14.452 Economic Growth: Lecture 2: The Solow Growth Model

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Review of the Discrete-Time Solow Model

- Per capita capital stock evolves according to

$$k(t+1) = sf(k(t)) + (1-\delta)k(t). \tag{1}$$

- The steady-state value of the capital-labor ratio k^* is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}. \tag{2}$$

- Consumption is given by

$$C(t) = (1-s)Y(t) \tag{3}$$

- And factor prices are given by

$$\begin{aligned} R(t) &= f'(k(t)) > 0 \text{ and} \\ w(t) &= f(k(t)) - k(t)f'(k(t)) > 0. \end{aligned} \tag{4}$$

Steady State Equilibrium

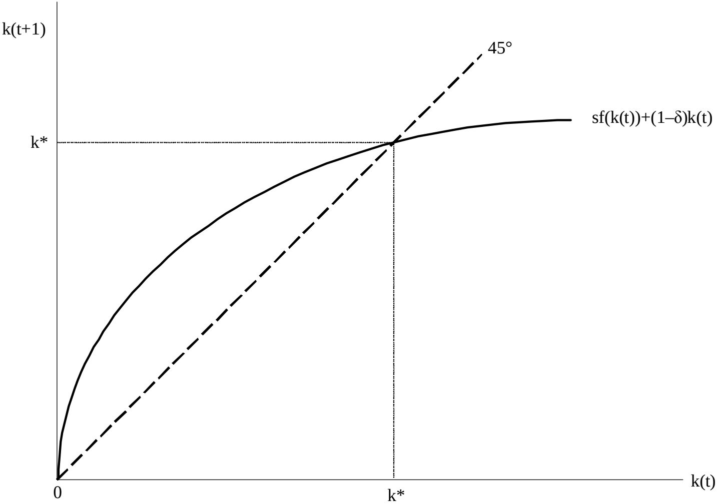

Figure 2.2 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Steady-state capital-labor ratio in the Solow model.

Transitional Dynamics

- *Equilibrium path*: not simply steady state, but entire path of capital stock, output, consumption and factor prices.
    - In engineering and physical sciences, equilibrium is point of rest of dynamical system, thus *the steady state equilibrium*.
    - In economics, non-steady-state behavior also governed by optimizing behavior of households and firms and market clearing.
- Need to study the "transitional dynamics" of the equilibrium difference equation (1) starting from an arbitrary initial capital-labor ratio $k(0) > 0$.
- Key question: whether economy will tend to steady state and how it will behave along the transition path.

Transitional Dynamics: Review I

- Consider the nonlinear system of autonomous difference equations,

$$\mathbf{x}(t+1) = \mathbf{G}(\mathbf{x}(t)), \tag{5}$$

- $\mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$.
- Let $\mathbf{x}^*$ be a fixed point of the mapping $\mathbf{G}(\cdot)$, i.e.,

$$\mathbf{x}^* = \mathbf{G}(\mathbf{x}^*).$$

- $\mathbf{x}^*$ is sometimes referred to as "an equilibrium point" of (5).
- We will refer to $\mathbf{x}^*$ as a stationary point or a *steady state* of (5).

Definition A steady state $\mathbf{x}^*$ is (locally) *asymptotically stable* if there exists an open set $B(\mathbf{x}^*) \ni \mathbf{x}^*$ such that for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$ to (5) with $\mathbf{x}(0) \in B(\mathbf{x}^*)$, we have $\mathbf{x}(t) \to \mathbf{x}^*$. Moreover, $\mathbf{x}^*$ is *globally asymptotically stable* if for all $\mathbf{x}(0) \in \mathbb{R}^n$, for any solution $\{\mathbf{x}(t)\}_{t=0}^{\infty}$, we have $\mathbf{x}(t) \to \mathbf{x}^*$.

Transitional Dynamics: Review II

Simple Result About Stability

- Let $x(t), a, b \in \mathbb{R}$, then the unique steady state of the linear difference equation $x(t+1) = ax(t) + b$ is globally asymptotically stable (in the sense that $x(t) \to x^* = b/(1-a)$) if $|a| < 1$.
- Suppose that $g : \mathbb{R} \to \mathbb{R}$ is differentiable at the steady state x^*, defined by $g(x^*) = x^*$. Then, the steady state of the nonlinear difference equation $x(t+1) = g(x(t))$, x^*, is locally asymptotically stable if $|g'(x^*)| < 1$. Moreover, if $|g'(x)| < 1$ for all $x \in \mathbb{R}$, then x^* is globally asymptotically stable.

Transitional Dynamics in the Discrete Time Solow Model

Proposition Suppose that Assumptions 1 and 2 hold, then the steady-state equilibrium of the Solow growth model described by the difference equation (1) is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t)$ monotonically converges to k^*.

Proof of Proposition: Transitional Dyamics I

- Let $g(k) \equiv sf(k) + (1-\delta)k$. First observe that $g'(k)$ exists and is always strictly positive, i.e., $g'(k) > 0$ for all k.
- Next, from (1),

$$k(t+1) = g(k(t)), \tag{6}$$

  with a unique steady state at k^*.
- From (2), the steady-state capital k^* satisfies $\delta k^* = sf(k^*)$, or

$$k^* = g(k^*). \tag{7}$$

- Recall that $f(\cdot)$ is concave and differentiable from Assumption 1 and satisfies $f(0) \geq 0$ from Assumption 2.

Proof of Proposition: Transitional Dyamics II

- For any strictly concave differentiable function,

$$f(k) > f(0) + kf'(k) \geq kf'(k), \tag{8}$$

- The second inequality uses the fact that $f(0) \geq 0$.
- Since (8) implies that $\delta = sf(k^*)/k^* > sf'(k^*)$, we have $g'(k^*) = sf'(k^*) + 1 - \delta < 1$. Therefore,

$$g'(k^*) \in (0, 1).$$

- The Simple Result then establishes local asymptotic stability.

Proof of Proposition: Transitional Dyamics III

- To prove global stability, note that for all $k(t) \in (0, k^*)$,

$$
\begin{aligned}
k(t+1) - k^* &= g(k(t)) - g(k^*) \\
&= -\int_{k(t)}^{k^*} g'(k)\, dk, \\
&< 0
\end{aligned}
$$

- First line follows by subtracting (7) from (6), second line uses the fundamental theorem of calculus, and third line follows from the observation that $g'(k) > 0$ for all k.

Proof of Proposition: Transitional Dyamics IV

- Next, (1) also implies

$$\begin{aligned} \frac{k(t+1)-k(t)}{k(t)} &= s\frac{f(k(t))}{k(t)}-\delta \\ &> s\frac{f(k^*)}{k^*}-\delta \\ &= 0, \end{aligned}$$

- Second line uses the fact that $f(k)/k$ is decreasing in k (from (8) above) andlast line uses the definition of k^*.
- These two arguments together establish that for all $k(t) \in (0, k^*)$, $k(t+1) \in (k(t), k^*)$.
- An identical argument implies that for all $k(t) > k^*$, $k(t+1) \in (k^*, k(t))$.
- Therefore, $\{k(t)\}_{t=0}^{\infty}$ monotonically converges to k^* and is globally stable.

Transitional Dynamics in the Discrete Time Solow Model III

- Stability result can be seen diagrammatically in the Figure:
  - Starting from initial capital stock $k(0) < k^*$, economy grows towards k^*, *capital deepening* and growth of per capita income.
  - If economy were to start with $k'(0) > k^*$, reach the steady state by decumulating capital and contracting.

Proposition Suppose that Assumptions 1 and 2 hold, and $k(0) < k^*$, then $\{w(t)\}_{t=0}^{\infty}$ is an increasing sequence and $\{R(t)\}_{t=0}^{\infty}$ is a decreasing sequence. If $k(0) > k^*$, the opposite results apply.

- Thus far Solow growth model has a number of nice properties, but no growth, except when the economy starts with $k(0) < k^*$.

Transitional Dynamics in Figure

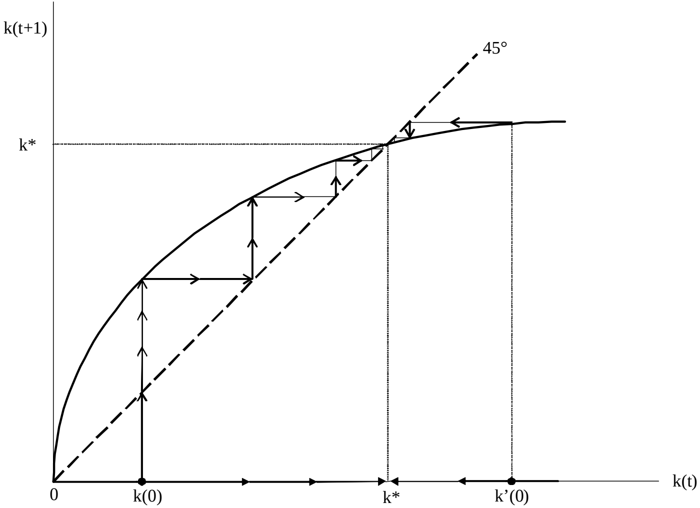

Figure 2.7 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Transitional dynamics in the basic Solow model.

From Difference to Differential Equations I

- Start with a simple difference equation

$$x(t+1) - x(t) = g(x(t)). \tag{9}$$

- Now consider the following approximation for any $\Delta t \in [0, 1]$,

$$x(t+\Delta t) - x(t) \simeq \Delta t \cdot g(x(t)),$$

- When $\Delta t = 0$, this equation is just an identity. When $\Delta t = 1$, it gives (9).
- In-between it is a linear approximation, not too bad if $g(x) \simeq g(x(t))$ for all $x \in [x(t), x(t+1)]$

From Difference to Differential Equations II

- Divide both sides of this equation by Δt, and take limits

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t) \simeq g(x(t)), \tag{10}$$

where

$$\dot{x}(t) \equiv \frac{dx(t)}{dt}$$

- Equation (10) is a differential equation representing (9) for the case in which t and $t+1$ is "small".

The Fundamental Equation of the Solow Model in Continuous Time I

- Nothing has changed on the production side, so (4) still give the factor prices, now interpreted as instantaneous wage and rental rates.
- Savings are again

$$S(t) = sY(t),$$

- Consumption is given by (3) above.
- Introduce population growth,

$$L(t) = \exp(nt) L(0). \tag{11}$$

- Recall

$$k(t) \equiv \frac{K(t)}{L(t)},$$

The Fundamental Equation of the Solow Model in Continuous Time II

- Implies

$$\begin{aligned} \frac{\dot{k}(t)}{k(t)} &= \frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}, \\ &= \frac{\dot{K}(t)}{K(t)} - n. \end{aligned}$$

- From the limiting argument leading to equation (10),

$$\dot{K}(t) = sF[K(t), L(t), A(t)] - \delta K(t).$$

- Using the definition of $k(t)$ and the constant returns to scale properties of the production function,

$$\frac{\dot{k}(t)}{k(t)} = s\frac{f(k(t))}{k(t)} - (n+\delta), \qquad (12)$$

The Fundamental Equation of the Solow Model in Continuous Time III

Definition In the basic Solow model in continuous time with population growth at the rate n, no technological progress and an initial capital stock $K(0)$, an equilibrium path is a sequence of capital stocks, labor, output levels, consumption levels, wages and rental rates $[K(t), L(t), Y(t), C(t), w(t), R(t)]_{t=0}^{\infty}$ such that $L(t)$ satisfies (11), $k(t) \equiv K(t)/L(t)$ satisfies (12), $Y(t)$ is given by the aggregate production function, $C(t)$ is given by (3), and $w(t)$ and $R(t)$ are given by (4).

- As before, *steady-state* equilibrium involves $k(t)$ remaining constant at some level k^*.

Steady State With Population Growth

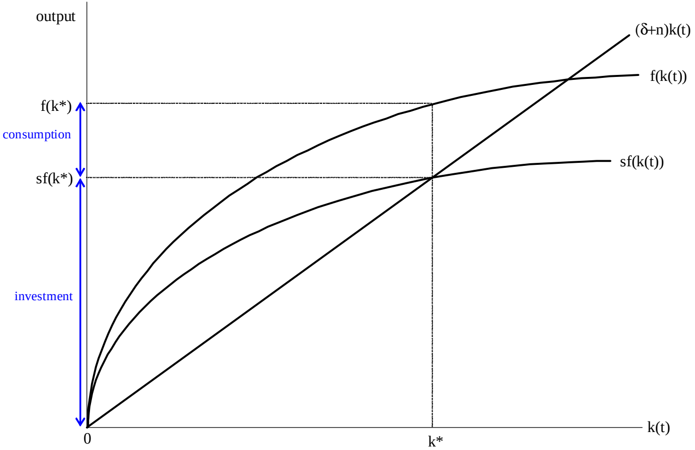

Figure 2.8 in Acemoglu, Daron. *Introduction to Modern Economic Growth*. Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Investment and consumption in the steady-state equilibrium with population growth.

Steady State of the Solow Model in Continuous Time

- Equilibrium path (12) has a unique *steady state* at k^*, which is given by a slight modification of (2) above:

$$\frac{f(k^*)}{k^*} = \frac{n+\delta}{s}. \tag{13}$$

Proposition Consider the basic Solow growth model in continuous time and suppose that Assumptions 1 and 2 hold. Then there exists a unique steady state equilibrium where the capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by (13), per capita output is given by

$$y^* = f(k^*)$$

and per capita consumption is given by

$$c^* = (1-s) f(k^*).$$

Steady State of the Solow Model in Continuous Time II

- Moreover, again defining $f(k) = a\tilde{f}(k)$, we obtain:

Proposition Suppose Assumptions 1 and 2 hold and $f(k) = a\tilde{f}(k)$. Denote the steady-state equilibrium level of the capital-labor ratio by $k^*(a, s, \delta, n)$ and the steady-state level of output by $y^*(a, s, \delta, n)$ when the underlying parameters are given by a, s and δ. Then we have

$$
\begin{aligned}
\frac{\partial k^*(\cdot)}{\partial a} &> 0, \ \frac{\partial k^*(\cdot)}{\partial s} > 0, \ \frac{\partial k^*(\cdot)}{\partial \delta} < 0 \text{ and } \frac{\partial k^*(\cdot)}{\partial n} < 0 \\
\frac{\partial y^*(\cdot)}{\partial a} &> 0, \ \frac{\partial y^*(\cdot)}{\partial s} > 0, \ \frac{\partial y^*(\cdot)}{\partial \delta} < 0 \text{and } \frac{\partial y^*(\cdot)}{\partial n} < 0.
\end{aligned}
$$

- New result is higher n, also reduces the capital-labor ratio and output per capita.
  - means there is more labor to use capital, which only accumulates slowly, thus the equilibrium capital-labor ratio ends up lower.

Transitional Dynamics in the Continuous Time Solow Model I

Simple Result about Stability In Continuous Time Model

- Let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function and suppose that there exists a unique x^* such that $g(x^*) = 0$. Moreover, suppose $g(x) < 0$ for all $x > x^*$ and $g(x) > 0$ for all $x < x^*$. Then the steady state of the nonlinear differential equation $\dot{x}(t) = g(x(t))$, x^*, is globally asymptotically stable, i.e., starting with any $x(0)$, $x(t) \to x^*$.

Simple Result in Figure

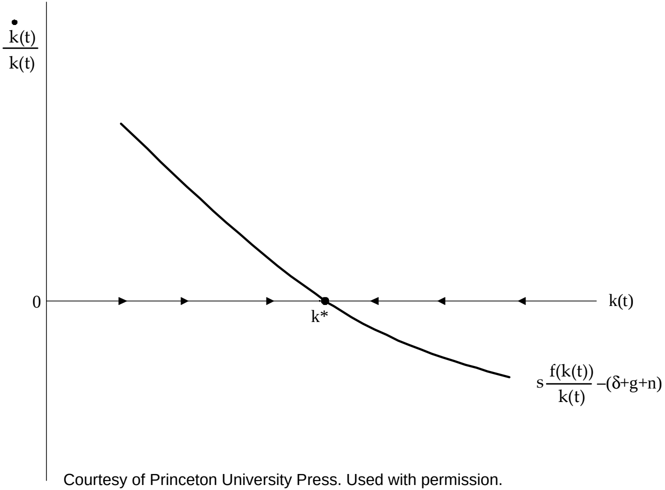

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Figure 2.9 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Dyna cap

Transitional Dynamics in the Continuous Time Solow Model II

Proposition Suppose that Assumptions 1 and 2 hold, then the basic Solow growth model in continuous time with constant population growth and no technological change is globally asymptotically stable, and starting from any $k(0) > 0$, $k(t) \rightarrow k^*$.

- **Proof:** Follows immediately from the Theorem above by noting whenever $k < k^*$, $sf(k) - (n + \delta) k > 0$ and whenever $k > k^*$, $sf(k) - (n + \delta) k < 0$.
- Figure: plots the right-hand side of (12) and makes it clear that whenever $k < k^*$, $\dot{k} > 0$ and whenever $k > k^*$, $\dot{k} < 0$, so k monotonically converges to k^*.

Dynamics with Cobb-Douglas Production Function I

- Return to the Cobb-Douglas Example

$$F[K, L, A] = AK^{\alpha}L^{1-\alpha} \text{ with } 0 < \alpha < 1.$$

- Special, mainly because elasticity of substitution between capital and labor is 1.
- Recall for a homothetic production function $F(K, L)$, the elasticity of substitution is

$$\sigma \equiv -\left[\frac{\partial \ln (F_K/F_L)}{\partial \ln (K/L)}\right]^{-1}, \tag{14}$$

- F is required to be homothetic, so that F_K/F_L is only a function of K/L.
- For the Cobb-Douglas production function $F_K/F_L = (\alpha/(1-\alpha)) \cdot (L/K)$, thus $\sigma = 1$.

Dynamics with Cobb-Douglas Production Function II

- Thus when the production function is Cobb-Douglas and factor markets are competitive, equilibrium factor shares will be constant:

$$\begin{aligned}\alpha_K(t) &= \frac{R(t)K(t)}{Y(t)} \\ &= \frac{F_K(K(t),L(t))K(t)}{Y(t)} \\ &= \frac{\alpha A[K(t)]^{\alpha-1}[L(t)]^{1-\alpha}K(t)}{A[K(t)]^{\alpha}[L(t)]^{1-\alpha}} \\ &= \alpha.\end{aligned}$$

- Similarly, the share of labor is $\alpha_L(t) = 1 - \alpha$.
- Reason: with $\sigma = 1$, as capital increases, its marginal product decreases proportionally, leaving the capital share constant.

Dynamics with Cobb-Douglas Production Function III

- Per capita production function takes the form $f(k) = Ak^{\alpha}$, so the steady state is given again as

$$A(k^*)^{\alpha-1} = \frac{n+\delta}{s}$$

or

$$k^* = \left(\frac{sA}{n+\delta}\right)^{\frac{1}{1-\alpha}},$$

- k^* is increasing in s and A and decreasing in n and δ.
- In addition, k^* is increasing in α: higher α implies less diminishing returns to capital.
- Transitional dynamics are also straightforward in this case:

$$\dot{k}(t) = sA[k(t)]^{\alpha} - (n+\delta)k(t)$$

with initial condition $k(0)$.

Dynamics with Cobb-Douglas Production Function IV

- To solve this equation, let $x(t) \equiv k(t)^{1-\alpha}$,

$$\dot{x}(t) = (1-\alpha) sA - (1-\alpha)(n+\delta) x(t),$$

- General solution

$$x(t) = \frac{sA}{n+\delta} + \left[x(0) - \frac{sA}{n+\delta} \right] \exp\left(-(1-\alpha)(n+\delta) t \right).$$

- In terms of the capital-labor ratio

$$k(t) = \left\{ \frac{sA}{n+\delta} + \left[[k(0)]^{1-\alpha} - \frac{sA}{\delta} \right] \exp\left(-(1-\alpha)(n+\delta) t \right) \right\}^{\frac{1}{1-\alpha}}.$$

Dynamics with Cobb-Douglas Production Function V

- This solution illustrates:
  - starting from any $k(0)$, $k(t) \to k^* = (sA/(n+\delta))^{1/(1-\alpha)}$, and rate of adjustment is related to $(1-\alpha)(n+\delta)$,
  - more specifically, gap between $k(0)$ and its steady-state value is closed at the exponential rate $(1-\alpha)(n+\delta)$.
- Intuitive:
  - higher α, less diminishing returns, slows down rate at which marginal and average product of capital declines, reduces rate of adjustment to steady state.
  - smaller δ and smaller n: slow down the adjustment of capital per worker and thus the rate of transitional dynamics.

Constant Elasticity of Substitution Production Function I

- Imposes a constant elasticity, σ, not necessarily equal to 1.
- Consider a vector-valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$.
- CES production function can be written as

$$\begin{aligned} Y(t) &= F[K(t), L(t), \mathbf{A}(t)] \\ &\equiv A_H(t)\left[\gamma (A_K(t) K(t))^{\frac{\sigma-1}{\sigma}} + (1-\gamma)(A_L(t) L(t))^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}} \end{aligned}$$

- $A_H(t) > 0$, $A_K(t) > 0$ and $A_L(t) > 0$ are three different types of technological change
- $\gamma \in (0, 1)$ is a distribution parameter,

Constant Elasticity of Substitution Production Function II

- $\sigma \in [0, \infty]$ is the elasticity of substitution: easy to verify that

$$\frac{F_K}{F_L} = \frac{\gamma A_K(t)^{\frac{\sigma-1}{\sigma}} K(t)^{-\frac{1}{\sigma}}}{(1-\gamma) A_L(t)^{\frac{\sigma-1}{\sigma}} L(t)^{-\frac{1}{\sigma}}},$$

- Thus, indeed have

$$\sigma = -\left[\frac{\partial \ln(F_K/F_L)}{\partial \ln(K/L)}\right]^{-1}.$$

Constant Elasticity of Substitution Production Function III

- As $\sigma \longrightarrow 1$, the CES production function converges to the Cobb-Douglas

$$Y(t) = A_H(t)(A_K(t))^{\gamma}(A_L(t))^{1-\gamma}(K(t))^{\gamma}(L(t))^{1-\gamma}$$

- As $\sigma \longrightarrow \infty$, the CES production function becomes linear, i.e.

$$Y(t) = \gamma A_H(t) A_K(t) K(t) + (1-\gamma) A_H(t) A_L(t) L(t).$$

- Finally, as $\sigma \longrightarrow 0$, the CES production function converges to the Leontief production function with no substitution between factors,

$$Y(t) = A_H(t) \min\{\gamma A_K(t) K(t); (1-\gamma) A_L(t) L(t)\}.$$

- Leontief production function: if $\gamma A_K(t) K(t) \neq (1-\gamma) A_L(t) L(t)$, either capital or labor will be partially "idle".

A First Look at Sustained Growth I

- Cobb-Douglas already showed that when α is close to 1, adjustment to steady-state level can be very slow.
- Simplest model of sustained growth essentially takes $\alpha = 1$ in terms of the Cobb-Douglas production function above.
- Relax Assumptions 1 and 2 and suppose

$$F\left[K\left(t\right),L\left(t\right),A\left(t\right)\right]=AK\left(t\right), \tag{15}$$

  where $A > 0$ is a constant.
- So-called "AK" model, and in its simplest form output does not even depend on labor.
- Results we would like to highlight apply with more general constant returns to scale production functions,

$$F\left[K\left(t\right),L\left(t\right),A\left(t\right)\right]=AK\left(t\right)+BL\left(t\right), \tag{16}$$

A First Look at Sustained Growth II

- Assume population grows at n as before (cfr. equation (11)).
- Combining with the production function (15),

$$\frac{\dot{k}(t)}{k(t)} = sA - \delta - n.$$

- Therefore, if $sA - \delta - n > 0$, there will be sustained growth in the capital-labor ratio.
- From (15), this implies that there will be sustained growth in output per capita as well.

A First Look at Sustained Growth III

Proposition Consider the Solow growth model with the production function (15) and suppose that $sA - \delta - n > 0$. Then in equilibrium, there is sustained growth of output per capita at the rate $sA - \delta - n$. In particular, starting with a capital-labor ratio $k(0) > 0$, the economy has

$$k(t) = \exp((sA - \delta - n)t)\, k(0)$$

and

$$y(t) = \exp((sA - \delta - n)t)\, Ak(0).$$

- Note no transitional dynamics.

Sustained Growth in Figure

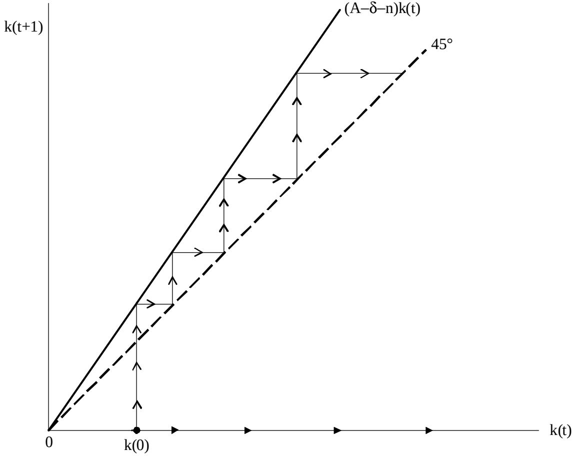

Figure 2.10 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Sustained growth with the linear AK technology with $sA - \delta - n > 0$.

A First Look at Sustained Growth IV

- Unattractive features:
  - Knife-edge case, requires the production function to be ultimately linear in the capital stock.
  - Implies that as time goes by the share of national income accruing to capital will increase towards 1.
  - Technological progress seems to be a major (perhaps the most major) factor in understanding the process of economic growth.

Balanced Growth I

- Production function $F[K(t), L(t), A(t)]$ is too general.
- May not have *balanced growth*, i.e. a path of the economy consistent with the *Kaldor facts* (Kaldor, 1963).
- Kaldor facts:
    - while output per capita increases, the capital-output ratio, the interest rate, and the distribution of income between capital and labor remain roughly constant.

Historical Factor Shares

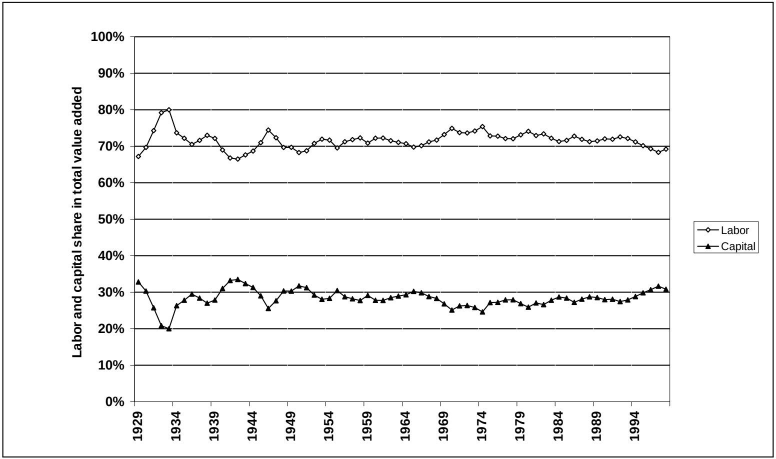

Figure 2.11 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
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Figure: Capital and Labor Share in the U.S. GDP.

Balanced Growth II

- Note capital share in national income is about $1/3$, while the labor share is about $2/3$.
- Ignoring land, not a major factor of production.
- But in poor countries land is a major factor of production.
- This pattern often makes economists choose $AK^{1/3}L^{2/3}$.
- Main advantage from our point of view is that balanced growth is the same as a steady-state in transformed variables
  - i.e., we will again have $\dot{k} = 0$, but the definition of k will change.
- But important to bear in mind that growth has many non-balanced features.
  - e.g., the share of different sectors changes systematically.

Types of Neutral Technological Progress I

- For some constant returns to scale function $\tilde{F}$:
  - *Hicks-neutral* technological progress:

$$\tilde{F}\left[K(t), L(t), A(t)\right] = A(t) F\left[K(t), L(t)\right],$$

    - Relabeling of the isoquants (without any change in their shape) of the function $\tilde{F}\left[K(t), L(t), A(t)\right]$ in the L-K space.
  - *Solow-neutral* technological progress,

$$\tilde{F}\left[K(t), L(t), A(t)\right] = F\left[A(t) K(t), L(t)\right].$$

    - Capital-augmenting progress: isoquants shifting with technological progress in a way that they have constant slope at a given labor-output ratio.
  - *Harrod-neutral* technological progress,

$$\tilde{F}\left[K(t), L(t), A(t)\right] = F\left[K(t), A(t) L(t)\right].$$

    - Increases output as if the economy had more labor: slope of the isoquants are constant along rays with constant capital-output ratio.

Isoquants with Neutral Technological Progress

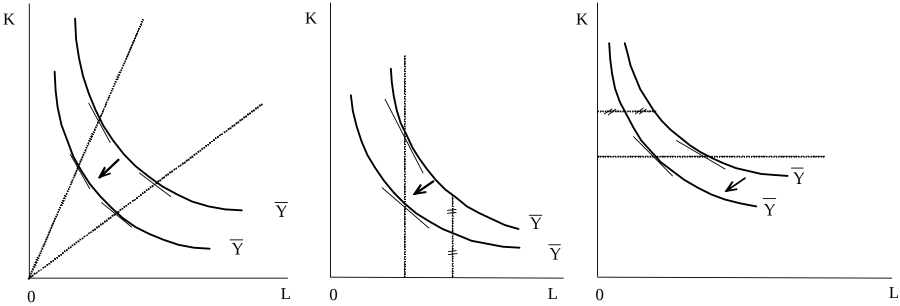

Figure 2.12 in Acemoglu, Daron. *Introduction to Modern Economic Growth*.
Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Hicks-neutral, Solow-neutral and Harrod-neutral shifts in isoquants.

Types of Neutral Technological Progress II

- Could also have a vector valued index of technology $\mathbf{A}(t) = (A_H(t), A_K(t), A_L(t))$ and a production function

$$\tilde{F}[K(t), L(t), \mathbf{A}(t)] = A_H(t) F[A_K(t) K(t), A_L(t) L(t)], \quad (17)$$

- Nests the constant elasticity of substitution production function introduced in the Example above.
- But even (17) is a restriction on the form of technological progress, $A(t)$ could modify the entire production function.
- Balanced growth necessitates that all technological progress be labor augmenting or Harrod-neutral.

Uzawa's Theorem I

- Focus on continuous time models.
- Key elements of balanced growth: constancy of factor shares and of the capital-output ratio, $K(t)/Y(t)$.
- By factor shares, we mean

$$\alpha_L(t) \equiv \frac{w(t)L(t)}{Y(t)} \text{ and } \alpha_K(t) \equiv \frac{R(t)K(t)}{Y(t)}.$$

- By Assumption 1 and Euler Theorem $\alpha_L(t) + \alpha_K(t) = 1$.

Uzawa's Theorem II

Theorem

(Uzawa I) *Suppose* $L(t) = \exp(nt) L(0)$,

$$Y(t) = \tilde{F}(K(t), L(t), \tilde{A}(t)),$$

$\dot{K}(t) = Y(t) - C(t) - \delta K(t)$, *and* $\tilde{F}$ *is CRS in* K *and* L. *Suppose for* $\tau < \infty$, $\dot{Y}(t)/Y(t) = g_Y > 0$, $\dot{K}(t)/K(t) = g_K > 0$ *and* $\dot{C}(t)/C(t) = g_C > 0$. *Then,*

- $g_Y = g_K = g_C$; *and*
- *for any* $t \geq \tau$, $\tilde{F}$ *can be represented as*

$$Y(t) = F(K(t), A(t) L(t)),$$

where $A(t) \in \mathbb{R}_+$, $F : \mathbb{R}_+^2 \to \mathbb{R}_+$ *is homogeneous of degree 1, and*

$$\dot{A}(t)/A(t) = g = g_Y - n.$$

Proof of Uzawa's Theorem I

- By hypothesis, $Y(t) = \exp(g_Y(t-\tau))\, Y(\tau)$, $K(t) = \exp(g_K(t-\tau))\, K(\tau)$ and $L(t) = \exp(n(t-\tau))\, L(\tau)$ for some $\tau < \infty$.
- Since for $t \geq \tau$, $\dot{K}(t) = g_K K(t) = I(t) - C(t) - \delta K(t)$, we have

$$(g_K + \delta)\, K(t) = Y(t) - C(t).$$

- Then,

$$\begin{aligned}(g_K + \delta)\, K(\tau) &= \exp((g_Y - g_K)(t-\tau))\, Y(\tau) \\ &\quad - \exp((g_C - g_K)(t-\tau))\, C(\tau)\end{aligned}$$

for all $t \geq \tau$.

Proof of Uzawa's Theorem II

- Differentiating with respect to time

$$\begin{aligned} 0 &= (g_Y - g_K) \exp\left((g_Y - g_K)(t-\tau)\right) Y(\tau) \\ &\quad - (g_C - g_K) \exp\left((g_C - g_K)(t-\tau)\right) C(\tau) \end{aligned}$$

  for all $t \geq \tau$.

- This equation can hold for all $t \geq \tau$
  1. if $g_Y = g_C$ and $Y(\tau) = C(\tau)$, which is not possible, since $g_K > 0$.
  2. or if $g_Y = g_K$ and $C(\tau) = 0$, which is not possible, since $g_C > 0$ and $C(\tau) > 0$.
  3. or if $g_Y = g_K = g_C$, which must thus be the case.
- Therefore, $g_Y = g_K = g_C$ as claimed in the first part of the theorem.

Proof of Uzawa's Theorem III

- Next, the aggregate production function for time $\tau' \geq \tau$ and any $t \geq \tau$ can be written as

$$\exp\left(-g_Y\left(t-\tau'\right)\right)Y\left(t\right) = \tilde{F}\left[\exp\left(-g_K\left(t-\tau'\right)\right)K\left(t\right),\exp\left(-n\left(t-\tau'\right)\right)L\left(t\right),\tilde{A}\left(\tau'\right)\right.$$

- Multiplying both sides by $\exp\left(g_Y\left(t-\tau'\right)\right)$ and using the constant returns to scale property of F, we obtain

$$Y\left(t\right) = \tilde{F}\left[e^{\left(t-\tau'\right)\left(g_Y-g_K\right)}K\left(t\right), e^{\left(t-\tau'\right)\left(g_Y-n\right)}L\left(t\right), \tilde{A}\left(\tau'\right)\right].$$

- From part 1, $g_Y = g_K$, therefore

$$Y\left(t\right) = \tilde{F}\left[K\left(t\right), \exp\left(\left(t-\tau'\right)\left(g_Y-n\right)\right)L\left(t\right), \tilde{A}\left(\tau'\right)\right].$$

Proof of Uzawa's Theorem IV

- Moreover, this equation is true for $t \geq \tau$ regardless of τ', thus

$$\begin{aligned} Y(t) &= F\left[K(t), \exp\left(\left(g_Y - n\right)t\right)L(t)\right], \\ &= F\left[K(t), A(t)L(t)\right], \end{aligned}$$

  with

$$\frac{\dot{A}(t)}{A(t)} = g_Y - n$$

  establishing the second part of the theorem.

Implications of Uzawa's Theorem

Corollary Under the assumptions of Uzawa Theorem, after time τ technological progress can be represented as Harrod neutral (purely labor augmenting).

- Remarkable feature: stated and proved without any reference to equilibrium behavior or market clearing.
- Also, contrary to Uzawa's original theorem, not stated for a balanced growth path but only for an asymptotic path with constant rates of output, capital and consumption growth.
- **But**, not as general as it seems;
  - the theorem gives only one representation.

Stronger Theorem

Theorem

(Uzawa's Theorem II) *Suppose that all of the hypothesis in Uzawa's Theorem are satisfied, so that* $\tilde{F} : \mathbb{R}_+^2 \times \mathcal{A} \to \mathbb{R}_+$ *has a representation of the form* $F(K(t), A(t)L(t))$ *with* $A(t) \in \mathbb{R}_+$ *and* $\dot{A}(t)/A(t) = g = g_Y - n$*. In addition, suppose that factor markets are competitive and that for all* $t \geq T$*, the rental rate satisfies* $R(t) = R^*$ *(or equivalently,* $\alpha_K(t) = \alpha_K^*$*). Then, denoting the partial derivatives of* $\tilde{F}$ *and* F *with respect to their first two arguments by* $\tilde{F}_K$*,* $\tilde{F}_L$*,* F_K *and* F_L*, we have*

$$\begin{aligned} \tilde{F}_K\left(K(t), L(t), \tilde{A}(t)\right) &= F_K\left(K(t), A(t)L(t)\right) \text{ and} \\ \tilde{F}_L\left(K(t), L(t), \tilde{A}(t)\right) &= A(t) F_L\left(K(t), A(t)L(t)\right). \end{aligned} \tag{18}$$

Moreover, if (18) holds and factor markets are competitive, then $R(t) = R^*$ *(and* $\alpha_K(t) = \alpha_K^*$*) for all* $t \geq T$*.*

Intuition

- Suppose the labor-augmenting representation of the aggregate production function applies.
- Then note that with competitive factor markets, as $t \geq \tau$,

$$
\begin{aligned}
\alpha_K(t) &\equiv \frac{R(t)K(t)}{Y(t)} \\
&= \frac{K(t)}{Y(t)} \frac{\partial F[K(t), A(t)L(t)]}{\partial K(t)} \\
&= \alpha_K^*,
\end{aligned}
$$

- Second line uses the definition of the rental rate of capital in a competitive market
- Third line uses that $g_Y = g_K$ and $g_K = g + n$ from Uzawa Theorem and that F exhibits constant returns to scale so its derivative is homogeneous of degree 0.

Intuition for the Uzawa's Theorems

- We assumed the economy features capital accumulation in the sense that $g_K > 0$.
- From the aggregate resource constraint, this is only possible if output and capital grow at the same rate.
- Either this growth rate is equal to n and there is no technological change (i.e., proposition applies with $g = 0$), or the economy exhibits growth of per capita income and capital-labor ratio.
- The latter case creates an asymmetry between capital and labor: capital is accumulating faster than labor.
- Constancy of growth requires technological change to make up for this asymmetry
- But this intuition does not provide a reason for why technology should take labor-augmenting (Harrod-neutral) form.
- But if technology did not take this form, an asymptotic path with constant growth rates would not be possible.

Interpretation

- Distressing result:
  - Balanced growth is only possible under a very stringent assumption.
  - Provides no reason why technological change should take this form.
- But when technology is endogenous, intuition above also works to make technology endogenously more labor-augmenting than capital augmenting.
- Not only requires labor augmenting asymptotically, i.e., along the balanced growth path.
- This is the pattern that certain classes of endogenous-technology models will generate.

Implications for Modeling of Growth

- Does not require $Y(t) = F[K(t), A(t)L(t)]$, but only that it has a representation of the form $Y(t) = F[K(t), A(t)L(t)]$.
- Allows one important exception. If,

$$Y(t) = [A_K(t)K(t)]^{\alpha} [A_L(t)L(t)]^{1-\alpha},$$

  then both $A_K(t)$ and $A_L(t)$ could grow asymptotically, while maintaining balanced growth.
- Because we can define $A(t) = [A_K(t)]^{\alpha/(1-\alpha)} A_L(t)$ and the production function can be represented as

$$Y(t) = [K(t)]^{\alpha} [A(t)L(t)]^{1-\alpha}.$$

- Differences between labor-augmenting and capital-augmenting (and other forms) of technological progress matter when the elasticity of substitution between capital and labor is not equal to 1.

Further Intuition

- Suppose the production function takes the special form $F\left[A_K\left(t\right)K\left(t\right), A_L\left(t\right)L\left(t\right)\right]$.
- The stronger theorem implies that factor shares will be constant.
- Given constant returns to scale, this can only be the case when $A_K\left(t\right)K\left(t\right)$ and $A_L\left(t\right)L\left(t\right)$ grow at the same rate.
- The fact that the capital-output ratio is constant in steady state (or the fact that capital accumulates) implies that $K\left(t\right)$ must grow at the same rate as $A_L\left(t\right)L\left(t\right)$.
- Thus balanced growth can only be possible if $A_K\left(t\right)$ is asymptotically constant.

The Solow Growth Model with Technological Progress: Continuous Time I

- From Uzawa Theorem, production function must admit representation of the form

$$Y(t) = F[K(t), A(t)L(t)],$$

- Moreover, suppose

$$\frac{\dot{A}(t)}{A(t)} = g, \tag{19}$$

$$\frac{\dot{L}(t)}{L(t)} = n.$$

- Again using the constant saving rate

$$\dot{K}(t) = sF[K(t), A(t)L(t)] - \delta K(t). \tag{20}$$

The Solow Growth Model with Technological Progress: Continuous Time II

- Now define $k(t)$ as the *effective capital-labor* ratio, i.e.,

$$k(t) \equiv \frac{K(t)}{A(t)L(t)}. \tag{21}$$

- Slight but useful abuse of notation.
- Differentiating this expression with respect to time,

$$\frac{\dot{k}(t)}{k(t)} = \frac{\dot{K}(t)}{K(t)} - g - n. \tag{22}$$

- Output per unit of effective labor can be written as

$$\begin{aligned} \hat{y}(t) &\equiv \frac{Y(t)}{A(t)L(t)} = F\left[\frac{K(t)}{A(t)L(t)}, 1\right] \\ &\equiv f(k(t)). \end{aligned}$$

The Solow Growth Model with Technological Progress: Continuous Time III

- Income per capita is $y(t) \equiv Y(t)/L(t)$, i.e.,

$$\begin{aligned} y(t) &= A(t)\hat{y}(t) \qquad (23) \\ &= A(t) f(k(t)). \end{aligned}$$

- Clearly if $\hat{y}(t)$ is constant, income per capita, $y(t)$, will grow over time, since $A(t)$ is growing.
- Thus should not look for "steady states" where income per capita is constant, but for *balanced growth paths*, where income per capita grows at a constant rate.
- Some transformed variables such as $\hat{y}(t)$ or $k(t)$ in (22) remain constant.
- Thus balanced growth paths can be thought of as steady states of a transformed model.

The Solow Growth Model with Technological Progress: Continuous Time IV

- Hence use the terms "steady state" and balanced growth path interchangeably.
- Substituting for $\dot{K}(t)$ from (20) into (22):

$$\frac{\dot{k}(t)}{k(t)} = \frac{sF[K(t), A(t)L(t)]}{K(t)} - (\delta + g + n).$$

- Now using (21),

$$\frac{\dot{k}(t)}{k(t)} = \frac{sf(k(t))}{k(t)} - (\delta + g + n), \qquad (24)$$

- Only difference is the presence of g: k is no longer the capital-labor ratio but the *effective* capital-labor ratio.

The Solow Growth Model with Technological Progress: Continuous Time V

Proposition Consider the basic Solow growth model in continuous time, with Harrod-neutral technological progress at the rate g and population growth at the rate n. Suppose that Assumptions 1 and 2 hold, and define the effective capital-labor ratio as in (21). Then there exists a unique steady state (balanced growth path) equilibrium where the effective capital-labor ratio is equal to $k^* \in (0, \infty)$ and is given by

$$\frac{f(k^*)}{k^*} = \frac{\delta + g + n}{s}. \tag{25}$$

Per capita output and consumption grow at the rate g.

The Solow Growth Model with Technological Progress: Continuous Time VI

- Equation (25), emphasizes that now total savings, $sf(k)$, are used for replenishing the capital stock for three distinct reasons:
  1. depreciation at the rate δ.
  2. population growth at the rate n, which reduces capital per worker.
  3. Harrod-neutral technological progress at the rate g.
- Now replenishment of effective capital-labor ratio requires investments to be equal to $(\delta + g + n)k$.

The Solow Growth Model with Technological Progress: Continuous Time VII

Proposition Suppose Assumptions 1 and 2 hold and let $A(0)$ be the initial level of technology. Denote the balanced growth path level of effective capital-labor ratio by $k^*(A(0), s, \delta, n)$ and the level of output per capita by $y^*(A(0), s, \delta, n, t)$. Then

$$\begin{aligned}
\frac{\partial k^*(A(0), s, \delta, n)}{\partial A(0)} &= 0, \ \frac{\partial k^*(A(0), s, \delta, n)}{\partial s} > 0, \\
\frac{\partial k^*(A(0), s, \delta, n)}{\partial n} &< 0 \text{ and } \frac{\partial k^*(A(0), s, \delta, n)}{\partial \delta} < 0,
\end{aligned}$$

and also

$$\begin{aligned}
\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial A(0)} &> 0, \ \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial s} > 0, \\
\frac{\partial y^*(A(0), s, \delta, n, t)}{\partial n} &< 0 \text{ and } \frac{\partial y^*(A(0), s, \delta, n, t)}{\partial \delta} < 0,
\end{aligned}$$

for each t.

The Solow Growth Model with Technological Progress: Continuous Time VIII

Proposition Suppose that Assumptions 1 and 2 hold, then the Solow growth model with Harrod-neutral technological progress and population growth in continuous time is asymptotically stable, i.e., starting from any $k(0) > 0$, the effective capital-labor ratio converges to a steady-state value k^* ($k(t) \rightarrow k^*$).

- Now model generates growth in output per capita, but entirely *exogenously*.

Comparative Dynamics I

- Comparative dynamics: dynamic response of an economy to a change in its parameters or to shocks.
- Different from comparative statics in Propositions above in that we are interested in the entire path of adjustment of the economy following the shock or changing parameter.
- For brevity we will focus on the continuous time economy.
- Recall

$$\dot{k}(t)/k(t) = sf(k(t))/k(t) - (\delta + g + n)$$

Comparative Dynamics in Figure

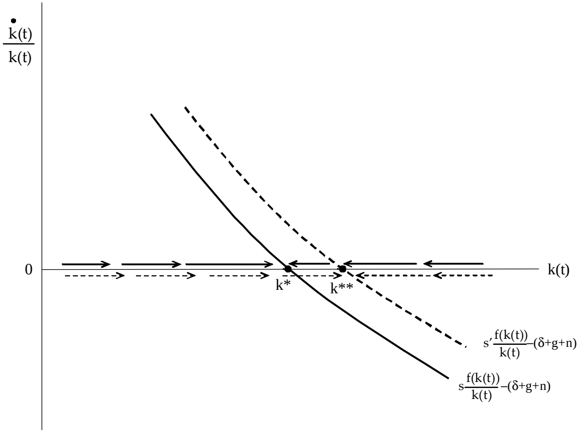

Figure 2.13 in Acemoglu, Daron. *Introduction to Modern Economic Growth.* Princeton, NJ: Princeton University Press, 2009. ISBN: 9780691132921.

Figure: Dynamics following an increase in the savings rate from s to s'. The solid arrows show the dynamics for the initial steady state, while the dashed arrows dynamics adv

Comparative Dynamics II

- One-time, unanticipated, permanent increase in the saving rate from s to s'.
  - Shifts curve to the right as shown by the dotted line, with a new intersection with the horizontal axis, k^{**}.
  - Arrows on the horizontal axis show how the effective capital-labor ratio adjusts gradually to k^{**}.
  - Immediately, the capital stock remains unchanged (since it is a *state* variable).
  - After this point, it follows the dashed arrows on the horizontal axis.
- s changes in unanticipated manner at $t = t'$, but will be reversed back to its original value at some known future date $t = t'' > t'$.
  - Starting at t', the economy follows the rightwards arrows until t'.
  - After t'', the original steady state of the differential equation applies and leftwards arrows become effective.
  - From t'' onwards, economy gradually returns back to its original balanced growth equilibrium, k^*.

Conclusions

- Simple and tractable framework, which allows us to discuss capital accumulation and the implications of technological progress.
- Solow model shows us that if there is no technological progress, and as long as we are not in the AK world, there will be no sustained growth.
- Generate per capita output growth, but only exogenously: technological progress is a blackbox.
- Capital accumulation: determined by the saving rate, the depreciation rate and the rate of population growth. All are exogenous.
- Need to dig deeper and understand what lies in these black boxes.

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