14.452 Recitation Notes: 1. Solow model with CES production function 2. Uzawa's Theorem (Recitation 1 on October 30, 2009)

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CES Production Function

- Consider the production function:

$$F(K,L) = \left(\gamma K^{(\sigma-1)/\sigma} + (1-\gamma)L^{(\sigma-1)/\sigma}\right)^{\sigma/(\sigma-1)} \quad (1)$$

- **Claim:** Elasticity of substitution between K and L is constant and equal to σ.
- **Proof of claim:** Elasticity of substitution is the percentage change in relative factor inputs K/L in response to a percentage change in relative factor prices, given by: $\frac{-d\log(K/L)}{d\log(F_K/F_L)}$.
- For the function in (1), we have

$$\begin{aligned} F_K &= \gamma K^{-1/\sigma} F^{1/\sigma}, \\ F_L &= (1-\gamma) L^{-1/\sigma} F^{1/\sigma}. \end{aligned} \quad (2)$$

- Using this we have:

$$\frac{F_K}{F_L} = \frac{\gamma}{1-\gamma}\left(\frac{K}{L}\right)^{-1/\sigma} => \frac{K}{L} = \left(\frac{F_K}{F_L}\Big/\frac{\gamma}{1-\gamma}\right)^{-\sigma}.$$

- In the last expression, think of $k = K/L$ as a function of $p = F_K/F_L$. It has the form $k = Cp^{-\sigma}$ for some constant C. This has constant elasticity.

Special Cases

- CES approximates linear production function as $\sigma \to \infty$. To see this, note:

$$\lim_{\sigma \to \infty} F(K, L) = \lim_{\sigma \to \infty} \gamma K + (1-\gamma) L.$$

- CES approximates the Leontieff function as $\sigma \to 0$. To see this, first suppose $K > L$. In this case, note:

$$\lim_{\sigma \to 0} \frac{F(K, L)}{L} = \lim_{\sigma \to 0} \left(\gamma \left(\frac{K}{L} \right)^{(\sigma-1)/\sigma} + (1-\gamma) \right)^{\sigma/(\sigma-1)}$$

$$= \lim_{\sigma \to 0} (1-\gamma)^{\sigma/(\sigma-1)} = 1,$$

where the second equality follows since $\frac{K}{L} > 1$ and $\frac{\sigma-1}{\sigma} \to -\infty$. This further implies that $\lim_{\sigma \to 0} F(K, L) = L$. Next suppose $K < L$ and note that a similar argument establishes $\lim_{\sigma \to 0} F(K, L) = K$ for this case. Combining these two results, note that

$$\lim_{\sigma \to 0} F(K, L) = \min(K, L),$$

which is the Leontieff production function.

Special Cases

- CES is the Cobb-Douglas function for $\sigma = 1$. To see this, note:

$$\begin{aligned}
\lim_{\sigma \to 1} \log F(K, L) &= \lim_{\sigma \to 1} \frac{\log\left(\gamma K^{(1-1/\sigma)} + (1-\gamma) L^{(1-1/\sigma)}\right)}{1 - 1/\sigma} \\
&= \lim_{\sigma \to 1} \frac{\frac{-\gamma K^{(1-1/\sigma)} \ln K/\sigma^2 - (1-\gamma) L^{(1-1/\sigma)} \ln L/\sigma^2}{\gamma K^{(1-1/\sigma)} + (1-\gamma) L^{(1-1/\sigma)}}}{-1/\sigma^2} \\
&= \gamma \ln K + (1-\gamma) \ln L,
\end{aligned} \tag{3}$$

where the second line uses L'Hospital's rule and the chain rule. This further implies that

$$\begin{aligned}
\lim_{\sigma \to 1} F(K, L) &= \exp\left(\gamma \ln K + (1-\gamma) \ln L\right) \\
&= K^{\gamma} L^{1-\gamma},
\end{aligned}$$

which is the Cobb-Douglas production function.

CES and Assumptions 1 and 2

- CES for any $\sigma \in (0, \infty)$ satisfies Assumption 1 in the textbook, that is, it is strictly increasing and strictly concave in each input. To check concavity, note:

$$F_K = \gamma K^{-1/\sigma} F^{1/\sigma} = \left(\left(\gamma + (1-\gamma) \left(\frac{L}{K} \right)^{(\sigma-1)/\sigma} \right)^{\sigma/(\sigma-1)} \right)^{1/\sigma}.$$

  This is strictly decreasing in K, that is, $F_{KK} < 0$. Similarly, $F_{LL} < 0$.
- CES for $\sigma = 1$ (Cobb-Douglas) also satisfies Assumption 2 Inada conditions.
- CES for any $\sigma \neq 1$ does **not** satisfy Assumption 2. To see this, by Eq. (2), note that:
  - For $\sigma > 1$:
  $$\lim_{K \to 0} F_K = \infty, \text{ but } \lim_{K \to \infty} F_K = \gamma^{1/(\sigma-1)} > 0.$$
  - For $\sigma < 1$:
  $$\lim_{K \to \infty} F_K = 0, \text{ but } \lim_{K \to 0} F_K = \gamma^{1/(\sigma-1)} > 0.$$

  It violates one part or the other of the Inada condition.

CES and Solow Model

- Consider CES with $\sigma \neq 1$. Despite the failure of Assumption 2, Solow model with CES has a simple characterization, but the equilibrium path may be qualitatively different than the baseline case.
- Consider Solow model in continuous time with population growth at rate n and no technological growth. Consider the accumulation of capital-labor ratio $k(t) = K(t)/L(t)$:

$$\frac{\dot{k}}{k} = s\frac{f(k)}{k} - (\delta + n), \tag{4}$$

where

$$f(k) = F(K,1) = \left(\gamma k^{(\sigma-1)/\sigma} + (1-\gamma)\right)^{\sigma/(\sigma-1)}.$$

CES and Solow Model

- Consider the limit of the average productivity as $k \to \infty$, and as $k \to 0$:

$$\text{For } \sigma > 1: \begin{cases} \lim_{k\to 0} \frac{f(k)}{k} = \infty \\ \lim_{k\to\infty} \frac{f(k)}{k} = \gamma^{\sigma/(\sigma-1)} \end{cases} \tag{5}$$

$$\text{and for } \sigma < 1: \begin{cases} \lim_{k\to 0} \frac{f(k)}{k} = \gamma^{\sigma/(\sigma-1)} \\ \lim_{k\to\infty} \frac{f(k)}{k} = 0 \end{cases} .$$

- How are these expressions different than the case in the textbook?
- Eqs. (4) and (5) also lead to a simple characterization of equilibrium.

CES and Solow Model

- For $\sigma > 1$, there are two cases. If
$$\gamma^{\sigma/(\sigma-1)} < \frac{\delta + n}{s}, \tag{6}$$
then there exists a unique k^* that solves $\frac{f(k^*)}{k^*} = \frac{\delta+n}{s}$, which is the steady state capital-labor ratio. The equilibrium is globally stable.
- If the opposite of condition (6) holds, then there is sustained growth. For any $k(t) > 0$, we have $\dot{k}(t) > 0$ (check this) and thus $k(t) \to \infty$. Moreover, by Eq. (4), $k(t)$ asymptotically grows at the constant rate $s\gamma^{\sigma/(\sigma-1)} - (\delta + n) \geq 0$.
- Intuition: When $\sigma > 1$, capital and labor are sufficiently substitutable so that sustained growth by capital accumulation is possible. As $k \to \infty$, CES with $\sigma > 1$ is qualitatively similar to the linear production function.

CES and Solow Model

- For $\sigma < 1$, there are also two cases. If

$$\gamma^{\sigma/(\sigma-1)} > \frac{\delta + n}{s}, \tag{7}$$

  then there exists a unique k^* that solves $\frac{f(k^*)}{k^*} = \frac{\delta+n}{s}$, which is the globally stable steady state.
- If the opposite of condition (7) holds, then $\dot{k}(t) < 0$ for any $k(t) > 0$. Capital-labor ratio asymptotes to 0.
- Intuition: When $\sigma < 1$, capital and labor are not substitutable. If productivity is low, then output is unable to replenish the diminished capital and capital falls. As $k \to 0$, average product increases but not sufficiently (in particular, $\lim_{k\to 0} \frac{f(k)}{k} < \infty$) because capital becomes the bottleneck. Thus capital falls towards zero.
- As $k \to 0$, CES with $\sigma < 1$ is qualitatively similar to the Leontieff production function.

Cobb-Douglas

- Recall that CES with $\sigma = 1$ gives $F(K, L) = K^{\gamma} L^{1-\gamma}$.
- This satisfies both sides of the Inada conditions, so there is always an interior steady-state k^*, which has a closed form solution.
- Cobb-Douglas is useful (at the same time very special) because it **always** has constant factor shares.
- To appreciate the generality of this result better, let us also introduce capital and labor-augmenting technology::

$$F(A_K K, A_L L) = (A_K K)^{\gamma} (A_L L)^{1-\gamma}$$

- Note that, taking the derivative of this expression with respect to K gives:

$$\frac{dF}{dK} = \frac{\gamma A_K F}{A_K K}.$$

Rewrite this to get

$$\frac{\frac{dF}{dK} K}{F} = \gamma.$$

The share of capital is always constant and equal to γ. Similarly, the share of labor is always constant and equal to $1 - \gamma$.

Cobb-Douglas

- Note that Cobb-Douglas has constant factor shares regardless of effective factor levels $A_K K$ and $A_L L$.
- Intuition: Elasticity of substitution is equal to 1. As the relative abundance of one factor increases by 1%, its price falls by 1%, the share of the factor remains constant.
- Note that this is not true for CES. For example as $(A_K K) / (A_L L) \to \infty$, it can be seen that:
    - Share of capital in CES with $\sigma > 1$ limits to 1.
    - Share of capital in CES with $\sigma < 1$ limits to 0.

  Intuition?

Uzawa's Theorem

- A representation theorem. Does not say that the production function must be of labor-augmenting form, but rather, that it has to have a representation of that form.
- Two versions of it. Formal statements in Section 2.7.3 of the textbook. I will provide a loose description and focus on intuition.
- **Version 1:** If $K(t), Y(t), C(t)$ grow at constant rates g_K, g_Y, g_C for each $t \geq T$, population grows at constant rate n, and the production function $\tilde{F}\left(K(t), L(t), \tilde{A}(t)\right)$ exhibits constant returns to scale in $K(t)$ and $L(t)$, then:
    1. $g_K = g_Y = g_C$.
    2. Production function has a Labor-augmenting representation, that is, there exists a technology term $A(t)$ that grows at rate $g \equiv g_Y - n$ and a production function $F : \mathbb{R}^2_+ \to \mathbb{R}$ such that

$$\tilde{F}\left(K(t), L(t), \tilde{A}(t)\right) = F(K(t), A(t) L(t)) \text{ for each } t \geq T.$$

- Summary: Balanced growth requires that the production function has a Labor-augmenting representation **along the equilibrium path**. Intuition on the next slide.

Uzawa's Theorem

- Intuition for part 1: If $K(t), Y(t), C(t)$ grow at constant rates, then they should grow at the same rates, otherwise they would get out of proportion and the resource constraint of the economy would be violated.
- Intuition for part 2: Consider the production function

$$Y(t) = \tilde{F}\left(K(t), L(t), \tilde{A}(t)\right). \quad (8)$$

  - Note that $Y(t)$ and $K(t)$ grow at the same rate, while $L(t)$ grows at rate n (suppose $n < g_Y = g_K$, which is the more reasonable case).
  - If technology were constant, Eq. (8) would be violated since $\tilde{F}$ exhibits CRS: This is because, left hand size grows at rate g_Y, while the inputs grow at rates $g_K = g_Y$ (capital) and $n < g_Y$ (labor). There is a slack in the labor input, so right hand side would fall behind.
  - Then, Eq. (8) implies that technology should make up for this slack => Production function can be rewritten (while preserving the CRS property) such that technology augments labor.
- This version does not ensure that the marginal returns $\tilde{F}_K, \tilde{F}_L$ are equal to the marginal returns F_K, F_L (see Exercise 2.19 for a counter-example). There could be an economic loss of generality in considering the Labor-augmenting representation F. A stronger version on the next slide.

Uzawa's Theorem

- **Version 2:** Make all assumptions of version 1, and in addition, assume the rental rate is constant: $R(t) = R^*$ for all $t \geq T$.
  - Note that this is equivalent to assuming capital has constant share, because share of capital is $\frac{R(t)K}{Y} = R^* \frac{K}{Y}$, and K and Y are growing at the same rate in view of Version 1.

  Under this additional assumption, there exists a representation $F(K(t), A(t) L(t))$ such that $F = \tilde{F}$, and in addition:

$$\tilde{F}_K(K(t), L(t), A(t)) = F_K(K(t), A(t)(L(t)))$$
$$\tilde{F}_L(K(t), L(t), A(t)) = \frac{dF(K(t), A(t)(L(t)))}{dL(t)}.$$

- Summary: Balanced growth and constant factor shares (Kaldor facts) require that the production function has a Labor-augmenting representation **in a neighborhood of the equilibrium path**. This is sufficient for most economic purposes, e.g. we can consider first order deviations without loss of generality.

Uzawa's Theorem: Intuition

- To get a better intuition for the second version, restrict attention to the production functions in which technology can be written in factor augmenting form, i.e. suppose there exists $\bar{F} : \mathbb{R}_+^2 \to \mathbb{R}$ and technology functions $A_K(t)$ and $A_L(t)$ such that

$$\tilde{F}\left(K(t), L(t), \tilde{A}(t)\right) = \bar{F}\left(A_K(t) K(t), A_L(t) L(t)\right).$$

- If effective factor ratio $A_K(t) K(t) / (A_L(t) L(t))$ changes over time, then the share of capital (and labor) would change for any production function (except for the Cobb-Douglas function). Thus, (loosely speaking) constant factor shares $=>$ effective factor proportions remain constant.
- Since effective factors grow at the same rate, $Y(t)$ must also grow at the same rate as effective factors because

$$Y(t) = \bar{F}\left(A_K(t) K(t), A_L(t) L(t)\right)$$

and $\bar{F}$ exhibits CRS. But recall that $Y(t)$ and $K(t)$ grow at the same rate (from resource constraints). Thus, $Y(t)$ and $A_K(t) K(t)$ can grow at the same rate only if $A_K(t)$ is constant. Hence, all technological progress should take labor-augmenting form.

Uzawa's Theorem and Cobb-Douglas

- The argument in the previous slide does note apply for the Cobb-Douglas production function, which has constant factor shares regardless of the relative ratio of effective factors.
- To complete the argument, consider the Cobb-Douglas production function allowing for all three kinds of technological progress:

$$A_H(t)\left(A_K(t)K(t)\right)^{\alpha}\left(A_L(t)L(t)\right)^{1-\alpha}$$

  and note that this always has a representation with Labor-augmenting technological progress:

$$K(t)^{\alpha}\left(A(t)L(t)\right)^{1-\alpha} \text{ where } A(t) \equiv A_H(t)^{1/(1-\alpha)}A_K(t)^{\alpha/(1-\alpha)}A_L(t).$$

- Intuitively, with Cobb-Douglas, all kinds of technological progress are qualitatively equivalent since the elasticity of substitution is equal to one. In particular, there is always a labor-augmenting representation.