H_0)$. Also compute $P('H_1' \mid H_0)$ and $P('H_1' \mid H_1)$ for this decision rule (where $'H_i'$ denotes the event that we decide in favor of H_i).



$$P(H_0) = \frac{1}{3}\ ,\ P(H_1) = \frac{2}{3}$$

We scale the densities by the appropriate probability, and sketch both on the same plot.



Thus, we pick our decision rule to be:

$$\text{`}H_0\text{'} \text{ for } |x| < 0.5,\ \text{`}H_1\text{'} \text{ othwerwise}$$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 7 (continued)

We must now find the probability of false alarm and the probability of detection.

$$P_{FA} = P(\text{`}H_1\text{'}|H_0) = (\frac{1}{2})(\frac{1}{2}) = \frac{1}{4}$$
$$P_D = P(\text{`}H_1\text{'}|H_1) = (\frac{3}{2})(\frac{1}{4})2 = \frac{3}{4}$$

Decision rule that yields minimum probability of error, assuming $P(H_1) = 2P(H_0)$:

$P('H_1' \,|\, H_0) =$

$P('H_1' \,|\, H_1) =$

PART II

Problem 8 (20 points)

Consider a causal, discrete-time LTI system with input $x[n]$ and output $y[n]$, described by a second-order state-space model of the form

$$\begin{aligned}\mathbf{q}[n+1] &= \mathbf{A}\mathbf{q}[n] + \mathbf{b}x[n] \\ y[n] &= \mathbf{c}^T\mathbf{q}[n] + \mathtt{d}x[n]\end{aligned}$$

(so $\mathbf{A}$ is a 2×2 matrix). You are given the following facts:

Fact 1: With $x[n] = 0$ for all $n \geq 0$ and with $\mathbf{q}[0]$ set at some particular value, the output for $n \geq 0$ is

$$y[n] = 4\,(0.5)^n + 2\,(-0.4)^n\ .$$

Fact 2: When $x[n]$ for $-\infty < n < \infty$ is a stationary random process that is equally likely to take the value $+1$ or -1 at each instant, independently of the values it takes at other instants, we find that

$$R_{yx}[m] = E\{y[n+m]x[n]\} = \delta[m] + 4\,(0.5)^{m-1}u[m-1]$$

where $E\{\,\cdot\,\}$ denotes the expected value, $\delta[m]$ is the unit sample function, and $u[m]$ is the unit step function.

(a) What are the eigenvalues of $\mathbf{A}$? Is the system asymptotically stable?

We can find the eigenvalues using fact 1 and the nature of the modal solution:

$$\lambda_1 = 0.5,\ \lambda_2 = -0.4$$

YES, the system is asymptotically stable, due to the fact that both eigenvalues have magnitude less than 1.

Eigenvalues of $\mathbf{A}$ are:

Is system asymptotically stable?

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 8 (continued)

(b) Is the system observable? Explain.

YES, since both modes appear at the output, even for the ZIR.

(c) For $x[n]$ in Fact 2, mean value $\mu_x = (1)\frac{1}{2} + (-1)\frac{1}{2} = 0$

Auto-correlation function $R_{xx}[m] = \delta[m]$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 8 (continued further)

(d) Unit-sample response $h[n]$ of the system from $x[n]$ to $y[n]$ (with $\mathbf{q}[0] = 0$), and system function $H(z)$.

$$R_{yx}[m] = h * R_{xx}[m] = h * \delta[m] = h[m]$$

$$h[m] = \delta[m] + 4(\frac{1}{2})^{m-1}u[m-1]$$

$$H(z) = 1 + \sum_{m=1}^{\infty} 4(\frac{1}{2})^{m-1}z^{-m}$$

$$H(z) = 1 + \frac{4z^{-1}}{1 - 0.5z^{-1}} = 1 + \frac{4}{z - 0.5} = \frac{z + 3.5}{z - 0.5}$$

$h[n] = \delta[m] + 4(\frac{1}{2})^{m-1}u[m-1]$

$H(z) = \frac{z+3.5}{z-0.5}$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 8 (continued still further)

(e) Is the system reachable? Explain.

NO. There is a hidden mode at $z = -0.4$. Since both modes are observable, this mode must be unreachable.

(f) With $x[n] = g_1 q_1[n] + g_2 q_2[n]$ for $n \geq 0$, specify for each of the following expressions whether or not it can be the output $y[n]$ for $n \geq 1$, for some choice of g_1, g_2 and $\mathbf{q}[0]$ (and explain your answer):

(i) $6\,(1.2)^n + (-0.4)^n$

(ii) $6\,(0.2)^n + (-0.4)^n$

(iii) $4\,(0.5)^n + 2\,(0.8)^n$

(iv) $4\,(1.2)^n + 2\,(0.8)^n$

(v) $7 + 4\,(0.5)^n + 2\,(-0.4)^n$

(vi) $8 + 2\,(-0.4)^n$

(vii) $3\,(-0.4)^n$

(Use this page and the next two)

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 8 (winding down)

The closed loop system is now undriven by any external input, so we only see the ZIR, with the new modes induced by state feedback. However, the mode at -0.4 is unreachable, and hence cannot be moved by state feedback.

(i) Fine, reachable mode moved from 0.5 to 1.2

(ii) Fine, reachable mode moved from 0.5 to 0.2

(iii) Not possible, $(-0.4)^n$ needs to be one of the terms, if both are excited

(iv) Not possible, $(-0.4)^n$ needs to be one of the terms, if both are excited

(v) Not possible, an undriven second order system can only have 2 modes

(vi) Fine, reachable mode is moved to 1

(vii) Fine, reachable mode is either is either moved to 0 by state fedback or it is unexcited

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 8 (conclusion)

(Continue to last problem on next side $\Longrightarrow\Longrightarrow$)

Problem 9 (20 points)

Suppose you are trying to decide with *minimum probability of error* between two hypotheses, H_0 and H_1, given measurements of L random variables $X_1, X_2, \cdots, X_L$. Under H_0, *all* of the L measurements are governed by the first set of relations below, while under H_1 *all* of them are governed by the second set of relations:

$$\begin{aligned} H_0 &: \quad X_i = W_i\,, & i = 1, 2, \cdots, L \\ H_1 &: \quad X_i = S_i + W_i\,, & i = 1, 2, \cdots, L \end{aligned}$$

All the W_i and S_i are zero-mean Gaussian random variables, and all are mutually independent, but the W_i all have the same variance σ_W^2, while the S_i all have the same variance σ_S^2. Denote the prior probabilities of H_0 and H_1 by p_0 and p_1 $(= 1 - p_0)$ respectively.

(a) Under hypothesis H_1, is X_i Gaussian (where i is any fixed integer between 1 and L)? Justify your answer.

YES, the sum of any two INDEPENDENT Gaussian random variables is also a Gaussian random variable (this can be proven by transform methods, for more information see 6.341)

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 9 (continued)

(b) Under hypothesis H_1, determine the correlation coefficient $\rho_{S_i X_i}$ of S_i and X_i, and the linear minimum-mean-square error (LMMSE) estimator for S_i, given X_i — denote this LMMSE estimator by $\widehat{S}_i(X_i)$. Also determine the associated minimum mean square error (MMSE). Can your LMMSE estimator of S_i under hypothesis H_1 be improved by allowing it to use measurements of not just X_i but also of X_ℓ for $\ell \neq i$?

$$\mu_x = 0$$
$$E(S_iX_i) = E(S_i(S_i + W_i)) = E(S_i^2 + S_iW_i) = E(S_i^2) + E(S_iW_i) = \sigma_s^2$$
$$E(X_i^2) = E((S_i + W_i)(S_i + W_i)) = E(S_i^2) + E(W_i^2) = \sigma_s^2 + \sigma_w^2$$
$$\rho_{S_iX_i} = \frac{\sigma_{S_iX_i}}{\sigma_{S_i}\sigma_{X_i}} = \frac{\sigma_2^2}{\sigma_s\sqrt{\sigma_s^2 + \sigma_w^2}}$$

$\rho_{S_iX_i} = \frac{\sigma_2^2}{\sigma_s\sqrt{\sigma_s^2+\sigma_w^2}}$

$\widehat{S}_i(X_i) = \mu_x + \frac{\rho_{S_iX_i}\sigma_s}{\sigma_x}(X_i) = \frac{\sigma_s^2}{\sigma_s^2+\sigma_w^2}X_i$

MMSE $= \sigma_s^2\left(1 - \frac{\sigma_s^2}{\sigma_s^2+\sigma_w^2}\right) = \frac{\sigma_s^2\sigma_w^2}{\sigma_s^2+\sigma_w^2}$

Can your LMMSE estimator of S_i under hypothesis H_1 be improved by also allowing measurements of X_ℓ for $\ell \neq i$? Explain.

NO - Since all of the measurements are independent of S_i and W_i and hence of each other, they will yield no additional information.

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 9 (continued further)

(c) Suppose we only had a single measurement, say $X_1 = x_1$. What is the optimal decision rule?

$$p_0 f_{X_1|H}(x_1|H_0) \underset{\text{`}H_1\text{'}}{\overset{\text{`}H_0\text{'}}{\gtrless}} p_1 f_{X_1|H}(x_1|H_1)$$

$$p_0 \frac{1}{\sqrt{2\pi\sigma_w^2}} e^{-\frac{x_1^2}{2\sigma_w^2}} \underset{\text{`}H_1\text{'}}{\overset{\text{`}H_0\text{'}}{\gtrless}} p_1 \frac{1}{\sqrt{2\pi(\sigma_w^2+\sigma_s^2)}} e^{-\frac{x_1^2}{2(\sigma_w^2+\sigma_s^2)}}$$

Take logs:

$$\ln\left(\frac{p_0}{p_1}\right) + \frac{1}{2}\ln\left(\frac{\sigma_w^2+\sigma_s^2}{\sigma_w^2}\right) \underset{\text{`}H_1\text{'}}{\overset{\text{`}H_0\text{'}}{\gtrless}} \frac{x_1^2}{2}\left(\frac{1}{\sigma_w^2} - \frac{1}{\sigma_s^2+\sigma_w^2}\right)$$

Rewrite as:

$$x_1\hat{S}_1(x_1) = \frac{\sigma_s^2}{\sigma_s^2+\sigma_w^2} x_i^2 \underset{\text{`}H_0\text{'}}{\overset{\text{`}H_1\text{'}}{\gtrless}} \sigma_w^2\left(2\ln\left(\frac{p_0}{p_1}\right) + \ln\left(\frac{\sigma_s^2+\sigma_w^2}{\sigma_w^2}\right)\right) = \gamma$$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 9 (continued still further)

(c) Optimal decision rule:

Rewrite the rule in a form that involves comparing the product $x_1 \widehat{S}_1(x_1)$ to a fixed threshold (this is trivial!):

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 9 (winding down)

(d) What is the optimal decision rule in the case where we have measurements of all L random variables, $X_1 = x_1, X_2 = x_2, \cdots, X_L = x_L$?

$$p_0 f_{\mathbb{X}|H}(\mathbf{x}|H_0) \underset{\text{`}H_1\text{'}}{\overset{\text{`}H_0\text{'}}{\gtrless}} p_1 f_{\mathbb{X}|H}(\mathbf{x}|H_1)$$

Since we assumed the measurements are all independent, the joint densities above will factor to the product of L individual gaussian PDFs. So, $\sum x_i^2$ will replace x_1^2 from part c, and $\left(\frac{\sigma_w^2+\sigma_s^2}{\sigma_w^2}\right)^L$ will replace $\frac{\sigma_w^2+\sigma_s^2}{\sigma_w^2}$ from the threshold expression in part c. We thus get the following expression:

$$\sum_{i=1}^{L} x_i \hat{S}_i(x_i) = \underset{\text{`}H_0\text{'}}{\overset{\text{`}H_1\text{'}}{\gtrless}} \sigma_w^2 \left(2\ln\left(\frac{p_0}{p_1}\right) + L\ln\left(\frac{\sigma_s^2+\sigma_w^2}{\sigma_w^2}\right)\right) = \gamma$$

(Continue this problem on next side $\Longrightarrow\Longrightarrow$)

Problem 9 (conclusion)

(d) Optimal decision rule:

Rewrite the rule in a form that involves comparing $\sum x_i \widehat{S}_i(x_i)$ with a fixed threshold. (Again, this rewriting is trivial.)