14.452 Recitation Notes:
1. Optimal Control and Neoclassical Model
2. Stability of Differential Systems

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Discounted Infinite Horizon Problem

- Consider a general formulation for the discounted infinite horizon problem

$$\max_{x(t),y(t)} W\left([x(t), y(t)]_t\right) \equiv \int_0^\infty \exp(-\rho t) f(x(t), y(t))\, dt, \tag{1}$$

$$\text{s.t. } \dot{x}(t) = g(t, x(t), y(t)),\ x(t) \in Int\mathcal{X}(t),\ y(t) \in Int\mathcal{Y}(t) \text{ for all } t,$$

$$x(0) = x_0, \text{ and } \lim_{t\to\infty} b(t)\, x(t) \geq x_1.$$

  Here, $b : \mathbb{R}_+ \to \mathbb{R}_+$ is a function such that $\lim_{t\to\infty} b(t) < \infty$. Economic problems typically fit into this framework.

- First order optimality conditions developed in Chapter 7 of the book in terms of the co-state variable $\lambda(t)$ (interpreted as the shadow value of the stock variable) and the Hamiltonian

$$H(t, x(t), y(t), \mu(t)) = \exp(-\rho t) f(x(t), y(t)) + \lambda(t)\, g(t, x(t), y(t)).$$

- In discounted problems, it is more convenient to work with the current co-state variable $\mu(t) = \exp(\rho t)\lambda(t)$ (interpreted as the current shadow value of stock variable) and the current-value Hamiltonian $\hat{H}$ as

$$\begin{aligned} \hat{H}(t, x(t), y(t), \mu(t)) &= \exp(\rho t)\, H(t, x(t), y(t), \mu(t)) \\ &= f(x(t), y(t)) + \mu(t)\, g(t, x(t), y(t)) \end{aligned} \tag{2}$$

Discounted Infinite Horizon Problem: FOCs

- The FOCs can be written in terms of $\hat{H}$ and $\mu(t)$, as

$$\hat{H}_y(t, \hat{x}(t), \hat{y}(t), \mu(t)) = 0 \text{ for all } t \tag{3}$$

$$\hat{H}_x(t, \hat{x}(t), \hat{y}(t), \mu(t)) = \rho\mu(t) - \dot{\mu}(t) \text{ for all } t \tag{4}$$

$$\lim_{t\to\infty} [\exp(-\rho t)\, \mu(t)\, \hat{x}(t)] = 0. \tag{5}$$

- We also have the stock evolution equation,

$$\dot{x}(t) = g(t, x(t), y(t)) \text{ with } x(0) = x_0. \tag{6}$$

- The line of attack: (i) Find a candidate path that satisfies FOCs and (6), (ii) use the sufficiency theorem to show the candidate is optimal.
- Eqs. (3) – (6) can typically be reduced to two differential equations in two variables, $x(t)$ and $\mu(t)$, with one initial condition (i.e., $x(0) = x_0$) and one end-value condition (i.e., the transversality condition (5)).
- The solution to these differential equations give a *candidate allocation* that satisfies the FOCs. But is this candidate allocation optimal? Are there other allocations that are also optimal? Sufficiency theorem takes care of all this.

Discounted Infinite Horizon Problem: Sufficiency Theorem

- **Theorem 7.14 (Sufficiency Conditions for Discounted Infinite-Horizon Problems)**: Consider problem (1) with f and g continuously differentiable. Define $\hat{H}(t, x, y, \mu)$ as the current-value Hamiltonian as in (2), and suppose that a solution $\hat{y}(t)$ and the corresponding path of state variable $\hat{x}(t)$ satisfy (3)-(5). Given the resulting current-value co-state variable $\mu(t)$, define $M(t, x, \mu) \equiv \max_{y(t) \in \mathcal{Y}(t)} \hat{H}(t, x, y, \mu)$. Suppose that $V(t, \hat{x}(t))$ exists and is finite for all t (where $V(t, x(t))$ is the value function formally defined in Eq. (7.38) of the textbook), that for any admissible pair $(x(t), y(t))$, $\lim_{t \to \infty} [\exp(-\rho t) \mu(t) x(t)] \geq 0$, and that $\mathcal{X}(t)$ is convex and $M(t, x, \mu)$ is concave in $x \in \mathcal{X}(t)$ for all t. Then, the pair $(\hat{x}(t), \hat{y}(t))$ achieves the global maximum of (1). Moreover, if $M(t, x, \mu)$ is strictly concave in x, $(\hat{x}(t), \hat{y}(t))$ is the unique solution to (1).

Sufficiency Theorem: What to Check

- Find a candidate path $[\hat{x}(t), \hat{y}(t), \mu(t)]_t$ that satisfies the FOCs and check, essentially, two requirements
  1. Concavity: check that the maximized Hamiltonian $M(t, x, \mu(t)) \equiv \max_y \hat{H}(t, x, y, \mu(t))$ is concave as a function of x (given the candidate co-state variable $\mu(t)$).
     A sufficient condition is that $\hat{H}(t, x, y, \mu(t)) = f(t, x, y) + \mu(t) g(t, x, y)$ is jointly concave in x and y. When the candidate $\mu(t)$ is positive, a sufficient condition is that f and g are both jointly concave in x and y.
  2. A budget constraint at infinity: check that any other admissible path $[x(t), y(t)]_t$ satisfies $\lim_{t\to\infty} \exp(-\rho t) \mu(t) x(t) \geq 0$ (given the candidate co-state variable $\mu(t)$).
- If these conditions hold, then the candidate path $[\hat{x}(t), \hat{y}(t), \mu(t)]_t$ is optimal. If Condition 1 holds with strict concavity, then the optimal path is unique.

Neoclassical Model: The Household's Problem

- Given the path of prices $[r(t), w(t)]_t$, households choose the path of per-capita consumption and assets $[c(t), a(t)]_{t=0}^{\infty}$ to solve

$$\max_{[c(t),a(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} e^{nt} u(c(t))\, dt = \int_0^{\infty} e^{-(\rho-n)t} u(c(t))\, dt$$
$$\text{s.t. } \dot{a}(t) = r(t)\, a(t) + w(t) - c(t) - na(t), \tag{7}$$
$$\lim_{t\to\infty} e^{-\int_0^t r(s)ds} e^{nt} a(t) = 0.$$

(Detour: how to derive the asset evolution equation. Aggregate assets, $\mathcal{A}(t)$, follow

$$\dot{\mathcal{A}}(t) = r(t)\,\mathcal{A}(t) + w(t)\, L(t) - c(t)\, L(t).$$

Hence Per-capita assets, $a(t) \equiv \mathcal{A}(t)/L(t)$ follow,

$$\frac{\dot{a}(t)}{a(t)} = \frac{\dot{\mathcal{A}}(t)}{\dot{\mathcal{A}}(t)} - n = r(t) + \frac{w(t)}{a(t)} - \frac{c(t)}{a(t)} - n$$

leading to Eq. (7)).

Neoclassical Model: Solving the Household's Problem

- CV Hamiltonian (with effective discount factor $\rho - n$),

$$\hat{H}(t, a, c, \mu) = u(c) + \mu \left(a(r(t) - n) + w(t) - c \right).$$

- FOCs:

$$\text{FOC-1: } \hat{H}_c = 0 \;\Rightarrow u'(c) = \mu \tag{8}$$

$$\text{FOC-2: } \hat{H}_a = (\rho - n)\,\mu - \dot{\mu} \Rightarrow \frac{\dot{\mu}}{\mu} = -(r(t) - \rho).$$

- The transversality condition is

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mu(t)\,a(t) = \lim_{t\to\infty} \mu(0)\,e^{-(\rho-n)t}\left(e^{-\int_0^t (r(s)-\rho)ds}\right) a(t)$$
$$= u'(c(0)) \lim_{t\to\infty} e^{-\int_0^t r(s)ds} e^{nt} a(t) = 0, \tag{9}$$

(where we use $\mu(t) = \mu(0)\,e^{\int_0^t -(r(s)-\rho)ds}$ which follows from FOC-2).

How to Use the Sufficiency Theorem: Constructing the Candidate Path

- Combining Eqs. (8) and eliminating $\mu(t)$ gives the Euler equation:

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\epsilon_u(c(t))}(r(t) - \rho), \tag{10}$$

where $\epsilon_u(c(t)) = -\left(\frac{u''(c(t))c(t)}{u'(c(t))}\right)$ is elasticity of marginal utility (and *inverse* elasticity of intertemporal substitution). With CES preferences, $c(t)^{1-\theta} / (1-\theta)$, we have $\epsilon_u(c(t)) = \theta$.

- Recall that we also have the asset evolution equation

$$\dot{a}(t) = r(t)a(t) + w(t) - c(t) - na(t). \tag{11}$$

- Two differential equations in $(a(t), c(t))$ with one initial condition for $a(0)$ and one transversality condition (9).
- These equations typically have a solution, which is our candidate path. Below, we see how to pictorially construct this path. For now, suppose we have constructed a candidate path $[\hat{a}(t), \hat{c}(t)]_t$. We now would like to apply the sufficiency theorem and show that this is the unique optimum.

How to Use the Sufficency Theorem: Conditions 1 and 2

- Let $[\hat{c}(t), \hat{a}(t)]_{t=0}^{\infty}$ be the candidate path (with the corresponding co-state variable $\mu(t) = u'(\hat{c}(t))$).
- Concavity condition (condition 1) checks since $\hat{H}(t, a, c, \mu(t)) = u(c) + \mu(t)(a(r(t) - n) + w(t) - c)$ is jointly concave in a and c. But not strictly concave.
- Boundary requirement for admissible paths (condition 2) also checks. To see this, consider some feasible path $[a(t), c(t)]_t$ and note that we need to check

$$\lim_{t\to\infty} a(t)\, e^{-(\rho-n)t}\mu(t) \geq 0.$$

Using FOC-2, we have $\mu(t) = \mu(0)\, e^{\int_0^t -(r(s)-\rho)ds}$ and we substitute this to get

$$\mu(0) \lim_{t\to\infty} a(t)\, e^{\int_0^t -(r(s)-n)ds} \geq 0,$$

which holds since any feasible $[a(t), c(t)]_t$ satisfies the no-Ponzi condition.

How to Use the Sufficency Theorem: Uniqueness?

- Hence, Theorem 7.14 applies and shows that the candidate $(\hat{a}(t), \hat{c}(t))_{t=0}^{\infty}$ solves the household's problem. Problem not strict concave, so Theorem 7.14 doesn't give uniqueness.
- See Exercise 8.11 for a proof that this is indeed the unique solution (as long as $u(.)$ is strictly concave).
- So far, we have characterized household behavior given the price sequence $[r(t), w(t)]_t$ (partial equilibrium analysis).
- Note that we do not yet know whether an equilibrium exists. Nor do we know whether it is unique. We only know that there is a unique solution to the household problem for any given price sequence.
- We next close the model and characterize the competitive equilibrium.

Neoclassical Model: Production Side and Equilibrium

- On the production side, final good producers take prices as given (competitive) and maximize output. Since capital and labor markets clear, this condition equivalently gives the competitive factor prices

$$R(t) = F_K(K(t), L(t)) = f'(k(t)) \text{ and } r(t) = R(t) - \delta \tag{12}$$
$$w(t) = F_L(K(t), L(t)) = f(k(t)) - f'(k(t))\, k(t).$$

- The competitive equilibrium is a path of allocations and prices $[c(t), a(t), k(t), w(t), r(t)]_{t=0}^{\infty}$ such that households choose $[c(t), a(t)]_t$ optimally, competitive firms optimize, and factor, good and asset markets clear (in particular, $a(t) = k(t)$ for all t).
- Conditions (12) capture firm optimization and factor market clearing, so the equilibrium definition in this model can be simplified by incorporating these conditions into the definition.

Neoclassical Model: Characterizing the Equilibrium Path

- Substitute competitive prices in household's Euler equation, budget constraint and the transversality condition (FOCs (10), (11) and (9) above) and use $a(t) = k(t)$ to get

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\epsilon_u(c(t))}\left(f'(k(t)) - \rho - \delta\right),$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t),$$

$$k(0) \text{ given, } \lim_{t\to\infty} e^{-\int_0^t \left(f'k(s)-\delta\right)ds} e^{nt} k(t) = 0.$$

  Two differential equations in $(c(t), k(t))$ with two conditions. Next steps:
  1. Find a candidate path $[c(t), k(t)]_t$ that satisfies these equations. Will be the saddle path on the phase diagram.
  2. Construct prices $R(t) = f'(k(t)), w(t) = f(k(t)) - k(t) f'(k(t))$ and $r(t) = R(t) - \delta$. These prices are consistent with equilibrium by construction.
  3. Construct the household allocation $[c(t), a(t) = k(t)]_t$ and note that it solves the household system (Dif. Eqs. (10), (11), and the transversality condition (9)), given equilibrium prices $[r(t), w(t)]_t$. I.e., this allocation is optimal for the household.
- It follows that the constructed path $[c(t), a(t), k(t), w(t), r(t)]_{t=0}^{\infty}$ is an equilibrium.

Neoclassical Model: Constructing the Equilibrium Path

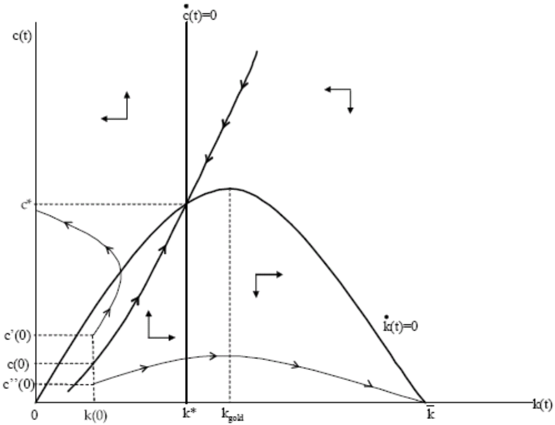

Figure: Transitional dynamics in the baseline neoclassical growth model

Figure 8.1 in Acemoglu, K. Daron. Introduction to Modern Economic Growth.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Neoclassical Model: Social Planner's Problem

- The social planner chooses $[k(t), c(t)]_t$ to solve

$$\max_{[c(t),k(t)]_{t=0}^{\infty}} \int_0^{\infty} e^{-(\rho-n)t} u(c(t))\, dt$$

$$\text{s.t. } \dot{k}(t) = f(k(t)) - c(t) - (n+\delta) k(t) \text{ and } k(t) \geq 0. \qquad (13)$$

- Note that the social planner solves *a different problem* than the household (that will turn out to have the same solution).
- The CV Hamiltonian for the planner's problem is $\hat{H}(t, k, k, \mu) = u(c) + \mu (f(k) - c - (n+\delta) k)$. The FOCs are

$$\hat{H}_c = 0 \Rightarrow u'(c) = \mu$$

$$\hat{H}_k = \mu ((f'(k(t)) - \delta - n)) = (\rho - n) \mu - \dot{\mu} \Rightarrow \frac{\dot{\mu}}{\mu} = -(f'(k(t)) - \delta - \rho).$$

- The transversality condition is

$$\lim_{t\to\infty} e^{-(\rho-n)t} \mu(t) k(t) = \lim_{t\to\infty} \mu(0) e^{-(\rho-n)t} \left(e^{-\int_0^t \left(f'(k(s)) - \delta - \rho\right) ds} \right) k(t)$$

$$= u'(c(0)) \lim_{t\to\infty} \left(e^{-\int_0^t \left(f'(k(s)) - \delta\right) ds} \right) e^{nt} a(t) = 0,$$

Neoclassical Model: Social Planner's Problem, FOCs

- Combining FOCs, we get the Euler equation. Combining this with the resource constraints, we get the system

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\epsilon_u(c(t))}\left(f'(k(t)) - \rho - \delta\right), \tag{14}$$

$$\dot{k}(t) = f(k(t)) - c(t) - (\delta + n)k(t),$$

$$k(0) \text{ given, } \lim_{t\to\infty} e^{-\int_0^t \left(f' k(s) - \delta\right) ds} e^{nt} k(t) = 0.$$

- Find a candidate path $\left[\hat{c}(t), \hat{k}(t)\right]_t$ that solves this system. Theorem 7.14 applies to the planner's problem since (1) $\hat{H}$ is jointly strictly concave in c and k, and (2) For any admissible $(k(t), c(t))$, we have $\lim_{t\to\infty} \exp(-(\rho - n)t)\mu(t)k(t) \geq 0$ (since $\mu(t) \geq 0$ and $k(t) \geq 0$). Theorem 7.14 shows that $\left[\hat{c}(t), \hat{k}(t)\right]_t$ is optimal, and moreover, it is unique due to strict concavity of CV Hamiltonian.

Neoclassical Model: Welfare Theorems, Uniqueness

- Note that the system (14) is identical to the equilibrium system. This shows that the equilibrium path of consumption and capital are identical to the social planner's choice of consumption and capital. Proves that the welfare theorems apply to this economy.
- Note also that looking at the social planner's problem proves that the equilibrium path must be unique. As the differential equation systems are identical, any equilibrium path $[c(t), k(t)]$ also solves the planner's problem. But the planner's problem has a unique solution from strict concavity. It follows that the equilibrium is unique.

Solving Autonomous Linear Differential Equations

- We first study stability of linear systems. Local stability of non-linear systems will be closely related.
- Let $y \in \mathbb{R}^n$, A be a non-singular matrix ($\det A \neq 0$) and consider the vector differential equation

$$\dot{y} = Ay + b, \text{ with } y(0) \text{ given.}$$

- Define $x = y + A^{-1}b$, the previous differential equation is equivalent to

$$\dot{x} = Ax, \text{ wth } x(0) \equiv y(0) + A^{-1}b \text{ given.}$$

- Hence we consider differential equations $\dot{x} = Ax$ without loss of generality. Note that the unique steady-state is $x^* = 0$. In other words, we have normalized the steady-state to 0.
- **Line of attack:** Represent x in the coordinates corresponding to eigenvectors of A, because, with this representation, the n dimensional differential equation $\dot{x} = Ax$ will decompose into n linear differential equations of dimension 1.

Some Linear Algebra

- Suppose A has linearly independent eigenvectors $\{v_1, .., v_n\}$ (one sufficient condition for this is that A has distinct eigenvalues $\{\lambda_1, .., \lambda_n\}$).
- Let the eigendecomposition of A be

$$A = PDP^{-1},$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

is the diagonal matrix of eigenvalues and

$$P = [v_1 \;\; v_2..v_n]$$

where v_i is the eigenvector corresponding to eigenvalue λ_i (that is, $Av_i = \lambda_i v_i$ for each i).
- P is the basis of eigenvectors, P^{-1} is the change of coordinate matrix that takes regular coordinates to eigen-coordinates (that is, weights on each eigenvector when we write the vector as a linear combination of eigenvectors). If we define $z = P^{-1}x$, then z is the eigencoordinates of the vector we represent as x with regular coordinates.

Example

- Let
$$A = \begin{bmatrix} 1/2 & -3/2 \\ -3/2 & 1/2 \end{bmatrix}.$$
- Eigenvalues are found as the solution to the following equation (why?):
$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 1/2 - \lambda & -3/2 \\ -3/2 & 1/2 - \lambda \end{bmatrix}\right) = 0,$$
which yields $\lambda_1 = -1$ and $\lambda_2 = 2$.
- Considering the equation $Ax = v_i x$, the corresponding eigenvectors can be derived as $v_1 = 1/\sqrt{2}\,[1\ 1]$, $v_2 = 1/\sqrt{2}\,[-1\ 1]$. Note that eigenvalues are uniquely determined, but eigenvectors are only determined up to a scalar, that is, for any $\alpha \neq 0$, αv_1 could also serve as an eigenvector corresponding to λ_1.
- An eigendecomposition of A is PDP^{-1}, where
$$P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix},\ P^{-1} = P^T, \Lambda = \begin{bmatrix} -1 & \\ & 2 \end{bmatrix}$$

Back to Linear Differential Equations

- Let $z \equiv P^{-1}x$, and rewrite $\dot{x} = Ax$ in terms of the eigencoordinates

$$\begin{aligned} \dot{x} &= Ax \\ \dot{x} &= PDP^{-1}x \\ P^{-1}\dot{x} &= Dz \\ \dot{z} &= Dz, \text{ with } z(0) = P^{-1}x(0). \end{aligned}$$

- The solution in eigencoordinates is

$$z^{i}(t) = z^{i}(0)\exp(\lambda_i t), \ \forall i \in \{1, .., n\}.$$

(Superscripts correspond to components of a vector, subscripts correspond to eigenvalues or eigenvectors)

Linear Differential Equations, Solution

- Then, going back to regular coordinates, we have

$$
\begin{aligned}
x &= Pz \\
x^j(t) &= \sum_{i=1}^{n} P_{ij} z^i(t) \\
&= \sum_{i=1}^{n} P_{ij} z^i(0) \exp(\lambda_i t), \text{ for each } j \in \{1, .., n\}
\end{aligned}
$$

- Hence, the solution is a linear combination of exponentials with eigenvalues. The weights could be found by solving for P and using the initial condition $x(0)$ to calculate

$$z(0) = P^{-1} x(0)$$

as the initial value in eigencoordinates.

Stability of Linear Systems

- Consider the linear system (with A non-singular)

$$\dot{x} = Ax.$$

- When is this system stable, i.e., when does it converge to the steady-state $x^* = 0$ from any starting point?
- Recall that the solution takes the form

$$x^j(t) = \sum_{i=1}^{n} P_{ij} z^i(0) \exp(\lambda_i t), \text{ for each } j \in \{1, .., n\}.$$

- This will be stable starting at any $z(0)$ (or, equivalently, any $x(0)$), when we have $\lambda_i < 0$ for all i (if the eigenvalue is complex, then the real part being less than zero is sufficient).
- Proves Theorem 2.4 in the book: The linear system is globally asymptotically stable if all eigenvalues of A have negative real parts.
- The analysis adds in addition: when some eigenvalues have positive real part, the system cannot be globally stable (starting with some initial conditions, the system diverges).

Saddle Path Stability

- What if not all eigenvalues have negative real parts? Suppose $m \leq n$ of the eigenvalues have negative real parts, call them $\{\lambda_1, .., \lambda_m\}$.
- Then, if the initial eigencoordinates $z(0)$ has the form $\left[z^1, ..., z^m, 0, .., 0\right]$, then the above system is stable. This set is equivalent to $\mathbb{R}^m \subset \mathbb{R}^n$, an $m-$dimensional sub-space of the larger space $\mathbb{R}^n$.
- Recall how $z(0)$ and $x(0)$ are related: $x(0) = Pz(0)$. Then, there is an m-dimensional subspace of x vectors for which, if we start with $x(0)$ in this space, then the system will asymptotically converge to 0.
- More specifically, if $x(0) = \sum_{i=1}^{m} z^i v_i$ for some $\left\{z^i\right\}_{i=1}^{m}$, that is, if x is in the sub-space of eigenvectors corresponding to "stable" eigenvalues, then the system converges.
- Proves Theorem 7.18: If m of the eigenvalues of A has negative real parts, then there exists $m-$dimensional subspace such that, starting from $x(0)$ in this space, $x(t) \to 0$.
- This analysis adds in addition: if $x(0)$ has a non-zero eigencoordinate for an eigenvector corresponding to an unstable eigenvalue (positive real part), then the system is unstable (diverges).

Back to Example, Pictorial Summary

- Consider the example before and consider the system $\dot{x} = Ax$. Eigenvectors: v_1 stable since $\lambda_1 = -1$, v_2 unstable since $\lambda_2 = 2$.
- Starting from $x(0)$ the system diverges, but starting from $x'(0)$ it converges to the steady state $(x^* = 0)$.

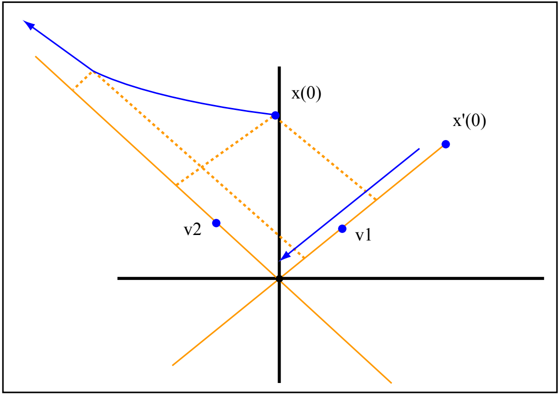

Image by MIT OpenCourseWare.

Nonlinear Differential Equations

- For some continously differentiable function $G : \mathbb{R}^n \to \mathbb{R}^n$, consider the system
$$\dot{x} = G(x).$$
- Let x^* denote a steady-state, i.e., suppose $G(x^*) = 0$.
- Apply Taylor's expansion around x^*:
$$\dot{x} \approx G(x^*) + \nabla G(x^*)(x - x^*)$$
$$\frac{d(x - x^*)}{dt} \approx \nabla G(x^*)(x - x^*)$$
- Locally, this system behaves like a linear system. Hence analogs of above theorems apply locally.
- Theorem 2.5: The non-linear system is locally asymptotically stable around steady state x^* if all eigenvalues of $\nabla G(x^*)$ has negative real parts. (There exists a neighborhood B of x^*, such that for all $x(0) \in B$, $x(t) \to x^*$.)
- Theorem 7.19: If m of the eigenvalues of $\nabla G(x^*)$ has negative real parts, then there exists a neighborhood B around x^* and an $m-$dimensional manifold $M \subset B$, such that for all $x(0) \in M$, $x(t) \to x^*$.

Stability in the Neoclassical Model

- The differential equation system in the neoclassical model can be written as:

$$\begin{bmatrix} \dot{c} \\ \dot{k} \end{bmatrix} = F(c,k) \equiv \begin{bmatrix} \frac{c}{\epsilon_{u_c}(c)} \left(f'(k) - \delta - \rho\right) \\ f(k) - (\delta + n) k - c \end{bmatrix}. \tag{15}$$

- The steady state (c^*, k^*) is characterized by

$$\begin{aligned} f'(k^*) &= \delta + \rho \\ c^* &= f(k^*) - (\delta + n) k^*. \end{aligned} \tag{16}$$

Steady State and Local Approximation

- A first-order approximation of the system in Eq. (15) around steady state gives

$$\frac{d}{dt}\begin{pmatrix} c-c^* \\ k-k^* \end{pmatrix} \approx \nabla F\left(c^*,k^*\right)\begin{pmatrix} c-c^* \\ k-k^* \end{pmatrix}, \tag{17}$$

  where $\nabla F\left(c^*,k^*\right)$ is the Jacobian of the vector-valued function $F\left(c,k\right)$ in (15).

- Local stability properties are governed by eigenvalues of $\nabla F\left(c^*,k^*\right)$. To calculate $\nabla F\left(c^*,k^*\right)$, first note that

$$\nabla F\left(c,k\right) = \begin{bmatrix} \frac{d\frac{c}{\epsilon_{u_c}(c)}}{dc}\left(f'\left(k\right)-\delta-\rho\right) & \frac{c}{\epsilon_{u_c}(c)}f''\left(k\right) \\ -1 & f'\left(k\right)-\delta-n \end{bmatrix}$$

  and use the steady state relations (16) to evaluate

$$\nabla F\left(c^*,k^*\right) = \begin{bmatrix} 0 & \frac{c^*}{\epsilon_{u_c}(c^*)}f''\left(k^*\right) \\ -1 & \rho-n \end{bmatrix}.$$

Eigenvalues and Saddle-Path Stability

- The eigenvalues of $\nabla F(c^*, k^*)$ are found as the roots of the polynomial $P(\lambda)$ given by

$$P(\lambda) = \det(\nabla F(c^*, k^*) - \lambda I) = \det\left(\begin{bmatrix} -\lambda & \frac{c^*}{\epsilon_{u_c}(c^*)} f''(k^*) \\ -1 & \rho - n - \lambda \end{bmatrix}\right)$$
$$= (\lambda + n - \rho)\lambda + \frac{c^*}{\epsilon_{u_c}(c^*)} f''(k^*).$$

- Note that, $P(\lambda)$ is a quadratic with positive coefficient on λ^2 which also satisfies

$$P(0) = \frac{c^*}{\epsilon_{u_c}(c^*)} f''(k^*) < 0,$$

hence $P(\lambda)$ has one negative and one positive root, that is:

$$\lambda_1 < 0 < \lambda_2.$$

- Hence, the system in (15) is not globally stable, but locally saddle path stable.

Shape of Saddle Path (Locally)

- We can check that components of v_1 (corresponding to the stable eigenvalue λ_1) has the same sign:

$$\begin{bmatrix} 0 & \frac{c^*}{\epsilon_{u_c}(c^*)} f''(k^*) \\ -1 & \rho - n \end{bmatrix} \begin{bmatrix} v_1^1 \\ v_1^2 \end{bmatrix} = \lambda_1 \begin{bmatrix} v_1^1 \\ v_1^2 \end{bmatrix}$$

- The first equation can be written as:

$$\frac{c^*}{\epsilon_{u_c}(c^*)} f''(k^*) v_1^2 = \lambda_1 v_1^1$$

Since $\frac{c^*}{\epsilon_{u_c}(c^*)} f''(k^*) < 0$ and $\lambda_1 < 0$, it follows that v_1^2 and v_1^1 are of the same sign.

- Confirms that the saddle path is increasing. Confirms the way we drew the picture.

Speed of Convergence

- Rate of convergence depends on the size of the stable eigenvalue, $|\lambda_1|$.
- Using the quadratic formula, the eigenvalues can be explicitly solved as:

$$\lambda_{1,2} = \frac{1}{2}\left(\rho - n \pm \sqrt{(\rho - n)^2 - 4\frac{c^* f''(k^*)}{\epsilon_{u_c}(c^*)}}\right).$$

- The smaller (and the negative) real root, λ_1, is given by

$$\lambda_1 = \frac{1}{2}(\rho - n)\left(1 - \sqrt{1 + 4\frac{c^* |f''(k^*)|}{(\rho - n)\,\epsilon_{u_c}(c^*)}}\right).$$

- Note that $\lambda_1 < 0$. The higher $|\lambda_1|$ (the lower λ_1), the faster the convergence.

- When $|f''(k^*)|$ is higher, convergence is faster. More concavity $=>$ faster convergence (also present in the Solow model).
- When $\epsilon_{u_c}(c^*)$ (inverse elasticity of substitution) is higher, convergence is slower. Lower elasticity of substitution $=>$ the less willing people are to give up consumption to invest $=>$ slower convergence to steady state.