14.452: Economic Growth
Problem Set 2

Due date: November 17, 2009 in Lecture.

Exercise 1: Consider an economy with $N < \infty$ goods, denoted by $j \in \{1, .., N\}$, and a set $\mathcal{H}$ of households. Suppose each household $h \in \mathcal{H}$ has total income w^h and CES preferences for the goods given by

$$U^h\left(x_1^h, ..., x_N^h\right) = \left[\sum_{j=1}^{N}\left(x_j^h - \xi_j^h\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}},$$

where $\sigma \in (0, \infty)$ and $\xi_j^h \in [-\xi, \xi]$.

1. Derive the utility maximizing demand and the indirect utility function for each household. Show that Theorem 5.2 of the textbook applies to this economy, and derive the indirect utility function of the representative household.

2. Consider a household with total income $w \equiv \int_{\mathcal{H}} w^h dh$ and CES preferences given by

$$U\left(x_1, ..., x_N\right) = \left[\sum_{j=1}^{N}\left(x_j - \xi_j\right)^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, \qquad (1)$$

   where $\xi_j = \int_{\mathcal{H}} \xi_j^h dh$. Derive the indirect utility function for this household and show that it agrees with the indirect utility function of the representative household obtained in part 1.

3. Now suppose that $U^h\left(x_1^h, ..., x_N^h\right) = \sum_{j=1}^{N}\left(x_j^h - \xi_j^h\right)^{\frac{\sigma-1}{\sigma}}$ with $\sigma > 1$. Repeat the same computations and verify that the resulting indirect utility function is homogeneous of degree 0 in p and w^h, but does not satisfy the Gorman form. Show, however, that a monotone transformation of the indirect utility function satisfies the Gorman form. Is this sufficient to ensure that the economy admits a representative household?

Exercise 2: Consider an economy consisting of N households each with utility function at time $t = 0$ given by

$$\sum_{t=0}^{\infty} \beta^t u\left(c^h(t)\right),$$

with $\beta \in (0, 1)$, where $c^h(t)$ denotes the consumption of household h at time t. Suppose that $u(0) = 0$. The economy starts with an endowment of $y > 0$ units of the final good and has access to no production technology. This endowment can be saved without depreciating or gaining interest rate between periods.

1. What are the Arrow-Debreu commodities in this economy?
2. Characterize the set of Pareto optimal allocations of this economy.
3. Prove that the Second Welfare Theorem (Theorem 5.7 in the textbook) can be applied to this economy.
4. Now consider an allocation of y units to the households, $\left\{y^h\right\}_{h=1}^N$, such that $\sum_{h=1}^N y^h = y$. Given this allocation, find the unique competitive equilibrium price vector and the corresponding consumption allocations.
5. Are all competitive equilibria Pareto optimal?
6. Now derive a redistribution scheme for decentralizing the entire set of Pareto optimal allocations?

Exercise 3: Consider the basic neoclassical growth model with CRRA preferences, but with consumer heterogeneity in initial asset holdings (you may assume no technological change if you wish). In particular, there is a set $\mathcal{H}$ of household and household $h \in \mathcal{H}$ starts with initial assets $a_h(0)$. Households are otherwise identical.

1. Characterize the competitive equilibrium of this economy and show that the behavior of per capita variables is identical to that in a representative household economy, with the representative household starting with assets $a(0) = |\mathcal{H}|^{-1} \int_{\mathcal{H}} a_h(0)\, dh$, where $|\mathcal{H}|$ is the measure (number) of households in this economy. Interpret this result and relate it to Theorem 5.2 of the textbook.
2. Show that if, instead of the natural debt limit or the no-Ponzi condition, we impose $a_h(t) \geq 0$ for all $h \in \mathcal{H}$ and for all t, then a different equilibrium allocation may result. In light of this finding, discuss whether (and when) it is appropriate to use a no-borrowing constraint instead of no-Ponzi condition.

Exercise 4: Consider a neoclassical growth model augmented with labor supply decisions. In particular, total population is normalized to 1 and all households have utility function $\int_0^\infty \exp(-\rho t)\, u(c(t), 1 - l(t))\, dt$, where $l(t) \in (0,1)$ is labor supply. In a symmetric equilibrium, employment $L(t)$ is equal to $l(t)$. Assume that the production function is $Y(t) = F(K(t), A(t) L(t))$ and $A(t) = \exp(gt) A(0)$.

1. Define a competitive equilibrium.

2. Set up the current-value Hamiltonian that each household solves taking wages and interest rates as given, and determine the necessary and sufficient conditions for the allocation of consumption over time and leisure-labor trade off.

3. Set up the current-value Hamiltonian for a planner maximizing the utility of the representative household, and derive the necessary and sufficient conditions for a solution.

4. Show that the two problems are equivalent given competitive markets.

5. Consider a BGP equilibrium that is consistent with Kaldor facts and that features a constant labor supply $l(t) \equiv l^*$. Show that along the BGP, the utility function needs to have a representation of the form

$$u(c(t), 1 - l(t)) = \begin{cases} \frac{Ac(t)^{1-\theta}}{1-\theta} h(1 - l(t)) & \text{for } \theta \neq 1, \\ A \log c(t) + Bh(1 - l(t)) & \text{for } \theta = 1, \end{cases}$$

   for some $h(\cdot)$ with $h'(\cdot) > 0$. [Hint: to simplify you may assume that the intertemporal elasticity of substitution for consumption, $\varepsilon_u \equiv -u_{cc}c/u_c$ is only a function of c]. Provide an intuition for this functional form in terms of income and substitution effects.

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