14.451 Recitation Notes:
1. Brief Review of Dixit Stiglitz Model
2. Endogenous Technology Models with Expanding Varieties

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Dixit-Stiglitz Framework

- Based on, Avinash Dixit and Joseph Stiglitz, "Monopolistic Competition and Optimal Product Diversity", AER 1977.
- Goal: Develop an equilibrium framework that features
  1. Love for variety (i.e. more variety of goods preferred by the consumer, or more variety of intermediate goods increase productivity in final good sector).
     - Relatedly, aggregate demand externalities (Not used in this course, but used in macro literature and will see in later courses).
  2. Monopolistic competition. Idea goes back to Chamberlain (1933).

Dixit-Stiglitz used to study optimal product diversity in market (where goods are close substitutes within the market, but may or may not be substitutes for the rest of the goods in the economy). Macroeconomists use since it provides a tractable framework that features 1-2.

Consumer Problem

- Consumers with given fixed income m choose consumption between bundle $[c_i]_{=0}^{N}$ in a particular market and other good y (simplification for all other goods in the economy). Other good y provided at fixed price (normalize $p^y = 1$).
- Consumers solve

$$\max_{[c_i]_{=0}^{N}, y} U\left([c_i]_{=0}^{N}, y\right) \equiv u(C, y)$$

$$\text{s.t.} \int_0^N p_i c_i di + y \leq m$$

$$\text{where } C \equiv \left(\int_0^N c_i^{\frac{\epsilon-1}{\epsilon}} di\right)^{\epsilon/(\epsilon-1)}.$$

Consumer's Sub-problem and Demand for Each Good

Consider the cost minimization problem

$$p(C) = \min_{[c_i]_{=0}^N} \int_0^N p_i c_i di$$

$$\text{s.t. } \left(\int_0^N c_i^{\frac{\epsilon-1}{\epsilon}} di \right)^{\epsilon/(\epsilon-1)} \geq C$$

FOCs for this problem are

$$p_i = \lambda c_i^{-1/\epsilon} C^{1/\epsilon}, \text{ except for a measure zero of } i. \tag{1}$$

Take to the power of $1-\epsilon$ and integrate over all i to get

$$\int_0^N p_i^{1-\epsilon} di = \lambda^{1-\epsilon} C^{(\epsilon-1)/\epsilon} C^{(1-\epsilon)/\epsilon} = \lambda^{1-\epsilon},$$

which implies

$$\lambda = \left(\int_0^N p_i^{1-\epsilon} di \right)^{1/(1-\epsilon)} \equiv P,$$

where the last line also defines the **ideal price index**. Plug in Eq. (1) to get

$$c_i = C \left(\frac{p_i}{P} \right)^{-\epsilon}. \tag{2}$$

The solution to cost minimization problem is $p(C) = PC$, hence P is the unit cost of production for the aggregated C.

Consumer Demand for the Aggregated Good

- Consumer's choice of y is now simplified to

$$\max U(C, y)$$
$$\text{s.t. } PC + y \leq m.$$

This gives FOC

$$\frac{\partial u(C, y)/\partial y}{\partial u(C, y)/\partial C} = \frac{1}{P}$$

which can be represented as

$$y = g(P, m),$$

for some demand function g. Now from budget constraint

$$C = \frac{m - g(P, m)}{P}. \tag{3}$$

- The bottom line is, for a given $\{p_i\}$ (and hence P), we can solve for the demand for y and C (and also for each c_i).

Monopolist's Pricing Decision

- Firm i holds patents to produce good i (is monopolist). It has fixed marginal cost ψ. Hence it solves

$$\max_{p_i} c_i (p_i) (p_i - \psi),$$

where c_i is given by the isoelastic demand Eq. (2). Optimal monopoly price is (learn this formula!)

$$p_i = p = \frac{\epsilon}{\epsilon - 1} \psi.$$

(Note that firm takes P as given, i.e. each firm is small so ignores its effect on the aggregate price index).

- Assume $\epsilon > 1$ for monopolistic competition (what happens if we don't assume this?)
- We then have

$$P = N^{-1/(\epsilon-1)} \frac{\epsilon}{\epsilon - 1} \psi,$$

Unit cost P is decreasing in N: love-for-variety effect.

- Recall that $c_i = (p_i/P)^{-\epsilon} C$. Then the profits are

$$\pi_i = \pi = N^{-\epsilon/(\epsilon-1)} C \frac{1}{\epsilon - 1} \psi \tag{4}$$

Aggregate Demand Externalities

- The level of C can now be calculated from Eq. (3) as

$$C = N^{1/(\epsilon-1)} \frac{\epsilon - 1}{\epsilon \psi} \left(m - g\left(N^{-1/(\epsilon-1)} \frac{\epsilon}{\epsilon - 1} \psi, m \right) \right) \tag{5}$$

  C will typically increase with N (when C is a normal good). This is due to the love-for-variety effect. Since consumers love variety, entry by a firm creates a positive spillover to other firms (through reducing P and increasing aggregate demand for C).

- Profits are

$$\pi = \frac{1}{\epsilon N} \left(m - g\left(N^{-1/(\epsilon-1)} \frac{\epsilon}{\epsilon - 1} \psi, m \right) \right).$$

  Profits might be increasing in N!

- Go back to Eq. (4) for the intuition. The N term represents the effect of more entrants decreasing profits (as expected). The C term represents the aggregate demand externality. The more goods there are, (typically) consumer consumes more C bundle than y good.
- This is a *pecuniary (price)* externality, since it works through P. Higher N leads to lower P leads to higher C.

- The number of varieties N in equilbrium are determined by free entry. Under free entry assumption, the profits in equilibrium must balance off the cost of entry

$$\pi = \frac{1}{\epsilon N}\left(m - g\left(N^{-1/(\epsilon-1)}\frac{\epsilon}{\epsilon-1}\psi, m\right)\right) = \mu,$$

  where μ is the cost of entry.
- So, there are profits from monopoly power in the static equilibrium, but there is no ex-ante profits. Static monopoly profits are just enough to meet entry costs. This is the essence of **monopolistic competition**.
- Will use this idea extensively in the endogenous technology models. There will be monopoly profits at any time, but there will be no dynamic profits (that is, profits will just meet the fixed entry or innovation costs the firm has incurred at some point in the past).
- (More for IO purposes): There might be over or under entry depending on ϵ and the form of g (intra and inter-elasticity of substitution) (and depending on how you define optimality).
  - Aggregate demand externality creates a force towards underentry.
  - Business stealing effect creates a force towards overentry (When one firm enters, reduces profits of other businesses, and does not take this into account).

Endogenous Technology Models

- Goal: Develop a framework in which growth is driven by technological change, and technology develops as a result of purposeful activities of individuals. Desired for a long time, accomplished by Romer (*JPE*, 1990). Led to the field of "New Growth Theory".
- The models in this tradition feature a more realistic growth process that responds to policy. Moroever, they allow to study issues related to intellectual property rights, investment in R&D, innovation etc. which we think are relevant to the growth process.

- Population constant at L, consumers solve

$$\max_{\{C(t)\}} \int_0^\infty \exp(-\rho t) \frac{C(t)^{1-\theta} - 1}{1-\theta} dt \tag{6}$$

$$\text{s.t. } \dot{A}(t) = r(t) A(t) + w(t) L - C(t), \tag{7}$$

$$\lim_{t\to\infty} A(t) \exp\left(-\int_0^t r(s)\, ds\right) \geq 0.$$

- Solution is characterized by the Euler equation

$$\frac{\dot{C}(t)}{C(t)} = \frac{1}{\theta}(r(t) - \rho) \tag{8}$$

and the budget constraint (7), along with the initial condition for $A(0)$ and the transversality condition

$$\lim_{t\to\infty} A(t) \exp\left(-\int_0^t r(s)\, ds\right) = 0. \tag{9}$$

Competitive Final Good Sector

- Final good production function

$$Y = \frac{1}{1-\beta} \tilde{X}(t)^{1-\beta} L^{\beta} \text{ where}$$

$$\tilde{X}(t) = \left(\int_0^{N(t)} x(\nu, t)^{(\epsilon_\beta - 1)/\epsilon_\beta} \right)^{\epsilon_\beta/(\epsilon_\beta - 1)} \quad \text{for } \epsilon_\beta = 1/\beta \text{ which implies}$$

$$Y = \frac{1}{1-\beta} \left[\int_0^{N(t)} x(\nu, t)^{1-\beta} \right] L^{\beta}. \tag{10}$$

- Price of final good normalized to 1. Final good firms solve

$$\max_{[x(\nu,t)]_t, L} \frac{1}{1-\beta} \left[\int_0^{N(t)} x(\nu, t)^{1-\beta} \, d\nu \right] L^{\beta} - \int_0^{N(t)} p(\nu, t) \, x(\nu, t) \, d\nu.$$

- Solving this problem shows that intermediate goods and labor are paid their marginal products:

$$p^x(\nu, t) = x(\nu, t)^{-\beta} L^{\beta} \tag{11}$$

$$w(t) = \beta \frac{Y}{L}. \tag{12}$$

- Each monopolist owns the blueprint to produce intermediate ν forever, hence maximizes

$$V(\nu,t) = \max_{[x(\nu,s),p(\nu,s)]_{s=t}^{\infty}} \int_t^{\infty} \exp\left(-\int_t^s r(t')\,dt'\right)\pi(\nu,s)\,ds \tag{13}$$

where

$$\pi(\nu,t) \equiv x(\nu,t)\left(p^x(\nu,t) - \psi\right)$$

- Here, $V(\nu,t)$ is the value function of the monopolist. Problem is separable, hence maximize $\pi(\nu,t)$ pointwise. By Eq. (11), the demand function is $x(\nu,t) = p(\nu,t)^{-1/\beta} L$, which is isoelastic. The monopoly price and the profits are

$$p^x(\nu,t) = \frac{\psi}{1-\beta},\ \pi^x(\nu,t) = \left(\frac{1-\beta}{\psi}\right)^{(1-\beta)/\beta} \beta L$$

- We take $\psi \equiv 1-\beta$ (simplification) which gives

$$p^x(\nu,t) = 1,\ x(\nu,t) = L,\ \pi^x(\nu,t) = \beta L. \tag{14}$$

- Value function of a monopolist (solution to (13)) also satisfies the HJB equation

$$r(t)\,V(\nu,t) = \dot{V}(\nu,t) + \pi(\nu,t). \tag{15}$$

R&D Sector (Monopolistically Competitive)

- R&D technology (deterministic for simplicity)

$$\dot{N}(t) = \eta Z(t) \tag{16}$$

where $Z(t)$ is the total final good invested in R&D. Investing 1 unit generates a flow rate η of machines for which you become monopolist.

- Free entry implies

$$\eta V(\nu, t) \leq 1 \text{ with equality if } Z(t) > 0. \tag{17}$$

This is Chamberlain's and Dixit-Stiglitz's monopolistic competition idea: each firm ex-post makes monopoly profits, but these rents are just enough cover their entry costs. There are no dynamic monopoly rents.

Equilibrium Definition I

Equilibrium is a time path of

$\left[C(t), A(t), X(t), Z(t), N(t), \{p^x(\nu,t), x(\nu,t), V(\nu,t)\}_{\nu \in N(t)}, r(t), w(t)\right]_{t=0}^{\infty}$ such that

- Consumers maximize, i.e., $[C(t), A(t)]_t$ solves the Euler equation (8) and the budget constraint (7), along with the initial condition $A(0)$ and the transversality condition (9).
- Final good firms choose quantities to maximize (taking prices given), i.e., $p^x(\nu,t)$ and $w(t)$ satisfy Eqs. (11).
- Intermediate good monopolists set prices to maximize, i.e., $p^x(\nu,t)$,$x(\nu,t)$ satisfy Eq. (14).
- Investment in R&D, $Z(t)$, is determined by free entry into R&D sector, i.e., value function $V(\nu,t)$ satisfies Eq. (17).
- Evolution of $N(t)$ is determined by R&D technology equation Eq. (16).
- Asset markets clear (here, shares of monopolistic firms are the only supply of assets)

$$A(t) = \int_0^{N(t)} V(\nu,t)\, d\nu. \tag{18}$$

- Final good market clears

$$Y(t) = C(t) + X(t) + Z(t). \tag{19}$$

- Claim: In equilibrium, consumer's budget constraint (7) is equivalent to the resource constraint (19). This essentially follows from Walras law.
- To prove explicitly, first take the time derivative of Eq. (18) to get

$$\dot{A}(t) = \int_0^{N(t)} \dot{V}(\nu, t)\, d\nu + \dot{N}(t)\, V(N(t), t).$$

In the equilibria we consider, Eq. (17) is satisfied with equality. Thus, $V(\nu, t) = 1/\eta$. Using this in the previous equation,

$$\dot{A}(t) = \eta \dot{N}(t) = Z(t), \tag{20}$$

where the second line uses the R&D technology equation (16).
In words: consumers' saving is equal to the investment in innovation.
- Second note that

$$r(t) A(t) = \int_0^{N(t)} r(t) V(\nu, t)\, d\nu = \int_0^{N(t)} \left(\dot{V}(\nu, t) + \pi(\nu, t)\right) d\nu$$
$$= \int_0^{N(t)} \pi(\nu, t)\, d\nu, \tag{21}$$

where the second equality uses the HJB equation, and the last equality uses $V(\nu, t) = 1/\eta$.
In words: consumers' return on assets is equal to intermediate monopolists' profit payments within the period.

- Third, using Eq. (10), we can show:

$$Y(t) = X(t) + \int_0^{N(t)} \pi(\nu, t)\, d\nu + w(t)\, L(t). \tag{22}$$

In words: net output either goes to workers or to intermediate monopolists (as profits).

- Finally, using Eqs. (20), (21) and (22), we have:

$$\begin{aligned} \dot{A}(t) &= r(t)\, A(t) + w(t)\, L(t) - C(t) \\ => Z(t) &= \int_0^{N(t)} \pi(\nu, t)\, d\nu + w(t)\, L(t) - C(t) \text{ (using (20) and (21))} \\ &= Y(t) - X(t) - w(t)\, L(t) + w(t)\, L(t) - C(t) \text{ (using (22))}. \end{aligned}$$

- Hence, consumer's budget constraint is equivalent to the resource constraint, proving the claim.
- In view of this result, we can forget about asset level $A(t)$ (and consumer's budget constraint) and simplify the definition of equilibrium.

Equilibrium Definition II

Equilibrium is a time path of

$$\left[C(t), X(t), Z(t), N(t), \{p^x(\nu,t), x(\nu,t), V(\nu,t)\}_{\nu \in N(t)}, r(t), w(t)\right]_{t=0}^{\infty}$$ such that:

- Consumers maximize, i.e., $C(t)$ satisfies the Euler equation (8), and the transversality condition holds:

$$\lim_{t\to\infty} \exp\left(-\int_0^t r(s)\,ds\right)\int_0^{N(t)} V(\nu,t)\,d\nu = 0.$$

- Final good firms choose quantities to maximize (taking prices given), i.e., $p^x(\nu,t)$ and $w(t)$ satisfy Eq. (11).
- Intermediate good monopolists set prices to maximize, i.e., $p^x(\nu,t)$,$x(\nu,t)$ satisfy Eq. (14).
- Investment in R&D, $Z(t)$, is determined by free entry into R&D sector, i.e. value function $V(\nu,t)$ satisfies Eq. (17).
- Evolution of $N(t)$ is determined by R&D technology equation Eq. (16).
- Resource constraints are satisfied:

$$Y(t) = C(t) + X(t) + Z(t) \text{ for each } t.$$

- We consider equilbria in which $Z(t) > 0$ for all t.
- By Eq. (17),

$$V(\nu, t) = \frac{1}{\eta} \text{ for all } \nu \text{ and } t. \tag{23}$$

- Then, $\dot{V} = 0$. Recall also that $\pi(\nu, t) = \beta L$ is constant [cf. Eq. (14)]. Hence the HJB Eq. (15) implies

$$r(t) = r^* = \eta\beta L \text{ for all } t. \tag{24}$$

  hence the interest rate is constant.
- Consumer optimization implies

$$\frac{\dot{C}(t)}{C(t)} = g_c \equiv \frac{1}{\theta}(\eta\beta L - \rho), \tag{25}$$

  i.e., consumption grows at a constant rate.
- Since we have started by assuming $Z(t) > 0$, we need positive growth. Thus, we need to make the parametric assumption

$$\eta\beta L > \rho. \tag{26}$$

Equilibrium Characterization, Cont'd

- Plugging in $x(\nu, t) = L$ from Eq. (14) into final good production, we have

$$Y(t) = \frac{1}{1-\beta} N(t) L, \tag{27}$$

i.e., output features increasing returns to scale. Also, the expenditure on machines is given by

$$X(t) = N(t) L (1-\beta) \tag{28}$$

- Recall also that $Z(t) = \dot{N}(t)/\eta$. Using this and Eqs. (25), (27), (28) in final good market clearing condition (19), we have

$$\frac{1}{1-\beta} N(t) L = C(0) \exp\left(\frac{1}{\theta}(\eta\beta L - \rho) t\right) + N(t)(1-\beta) + \dot{N}(t)/\eta, \tag{29}$$

where $N(0)$ is given. First order linear ODE in $N(t)$. Can solve for any given $C(0)$.

- There is a unique value of $C(0)$ (that depends on $N(0)$ and the parameters) that makes $N(t)$ also grow at rate g_c. In fact, this $C(0)$ can be explicitly solved for after substituting $\dot{N}(0) = \eta N(0)$ in Eq. (29).

- With this level of $C(0)$, the resulting allocation satisfies the transversality condition:

$$\lim_{t\to\infty} \exp(-r^* t) \int_0^{N(t)} V(\nu, t)\, d\nu = \lim_{t\to\infty} \exp(-r^* t) N(t) \frac{1}{\eta} = \lim_{t\to\infty} \exp(-r^* t) N(0) \text{ex}$$

when the parameters are such that $g_c < r^*$. That is, when

$$\eta\beta L(1-\theta) < \rho. \tag{30}$$

- Hence, under this parametric restrictions (26) and (30), the path that we have characterized above is an equilibrium. In equilibrium, $r(t)$ is given by (24), $C(0)$ is determined from Eq. (29), and $C(t), N(t), Y(t)$ grow at the constant rate g_c given in (25). Other equilibrium variables are derived as above. Check that this path satisfies all equilibrium conditions.
- Note that there are no transitional dynamics in this equilibrium. Intuition?

- Going back to above steps, note that we only conjectured $Z(t) > 0$ for all t, which lead to Eq. (23), which is the free entry condition with equality. Given Eq. (23), everything else is uniquely determined.
- Hence, the uniqueness of equilibrium follows if we can show that Eq. (23) holds for all t. Then, the proof for uniqueness follows in two additional steps:
  - Note that $Z(t) = 0$ for all t cannot be an equilibrium, because of the parametric condition (26).
  - Then, $Z(t) > 0$ at least in an interval $(t' - \varepsilon, t' + \varepsilon)$. This implies $\eta V(\nu, t) = 1$ for all $(t' - \varepsilon, t' + \varepsilon)$. Using the HJB equation in equilibrium:

    $$r(t)\, V(\nu, t) = \beta L + \dot{V}(\nu, t).$$

    This further implies $\eta V(\nu, t) = 1$ for all t, which is Eq. (23).
- Given Eq. (23), the HJB equation shows that $r(t) = r^*$ for all t, and the rest of the variables are also uniquely determined. There is a unique equilibrium without transitional dynamics.

(In)efficiency of Equilibrium, Planner's Problem

Social planner's problem can be split into a static and a dynamic problem

1. Static problem: Given level of machines $N^S(t)$, social planner maximizes net output

$$\max_{[x(\nu,t)]_{\nu\in[0,N(t)]}} \tilde{Y}^S(t) \equiv Y^S(t) - X^S(t)$$

$$\text{s.t. } Y^S(t) = \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu,t)^{1-\beta}\, d\nu\right] L^{\beta}, \text{ and } X^S(t) = \int_0^{N(t)} \psi x(\nu,t)\, d\nu,$$

which gives

$$x^S(\nu,t) = (1-\beta)^{-1/\beta} L,\ Y^S(t) = (1-\beta)^{-1/\beta} N^S(t) L$$

$$\tilde{Y}^S(t) = (1-\beta)^{-1/\beta} \beta N^S(t) L$$

Social planner produces more net output than the equilibrum for the same level of $N(t)$, since she avoids monopoly distortions.

2. Dynamic Problem: The social planner then chooses $\{N^S(t)\}_{t=0}^{\infty}$ so as to maximize consumer utility. Tradeoff is splitting net output $\tilde{Y}^S(t)$ between consumption now and investment in technology.

$$\max_{\{N^S(t),C(t)\}} \int_0^{\infty} \exp(-\rho t)\, \frac{C(t)^{1-\theta}-1}{1-\theta} dt$$

$$\text{s.t. } \dot{N}^S(t) = \eta\left[(1-\beta)^{-1/\beta} \beta N^S(t) L - C(t)\right].$$

(In)efficiency of Equilibrium, Solution to Planner's Problem

Standard optimal control problem. Solution yields

$$\frac{\dot{C}^S(t)}{C^S(t)} = g^S \equiv \frac{1}{\theta}\left(\eta (1-\beta)^{-1/\beta} \beta L - \rho\right).$$

- $N^S(t)$ also grows at rate g^S.
- Equilibrium is sub-optimal. Social planner chooses a faster growth rate for consumption (and technology) than equilibrium, since

$$(1-\beta)^{-1/\beta} \eta\beta L > \eta\beta L.$$

- Intuition:
  - Since social planner fixes monopoly distortions, she utilizes a given level of technology better and attains a higher level of net output.
  - Social planner considers the positive effect of new technology on total net output, which is split between wages and profits. Equilibrium entry consider only the effect on profits.

  For both of these reasons, the value of new technology is higher for social planner than equilibrium firms, hence social planner chooses a higher level of investment in R&D and attains a higher growth rate.
- Remark: More growth is not always better for welfare. In these models, more growth might come in expense of initial consumption (since growth comes from investment in technology) so there is a tradeoff between growth and initial consumption. But in this model (for the reasons stated above) the optimal solution features more growth.

- Social planner can provide the monopolists with a linear subsidy τ (applied to their sales), financed by lump-sum taxes on consumers.
- Monopolist now sets price p such that $p(1+\tau) = 1$, which gives $p = \frac{1}{1+\tau}$, and produces $x(\nu, t \mid \tau) = Lp^{-1/\beta} = L(1+\tau)^{1/\beta}$.
- With the right choice of subsidy (solved from $1+\tau = \frac{1}{1-\beta}$), social planner can get the price drop to marginal cost and the output increase to socially optimal level, that is

$$x(\nu, t \mid \tau = \beta) \equiv L(1-\beta)^{-1/\beta} = x^S(\nu, t).$$

- Monopolist profits are $\pi(\nu, t \mid \tau = \beta) = \beta L(1-\beta)^{-1/\beta}$. From the free entry condition, interest rate is pinned down as $r = \eta \beta L(1-\beta)^{-1/\beta}$. The growth rate with subsidies is then

$$g^{eq}(\tau = \beta) = \frac{1}{\theta}\left(\eta \beta L(1-\beta)^{-1/\beta} - \rho\right) = g^S.$$

- Hence, social planner can replicate the optimal outcome by only using a linear subsidy **financed by a lump sum tax on consumers**.
- The subsidy should be financed by a lump sum tax on consumers (not the monopolists) since we need to provide the monopolists with the right incentives to enter.

Expanding Variety Model with Scarce Factors and Knowledge Spillovers.

- Everything is the same except for R&D production technology, which uses labor (scarce factor) instead of the final good (hence there is no $Z(t)$ in this version). Labor is used both in final good and R&D sectors so labor allocation between these sectors will be determined in equilibrium.
- Final good production

$$Y(t) = \frac{1}{1-\beta}\left[\int_0^{N(t)} x(\nu, t)^{1-\beta}\right] L_E(t)^{\beta}$$

- R&D production

$$\dot{N}(t) = \eta N(t) L_R(t)$$

  The term $N(t)$ represents spillovers (technological externalities). Think scientists are standing on the shoulder of previous scientist, so they get more productive as economy develops. Why do we have to add spillovers - we didn't have them before? (answer is on the next slide)
- Labor market clearing

$$L_E(t) + L_R(t) \leq L(t).$$

- The rest of the model is identical to the baseline. Equilibrium defined similarly.

- As before, we have for output

$$Y(t) = \frac{1}{1-\beta} N(t) L_E(t)$$

and wages

$$w(t) = \frac{\beta}{1-\beta} \frac{Y(t)}{L_E(t)} = \frac{\beta}{1-\beta} N(t)$$

- The free entry condition is (why?):

$$\eta N(t) V(\nu, t) = w(t) = \frac{\beta}{1-\beta} N(t),$$

hence, externalities carefully chosen such that

$$V(\nu, t) = \frac{\beta}{(1-\beta)\eta} \tag{31}$$

is constant as in the baseline model.

- Intuition: R&D is using a scarce factor (labor) so innovation becomes costlier as economy develops (wages are increasing). We introduce spillovers so that a unit of labor employed in R&D generates more machines when economy is more developed. The spillovers are chosen such that the cost of producing one machine remains constant (essential to have sustained growth in this model).

Equilbrium Characterization, Con't

- Monopolist profits are (as in the baseline model) given by

$$\pi(\nu, t) = \beta L_E(t) \tag{32}$$

- We conjecture an equilibrium in which $r(t) = r^*$ and $L_E(t) = L_E^*$ are constant. This implies that profits are constant and value function satisfies

$$V(\nu, t) = \frac{\beta L_E^*}{r^*} = \frac{\beta}{(1-\beta)\eta},$$

which gives

$$r^* = (1-\beta)\eta L_E^*.$$

- As in baseline model, Euler equation implies

$$\frac{\dot{C}}{C} = g_C \equiv \frac{1}{\theta}\left((1-\beta)\eta L_E^* - \rho\right). \tag{33}$$

- From final good clearing (remember no $Z(t)$ in this version, since R&D sector uses labor as input not final good)

$$Y(t) = C(t) + X(t)$$

$$\frac{1}{1-\beta} N(t) L_E^* = C(t) + (1-\beta) L_E^* N(t). \tag{34}$$

Hence, $N(t)$ also grows at the constant rate g_C.

- The R&D technology is

$$\dot{N}(t) = \eta N(t) L_R(t) \text{ which gives } g_N = \frac{\dot{N}(t)}{N(t)} = \eta (L - L_E^*). \quad (35)$$

  Hence L_E^* is solved by equating the growth expressions in (33) and (35).
- The growth rate is solved as:

$$g_C = g_N = \frac{1}{\theta}((1-\beta)\eta L_E^* - \rho)$$

  where

$$L_E^* = \frac{\theta\eta L + \rho}{(1-\beta)\eta + \theta\eta}.$$

- This corresponds to an equilibrium under parametric restrictions
  - $(1-\beta)\eta L_E^* > \rho$, so growth is positive
  - $(1-\theta)(1-\beta)\eta L_E^* < \rho$, so that the transversality condition holds.
- No transitional dynamics: equilibrium $C(t)$ and $N(t)$ grows exactly at rate g starting time 0.

Expanding Variety Model with Scarce Factors and Limited Knowledge Spillovers.

- Consider the same model as the last one, except that the R&D production technology is

$$\dot{N}(t) = \eta N(t)^{\phi} L_R(t) \text{ where } \phi < 1.$$

- Assume first population is constant as before. Consider a BGP in which $r(t) = r^*$ and $L_E(t) = L_E^*$. The free entry condition now implies

$$\eta N(t)^{\phi} \frac{\beta L_E^*}{r^*} = w(t) = \frac{\beta N(t)}{1-\beta}.$$

- Clearly, we will not have a BGP. Spillovers just not enough to sustain growth.
- Add population growth at rate n to the same model. This model is also known as the semi-endogenous growth model (reasons to be clear in a second) or endogenous growth model without scale effects (by Chad Jones).

(Semi-endogenous) Growth without Scale Effects

- We conjecture a BGP equilibrium in which $r(t) = r^*$ is constant and $L_E(t) = l_E^* L(t)$ (share of labor in industry and R&D is constant).
- This time, profits are $\pi(\nu, t) = \beta L_E(t)$, growing at rate n. Hence the value function

$$V(\nu, t) = \frac{\beta L_E(t)}{r^* - n}$$

  is also growing at rate n (also understand why we divide by $r^* - n$ instead of just n).
- The free entry condition now implies

$$\eta N(t)^{\phi} \frac{\beta l_E^* L(t)}{r^* - n} = w(t) = \frac{\beta N(t)}{1 - \beta}. \tag{36}$$

- Differentiating the previous equation gives

$$(1 - \phi) \frac{\dot{N}}{N} + \frac{\dot{L}}{L} = 0$$

  Hence,

$$g_N = \frac{n}{1 - \phi}, \tag{37}$$

  is the only growth rate consistent with a BGP.

BGP Equilibrium, Characterization

- Market clearing condition this time implies

$$\frac{1}{1-\beta} N(t) l_E^* L(t) = c(t) L(t) + (1-\beta) N(t) l_E^* L(t). \tag{38}$$

  hence per-capita consumption $c(t)$ must also grow at the same rate as N, that is $g_c = g_N$.

- Interest rate is pinned down by Euler equation,

$$r^* = \theta g_c + \rho = \theta \frac{n}{1-\phi} + \rho.$$

- Evolution of $N(t)$ is given by $\dot{N}(t) = \eta N(t)^{\phi} (1 - l_E^*) L(t)$, which implies

$$g_N = \frac{n}{1-\phi} = \eta (1 - l_E^*) \frac{L(t)}{N(t)^{1-\phi}}.$$

- Also, Eq. (36) implies

$$\frac{N(t)^{1-\phi}}{L(t)} = \frac{(1-\beta) l_E^*}{r^* - n}.$$

- The last three displayed equations are three equations in three unknowns $\frac{N(t)^{1-\phi}}{L(t)}$, r^*, and l_E^*. This can be solved. The resulting path is a BGP equilibrium. There are transitional dynamics (why?)

Semi-Endogenous Growth, Results

- Growth without scale effects (growth rate does not increase with population). More in line with data. But growth depends on n. Scale effects in a different guise.
- Why semi-endogenous (hint, does the growth rate in Eq. (37) respond to policy)?. Is this a useful model to study policy issues?
- One plus for the model: $L_R(t)$ increases over time, that is, amount of (scarce) resources allocated to R&D sector increases. This is in line with data.

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