14.471/Fall 2012: PS 1: Solutions*

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Question 1

Assume that a consumer's utility function is given by $u(x_1, x_2, x_3) = \beta_1 \log(x_1 - \alpha_1) + \beta_2 \log(x_2 - \alpha_2) + \beta_3 \log(x_3)$ and that the consumer faces consumer prices q_1 and q_2, with the price of good three normalized to unity. The consumer's endowment y is measured in units of good three.

(a) Indirect utility function and expenditure function? We will solve the consumer's problem, find Marshallian demands, plug those into the utility function, find the indirect utility function to finally get the expenditure function.

The Lagrangian corresponding to the consumer's constrained utility maximization problem is given by:

$$L = \beta_1 \log(x_1 - \alpha_1) + \beta_2 \log(x_2 - \alpha_2) + \beta_3 \log(x_3) + \mu\left(y - (q_1 x_1 + q_2 x_2 + x_3)\right)$$

The FOC's are respectively:

$$\frac{\beta_1}{x_1 - \alpha_1} = \mu q_1$$

$$\frac{\beta_2}{x_2 - \alpha_2} = \mu q_2$$

$$\frac{\beta_3}{x_3} = \mu$$

$$y - (q_1 x_1 + q_2 x_2 + x_3) = 0$$

Let us solve for x_1. The multiplier is:

$$\mu = \frac{\beta_1}{q_1(x_1 - \alpha_1)}$$

Let us eliminate $x_2 q_2$ from the B.C. using the 2nd FOC from above:

$$x_2 = \alpha_2 + \frac{\beta_2}{\mu q_2} = \alpha_2 + \frac{\beta_2 q_1 (x_1 - \alpha_1)}{q_2 \beta_1}$$

*Thanks to Greg Leierson since proposed solutions build further on his solutions.

Therefore, we have:

$$q_2x_2 = q_2\alpha_2 + \frac{\beta_2}{\mu} = q_2\alpha_2 + \frac{\beta_2 q_1(x_1 - \alpha_1)}{\beta_1}$$

Similarly we can eliminate x_3 from the B.C.:

$$x_3 = \frac{\beta_3}{\mu} = \frac{\beta_3 q_1(x_1 - \alpha_1)}{\beta_1}$$

The B.C. is thus:

$$y = q_1x_1 + q_2\alpha_2 + \frac{\beta_2 q_1(x_1 - \alpha_1)}{\beta_1} + \frac{\beta_3 q_1(x_1 - \alpha_1)}{\beta_1} = x_1\left(q_1 + \frac{\beta_2 q_1}{\beta_1} + \frac{\beta_3 q_1}{\beta_1}\right) + q_2\alpha_2 - \alpha_1\left(\frac{\beta_2 q_1}{\beta_1} + \frac{\beta_3 q_1}{\beta_1}\right)$$

Using now that $\sum \beta_i = 1$, we get:

$$y = \frac{q_1x_1}{\beta_1} + q_2\alpha_2 - \alpha_1 q_1 \frac{1-\beta_1}{\beta_1}$$

Thus:

$$x_1 = \beta_1 \frac{y}{q_1} - \frac{q_2}{q_1}\alpha_2\beta_1 + \alpha_1(1-\beta_1) = \alpha_1 + \frac{\beta_1}{q_1}(y - q_1\alpha_1 - q_2\alpha_2)$$

Generalizing this formula also gives us the Marshallian demands x_2:

$$x_2 = \alpha_2 + \frac{\beta_2}{q_2}(y - q_1\alpha_1 - q_2\alpha_2)$$

Let us plug in the Marshallian demands into the utility function to get the indirect utility function:

$$v(q,y) = \beta_1 \log(\frac{\beta_1}{q_1}(y - q_1\alpha_1 - q_2\alpha_2)) + \beta_2 \log(\frac{\beta_2}{q_2}(y - q_1\alpha_1 - q_2\alpha_2)) + \beta_3 \log(\frac{\beta_3}{q_3}(y - q_1\alpha_1 - q_2\alpha_2))$$

Using that $\log(xy) = \log x + \log y$, we get the indirect utility function:

$$v(q,y) = \log(y - q_1\alpha_1 - q_2\alpha_2) - \beta_1 \log q_1 - \beta_2 \log q_2 + \sum \beta_i \log \beta_i$$

Replacing $y = e(q,u)$ in the indirect utility function allows us to find the expenditure function:

$$u = \log(e(q,u) - q_1\alpha_1 - q_2\alpha_2) - \beta_1 \log q_1 - \beta_2 \log q_2 + \sum \beta_i \log \beta_i$$

Solving for the expenditure function gives:

$$e(q,u) = \exp\left(u + \beta_1 \log q_1 + \beta_2 \log q_2 - \sum \beta_i \log \beta_i\right) + q_1\alpha_1 + q_2\alpha_2 = q_1\alpha_1 + q_2\alpha_2 + e^u q_1^{\beta_1} q_2^{\beta_2} \beta_1^{-\beta_1} \beta_2^{-\beta_2} \beta_3^{-\beta_3}$$

(b) CV: The compensating variation equals the change in the expenditures assessed at initial utility level:

$$CV = e(q', u) - e(q, u) = (q_1' - q_1)\,\alpha_1 + (q_2' - q_2)\,\alpha_2 + e^u \beta_1^{-\beta_1} \beta_2^{-\beta_2} \beta_3^{-\beta_3} \left(q_1'^{\beta_1} q_2'^{\beta_2} - q_1^{\beta_1} q_2^{\beta_2} \right) \tag{1}$$

This expression only depends on known parameters except for e^u which can be expressed from the expenditure function as a function of income $y = e(q, u)$:

$$q_1^{-\beta_1} q_2^{-\beta_2} \beta_1^{+\beta_1} \beta_2^{+\beta_2} \beta_3^{+\beta_3} \left(y - q_1\alpha_1 - q_2\alpha_2\right) = e^u$$

Combining the last 2 expressions gives:

$$CV = (q_1' - q_1)\,\alpha_1 + (q_2' - q_2)\,\alpha_2 + q_1^{-\beta_1} q_2^{-\beta_2} \beta_1^{+\beta_1} \beta_2^{+\beta_2} \beta_3^{+\beta_3} \left(y - q_1\alpha_1 - q_2\alpha_2\right) \beta_1^{-\beta_1} \beta_2^{-\beta_2} \beta_3^{-\beta_3} \left(q1^{\beta_1} q2^{\beta_2} - q_1^{\beta_1} q_2^{\beta_2} \right)$$

This simplifies into:

$$CV = (q_1' - q_1)\,\alpha_1 + (q_2' - q_2)\,\alpha_2 + \left(y - q_1\alpha_1 - q_2\alpha_2\right) \left(\left(\frac{q_1'}{q_1}\right)^{\beta_1} \left(\frac{q_2'}{q_2}\right)^{\beta_2} - 1 \right)$$

Let us plug in the prices and income, to get $CV(q, q', y) = 0.5 + 0.25 + (4)\left((2)^{0.4}(1.5)^{0.4} - 1\right) = 0.75 + 4 * (3^{\frac{4}{10}} - 1) = 2.9574$

The EV would be given by the same formula as (1) but with u' rather than u. Since prices rise, utility drops $u' < u$ and hence the EV is smaller than the CV.

Without referencing to the formula, we could use that:

1. CV is the area to the left of the Hickian demand curve related to old utility/EV is the area to the left of the Hickian demand curve related to new utility
2. Demanded quantities increase in utility (normality)
3. Utility drops

to conclude that the CV is bigger than the EV. To find the efficiency cost, note that the compensated revenue equals

From Shephard's lemma we know that Hicksian demand equals the price derivative of the expenditure function:

$$h_1(q, u) = \alpha_1 + e^u \left(\frac{q_1}{\beta_1}\right)^{\beta_1 - 1} \left(\frac{q_2}{\beta_2}\right)^{\beta_2} \left(\frac{1}{\beta_3}\right)^{\beta_3} = 1.73$$

$$h_2(q, u) = \alpha_2 + e^u \left(\frac{q_1}{\beta_1}\right)^{\beta_1} \left(\frac{q_2}{\beta_2}\right)^{\beta_2 - 1} \left(\frac{1}{\beta_3}\right)^{\beta_3} = 2.135$$

The compensated revenue equals the tax times the expenditures is thus equal to $R(q^1, u_0) = 1.73 + 0.5 * 2.135 = 2.81$ and since CV=2.96 we find an efficiency cost of 0.13826.

Question 2

Assume that cross-sectional data on household demand for gasoline g_i, with households located in different states and facing different gasoline tax burdens, suggests a demand curve of the form:

$$\ln g_i = -1.5 - 0.4 \ln q_i + 0.8 \ln y_i$$

where q_i denotes the tax-inclusive price of gasoline facing household i and y_i denotes household i 's income in dollars. Assume that the producer price of gasoline is \$3.00.

(a) Do estimates make sense? They make sense: a 1% increase in gas prices decreases gas consumtion by 0.4% and a 1% increase inincome increases gas consumption by 0.8%. To infer the units, the relatively large price elasticity suggest we are in the medium to long run. Let us plug in some reasonable numbers for the consume price ($q_i = 5\$$) and income ($y_i = 50,000\$/year$) to get $g_i = 673$ per year. This suggests the units are gallons per year.

(b) Harberger DWL from specific tax? Demand drops from $g_0 = 825.8$ to $g_1 = 736.1$. Hence the simple triangular DWL is:

$$DWL = \frac{1}{2}(825.8 - 736.1) = 44.87$$

Note, we could also use $DWL = \eta_D \frac{g_1 \tau^2}{p_1 2}$ which gives 49.07

(c) Exact welfare loss? Demand has a constant elasticity. Thus we follow Hausman (1981). He solves the PDE implied by Roy's identity to find the indirect utility function (see equation (21)):

$$v(p_1, y) = -e^{zy} \frac{p_1^1 + \alpha}{1+\alpha} + \frac{y^{1-\delta}}{1-\delta}$$

where $\alpha = -0.4$ and $\delta = 0.8$.

Inverting the indirect utility function gives the expenditure function (equation (22)):

$$e(p_1, \bar{u}) = \left[(1-\delta) \left(\bar{u} + e^{zy} \frac{p_1^{1+\alpha}}{1+\alpha} \right) \right]^{\frac{1}{1-\delta}}$$

Utility at the old prices is $v(3, 50000) = 29.09$. Hence we can compute CV as

$$CV = e(4, 29.09) - e(3, 29.09) = 783$$

To compute compensated revenue (as in Diamond and McFadden), we get the Hicskian demand:

$$h(4, 29.09) = x(4, e(4, 29.09)) = 745$$

This, the DWL thus equals:

$$DWL = CV - R(4, 29.09) = CV - 1 * h(4, 29.09) = 783 - 745 = 38$$

(d) Standard deviation? Given the point estimates of the demand parameters, The DWL estimate is given by $CV(\alpha, \delta, \gamma) - R(\alpha, \delta, \gamma) = f(\alpha, \delta, \gamma)$. With the covariance matrix and for instance the delta-method, we can now estimate the joint probability distribution/pdf of $f(\alpha, \delta, \gamma)$. With the pdf, we then simulate standard errors of DWL.

Question 3

Consider a consumer endowed with an exogenous income y. If there is an initial price vector p^0 and a post-reform price vector p^1, we know that both the compensating variation $CV\left(p^0, p^1, y\right)$ and the equivalent variation $EV\left(p^0, p^1, y\right)$ provide a meaningful welfare ranking of p^0 and p^1. Suppose now, however, that the status quo p^0 is being compared with two possible price vectors p^1 and p^2. For instance, the government considers which goods to tax, starting from a no tax situation.

(a) Show that the consumer is better off under p^1 than under p^2 if and only if $EV\left(p^0, p^1, y\right) < EV\left(p^0, p^2, y\right)$. Thus, the measures $EV\left(p^0, p^1, y\right)$ and $EV\left(p^0, p^2, y\right)$ can be used not only to compare these two price vectors with p^0, but also to determine which of them is better for the consumer.

We want to compare $u^1 \equiv v\left(p^1, y\right)$ and $u^2 \equiv v\left(p^2, y\right)$. Using the equivalent variation, we have

$$\begin{aligned} EV\left(p^0, p^1, y\right) - EV\left(p^0, p^2, y\right) &= \left(e\left(p^1, u^1\right) - e\left(p^0, u^1\right)\right) - \left(e\left(p^2, u^2\right) - e\left(p^0, u^2\right)\right) \\ &= y - e\left(p^0, u^1\right) - y + e\left(p^0, u^2\right) \\ &= e\left(p^0, u^2\right) - e\left(p^0, u^1\right). \end{aligned}$$

Since the expenditure function is increasing in utility, a comparison of equivalent variations will provide the correct welfare ranking.

(b) By constructing an example, show that a comparison of $CV\left(p^0, p^1, y\right)$ and $CV\left(p^0, p^2, y\right)$ does not necessarily rank the consumer's welfare under p^1 and p^2 correctly. Interpret this result.

The same algebra as above with compensating variations yields

$$\begin{aligned} CV\left(p^0, p^1, y\right) - CV\left(p^0, p^2, y\right) &= \left(e\left(p^1, u^0\right) - e\left(p^0, u^0\right)\right) - \left(e\left(p^2, u^0\right) - e\left(p^0, u^0\right)\right) \\ &= e\left(p^1, u^0\right) - e\left(p^2, u^0\right). \end{aligned}$$

Observe that we are now comparing the cost of the same utility bundle at different price vectors, rather than different utility levels at the same price. In general this will not provide a valid welfare ranking. Consider the preferences from question 1 and the price vectors $p^1 = (1.0, 1.9)$ and $p^2 = (0.9, 2.0)$. Then $CV\left(p^0, p^1, y\right) = 2.082 > 2.063 = CV\left(p^0, p^2, y\right)$ but $u^1 = -1.1719 > -1.1947 = u^2$. This will be a possibility when the consumer's willingness to substitute between goods differs across utility levels (i.e. preferences are not homothetic).

(c) Assume a linear production technology so that the producer price vector p is fixed. In this setting, the resource cost of providing the consumer with a welfare level u_0 is equal to $e_0 = \sum_i p_i h_i\left(p, u_0\right)$, where h_i denotes the consumer's compensated demand function for good i. If we distort consumer prices to a new price vector q (for instance by imposing a tax vector $\tau = q - p$), the resource cost necessary to provide the consumer with utility u_0 becomes $e_1 = \sum_i p_i h_i\left(q, u_0\right)$. The deadweight loss of moving from consumer prices p to consumer prices q is given by $e_1 - e_0 = CV\left(p, q, u_0\right) - \sum_i \left(q_i - p_i\right) h_i\left(q, u_0\right)$. This simply states that the deadweight loss can be interpreted as the additional resource cost that is needed to provide utility u_0 to the consumer under the distorted consumer price vector q relative to the price vector p. How would you extend the analysis to incorporate a population of consumers with heterogeneous preferences?

Suppose there are J consumers indexed by $j = 1, \ldots, J$ with different preferences and income levels so that, under the undistorted producer price vector p, we have $u_0^j \equiv v^j\left(p, u^j\right)$. Let $h^j\left(p, u_0^j\right)$ denote the vector of compensated demands for consumer j given prices p and the initial utility level u_0^j. Then the efficiency cost of moving from price vector p to another price vector q, for instance by imposing a tax $\tau = q - p$, is given by the additional resource cost necessary to provide utility u_0^j to each consumer j under the distorted consumer price vector q :

$$\begin{aligned}
DWL &= \sum_j p \cdot h^j \left(q, u_0^j\right) - \sum_j p \cdot h^j \left(p, u_0^j\right) \\
&= \sum_j p \cdot h^j \left(q, u_0^j\right) - \sum_j q \cdot h^j \left(q, u_0^j\right) + \sum_j q \cdot h^j \left(q, u_0^j\right) - \sum_j p \cdot h^j \left(p, u_0^j\right) \\
&= -\sum_j (q-p) \cdot h^j \left(q, u_0^j\right) + \sum_j \left[q \cdot h^j \left(q, u_0^j\right) - p \cdot h^j \left(p, u_0^j\right)\right] \\
&= \sum_j CV \left(p, q, u_0^j\right) - (q-p) \cdot \sum_j h^j \left(q, u_0^j\right).
\end{aligned}$$

After manipulating the expressions, we see that this yields the usual compensating variation formula for the deadweight loss summed across consumers.

(d) Continuing with the linear technology case, suppose that the government does not impose taxes, but instead imposes quantity restrictions, as in the quotas and rations imposed during war time. How would you compute the deadweight loss? How would heterogeneous consumers affect your analysis? Consider separately the cases where they can trade their quotas or rations and a situation where they cannot.

We start with heterogeneous consumers, as homogeneous consumers just represent a special case. Suppose the government imposes a consumption vector x^j for each individual j (i.e. all quotas are binding). The resource cost of this allocation is given by $\sum_j p \cdot x^j$. Let $u_1^j = u^j \left(x^j\right)$ denote the utility that individual j derives from consuming her assigned bundle x^j. If we let consumers face producer prices p and choose their consumption optimally, the resource cost of providing the same utility level u_1^j to each individual j is $\sum_j p \cdot h^j \left(p, u_1^j\right)$. Hence the deadweight loss associated with the quotas can be written as

$$DWL = \sum_j p \cdot x^j - \sum_j p \cdot h^j \left(p, u_1^j\right).$$

We can further transform this expression if we find the price vectors q^j that leads consumer j to choose the bundle $x^j : h^j \left(q^j, u_1^j\right) = x^j$. Then

$$\begin{aligned}
DWL &= \sum_j p \cdot h^j \left(q^j, u_1^j\right) - \sum_j p \cdot h^j \left(p, u_1^j\right) \\
&= \sum_h EV^j \left(q^j, p, u_1^j\right) - (q-p) \cdot \sum h^j \left(q^j, u_1^j\right).
\end{aligned}$$

This shows that there are now two kinds of distortions: first, consumers do not face the producer price vector p and therefore their demand is distorted as before. However, there is now an additional distortion from the fact that different consumer prices q^j are required to make consumer j consume the assigned bundle x^j. Hence, not only is there a wedge between consumers' marginal rate of substitution and the marginal rate of transformation, but also the marginal rates of substitution are not equated across consumers. Clearly, the latter distortion vanishes when consumers can trade their rations. In that case, there will be one (market clearing) consumer price vector q (not necessarily equal to p), and the efficiency cost reduces to

$$DWL = \sum_j EV^j \left(p, q, u_1^j\right) - (q-p) \cdot \sum_j h^j \left(q, u_1^j\right).$$

Similarly, if consumers are homogeneous, there is one price q^j which will induce all consumers to consume their rations and therefore there are no distortions due to a poor allocation across consumers. Intuitively, all consumers would like to change their ration in the same way, so there is no scope for trade.

Question 4

Consider a world economy consisting of N identical countries, each endowed with one unit of land. The world contains one unit of capital, which is freely mobile between countries. Land, in contrast, is immobile. All countries have identical production technologies, $Y = K^{.25}L^{.75}$ where K and L denote capital and land, respectively. The output price is normalized to unity.

(a) Find the equilibrium interest rate and total income received by capitalists and landowners when none of the jurisdictions tax either capital or land. Evaluate these quantities for $N = 2$ and $N = 100$. Note that the $N = 100$ economy has more total land than the $N = 2$ economy.

Firm profit maximization requires that the interest rate equal the marginal product of capital. Income maximization on the part of the capitalists requires that the interest rate be equal across all jurisdictions. Therefore, since the marginal product of capital is a function solely of the amount of capital in each country, the capital will spread evenly across countries such that each country i has $1/N$ of the world's capital. The interest rate will be

$$r = F_K = \frac{1}{4}K^{-\frac{3}{4}}L^{\frac{3}{4}} = \frac{1}{4}\left(\frac{1}{N}\right)^{-\frac{3}{4}} = \frac{1}{4}N^{\frac{3}{4}}$$

and the cost of land

$$w = F_L = \frac{3}{4}K^{\frac{1}{4}}L^{-\frac{1}{4}} = \frac{3}{4}N^{-\frac{1}{4}}.$$

The income of the capitalists will be

$$\sum rK_i = r = \frac{1}{4}N^{\frac{3}{4}}.$$

and the income of the landowners

$$\sum wL_i = wN = \frac{3}{4}N^{\frac{3}{4}}.$$

When $N = 2$, $r = 0.4204$, capitalist income is 0.4204, $w = 0.6307$, and landowner income is 1.2613. When $N = 100$, $r = 7.9057$, capitalist income is 7.9057, $w = 0.2372$, and landowner income is 23.7171. The marginal product of capital increases as land increases (since the stock of capital is fixed). Total landowner income increases, but income per unit of land decreases since there is less capital in each country.

(b) Now consider the impact of a tax at rate θ on capital income in country 1. Assume that revenues are used to purchase good Y, and that the government's purchases do not affect the production technology used in country 1. The after-tax rate of return to capital invested in country 1 is now $(1-\theta)F_K$. Find new expressions for the after tax interest rate, total landowner and capitalist income, and government revenue in country 1 as functions of N and θ.

As before firm profit maximization and capitalist income maximization require the existence of a single interest rate that equates the after-tax marginal product of capital across all countries. As before, capital will be spread evenly across all countries without a tax. Denote by K_1 the capital in country 1. Then capital in country $i \neq 1$ must be $\frac{1-K_1}{N-1}$. The capital level must satisfy

$$(1-\theta)F_K(K_1) = F_K\left(\frac{1-K_1}{N-1}\right).$$

Solving for K_1 :

$$K_1 = \frac{(1-\theta)^{\frac{4}{3}}}{(N-1)+(1-\theta)^{\frac{4}{3}}}$$

which implies

$$K_{i\neq 1} = \frac{1}{(N-1)+(1-\theta)^{\frac{4}{3}}}$$

The interest rate is then given by the after-tax marginal product of capital in any of the N countries

$$r = \frac{1}{4}\left((N-1)+(1-\theta)^{\frac{4}{3}}\right)^{\frac{3}{4}}.$$

Capitalist income in country 1 is

$$rK_1 = \frac{1}{4}\frac{(1-\theta)^{\frac{4}{3}}}{\left((N-1)+(1-\theta)^{\frac{4}{3}}\right)^{\frac{1}{4}}},$$

landowner income in country 1 is

$$w_1 L_1 = \frac{3}{4}\frac{(1-\theta)^{\frac{1}{3}}}{\left((N-1)+(1-\theta)^{\frac{4}{3}}\right)^{\frac{1}{4}}},$$

and government revenue is

$$\theta F_{K_1} K_1 = \frac{\theta r K_1}{1-\theta} = \frac{1}{4}\frac{\theta\,(1-\theta)^{\frac{1}{3}}}{\left((N-1)+(1-\theta)^{\frac{4}{3}}\right)^{\frac{1}{4}}}.$$

(c) For $\theta = 0.25$, find the change in total capital and total land income, and the revenue raised in country 1 if $N = 2$ and $N = 100$. What happens to the pretax marginal product of capital in the countries without taxes? How do landowners in country 1 fare as a result of the tax? What do these examples suggest about the usefulness of the "small open economy" assumption that world interest rates are fixed, so capital taxes are shifted to land?

Assuming $\theta = 0.25$, we obtain the values in the first table below.

	r	w_1	$w_{j\neq 1}$	K_1	$K_{j\neq 1}$	rK_1	$rK_{j\neq 1}$	$r\sum K_j$	$w\sum L_j$	$\theta F_{K_1} K_1$
$N=2$	0.369	0.598	0.659	0.4050	0.595	0.1490	0.219	0.369	1.257	0.0499
$N=100$	7.890	0.216	0.237	0.0068	0.010	0.0539	0.791	7.890	23.679	0.0179

Assuming instead $\theta = 0$, we obtain the values in the second table.

	r	w_1	$w_{j\neq 1}$	K_1	$K_{j\neq 1}$	rK_1	$rK_{j\neq 1}$	$r\sum K_j$	$w\sum L_j$	$\theta F_{K_1} K_1$
$N=2$	0.420	0.631	0.631	0.5000	0.500	0.2100	0.210	0.420	1.261	0.0000
$N=100$	7.910	0.237	0.237	0.0100	0.010	0.0539	0.791	7.910	23.717	0.0000

In the $N = 2$ case the pre-tax marginal product of capital falls in countries without taxes, but in the $N = 100$ case it is essentially unaffected. Landowners in country 1 suffer in both cases, but suffer relatively more in the $N = 100$ case.

The small open economy conclusion that world interest rates are fixed so capital taxes are shifted to land depends on the quality of the smallness assumption. In the $N = 100$ case this is what happens, but in the $N = 2$ case the burden is shared (the smallness assumption is invalid).

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