14.451 Lecture Notes 10

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1 Continuous time: finite horizon

Time goes from 0 to T. Instantaneous payoff:

$$f(t, x(t), y(t)),$$

(the time dependence includes discounting), where $x(t) \in X$ and $y(t) \in Y$.

Constraint:

$$\dot{x}(t) = g(t, x(t), y(t)). \tag{1}$$

Relative to discrete time, we are now using the state variable/control variable approach (x is the state, y is the control). We assume throughout that f and g are continuously differentiable functions.

Problem is to choose the functions $x(.)$ and $y(.)$ to maximize

$$\int_0^T f(t, x(t), y(t)) \, dt$$

subject to constraint (1) for each $t \in [0, T]$ and the initial condition: $x(0)$ given. We will use two approaches, a variational approach and a dynamic programming approach. Both approaches will be used as heuristic arguments for the Principle of Optimality.

1.1 Variational approach

1.1.1 Necessity argument

Suppose you know that optimal $x^*(t)$ and $y^*(t)$ exist and are interior for each t. We use a variational approach to derive necessary conditions for optimality.

Take any continuous function $h(t)$ and let

$$y(t, \varepsilon) = y^*(t) + \varepsilon h(t).$$

To obtain the perturbed version of x, solve the differential equation

$$\dot{x}(t, \varepsilon) = g(t, x(t, \varepsilon), y(t, \varepsilon)) \tag{2}$$

with initial condition $x(0,\varepsilon) = x(0)$. For $|\varepsilon|$ small enough both $y(t,\varepsilon)$ and $x(t,\varepsilon)$ are, respectively, in Y and X.

Optimality implies that

$$W(\varepsilon) \equiv \int_0^T f(t, x(t,\varepsilon), y(t,\varepsilon))\, dt \leq \int_0^T f(t, x^*(t), y^*(t))\, dt = W(0)$$

for all ε in a neighborhood of 0, and, supposing, W is differentiable this implies

$$W'(0) = 0.$$

Moreover, (2) implies that

$$W(\varepsilon) = \int_0^T f(t, x(t,\varepsilon), y(t,\varepsilon))\, dt + \int_0^T \lambda(t) [g(t, x(t,\varepsilon), y(t,\varepsilon)) - \dot{x}(t,\varepsilon)]\, dt$$

for any continuous function $\lambda(t)$. Integrating by parts, this can be rewritten as

$$\begin{aligned} W(\varepsilon) &= \int_0^T [f(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t) g(t, x(t,\varepsilon), y(t,\varepsilon))]\, dt + \\ &\quad -\lambda(T) x(T,\varepsilon) + \lambda(0) x(0,\varepsilon) + \int_0^T \dot{\lambda}(t) x(t,\varepsilon)\, dt \end{aligned}$$

Suppose we can differentiate $x(t,\varepsilon)$ and $y(t,\varepsilon)$, we have that

$$\begin{aligned} W'(\varepsilon) &= \int_0^T \left[f_x(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t) g_x(t, x(t,\varepsilon), y(t,\varepsilon)) + \dot{\lambda}(t)\right] x_\varepsilon(t,\varepsilon)\, dt \\ &\quad + \int_0^T [f_y(t, x(t,\varepsilon), y(t,\varepsilon)) + \lambda(t) g_y(t, x(t,\varepsilon), y(t,\varepsilon))] y_\varepsilon(t,\varepsilon)\, dt \\ &\quad -\lambda(T) x_\varepsilon(T,\varepsilon) \end{aligned}$$

(given that $x(0,\varepsilon) = x(0)$). Now we will choose the function λ so that all the derivatives $x_\varepsilon(T,\varepsilon)$ (which are hard to compute since they involve the differential equation (2)) vanish.

In particular set $\lambda(T) = 0$ and let $\lambda(t)$ solve the differential equation

$$\dot{\lambda}(t) = -f_x(t, x^*(t), y^*(t)) - \lambda(t) g_x(t, x^*(t), y^*(t)).$$

Moreover $y_\varepsilon(t,\varepsilon) = h(t)$ so putting together our results we have:

$$W'(0) = \int_0^T [f_y(t, x^*(t), y^*(t)) + \lambda(t) g_y(t, x^*(t), y^*(t))] h(t)\, dt = 0.$$

Since this must be true for any choice of $h(t)$ it follows that a necessary condition for this to be satisfies is

$$f_y(t, x^*(t), y^*(t)) + \lambda(t) g_y(t, x^*(t), y^*(t)) = 0.$$

We can summarize this result defining the Hamiltonian

$$H(t, x(t), y(t), \lambda(t)) = f(t, x(t), y(t)) + \lambda(t) g(t, x(t), y(t)),$$

and we have the following:

Theorem 1 *If x^* and y^* are optimal, continuous and interior then there exists a continuously differentiable function $\lambda(t)$ such that*

$$H_y(t, x^*(t), y^*(t), \lambda(t)) = 0 \tag{3}$$

$$\begin{aligned} \dot{\lambda}(t) &= -H_x(t, x^*(t), y^*(t), \lambda(t)) &(4) \\ \dot{x}^*(t) &= H_\lambda(t, x^*(t), y^*(t), \lambda(t)) &(5) \end{aligned}$$

and $\lambda(T) = 0$.

1.1.2 Sufficiency argument

Now suppose we have found candidate solutions $x^*(t)$ and $y^*(t)$ which satisfy (3)-(5) for some continuous function $\lambda(.)$. We want to check if these conditions are sufficient for an optimum. For this, we need an additional concavity assumption.

Let

$$M(t, x(t), \lambda(t)) = \max_y H(t, x(t), y, \lambda(t)) \tag{6}$$

and suppose that M *is concave in* x.

Consider any other feasible path $x(t)$ and $y(t)$. Then

$$\begin{aligned} & f(t, x(t), y(t)) + \lambda g(t, x(t), y(t)) \leq \\ & M(t, x(t), \lambda(t)) \leq M(t, x^*(t), \lambda(t)) + M_x(t, x^*(t), \lambda(t))(x(t) - x^*(t)) \end{aligned}$$

or

$$\begin{aligned} & f(t, x(t), y(t)) + \lambda(t) g(t, x(t), y(t)) \leq \\ & f(t, x^*(t), y^*(t)) + \lambda(t) g(t, x^*(t), y^*(t)) + M_x(t, x^*(t), \lambda(t))(x(t) - x^*(t)) \end{aligned}$$

Now

$$M_x(t, x^*(t), \lambda(t)) = H_x(t, x^*(t), y^*(t), \lambda(t)) = -\dot{\lambda}(t)$$

the first from an envelope argument, and the second from (4). So we have

$$f(t, x(t), y(t)) \leq f(t, x^*(t), y^*(t)) + \lambda(t)(\dot{x}^*(t) - \dot{x}(t)) + \dot{\lambda}(t)(x^*(t) - x(t)) \tag{7}$$

Integrating by parts and using $\lambda(T) = 0$ and $x(0) = x^*(0)$ we have

$$\int_0^T f(t, x(t), y(t))\, dt \leq \int_0^T f(t, x^*(t), y^*(t))\, dt.$$

We have thus proved a converse to Theorem 1:

Theorem 2 *If x^* and y^* are two continuous functions that satisfy (3)-(5) for some continuous function $\lambda(.)$ with $\lambda(T) = 0$, X is a convex set and $M(t, x, \lambda(t))$ is concave in x for all $t \in [0, T]$ then x^* and y^* are optimal.*

1.2 Dynamic programming

We now use a different approach to characterize the same problem.

Define the value function (sequence problem):

$$\begin{aligned} V(t, x(t)) = \max_{x,y} \quad & \int_t^T f(t, x(\tau), y(\tau))\, d\tau \\ s.t. \quad & \dot{x}(\tau) = g(\tau, x(\tau), y(\tau)) \\ & x(t) \text{ given} \end{aligned}$$

The principle of optimality can be applied to obtain

$$\begin{aligned} V(t, x(t)) &= \max_{x,y} \left\{ \int_t^s f(\tau, x(\tau), y(\tau))\, d\tau + V(s, x(s)) \right\} \\ s.t. \quad & \dot{x} = g(\tau, x(\tau), y(\tau)) \text{ and } x(t) \text{ given} \end{aligned}$$

rewrite the maximization problem as

$$\max_{x,y} \left(\int_t^s f(\tau, x(\tau), y(\tau))\, d\tau + V(s, x(s)) - V(t, x(t)) \right) = 0$$

and taking limits we have

$$\lim_{s \to t} \frac{1}{s-t} \max_{x,y} \left(\int_t^s f(\tau, x(\tau), y(\tau))\, d\tau + V(s, x(s)) - V(t, x(t)) \right) = 0$$

Suppose now that: (1) we can invert taking the limit and taking the maximum, and (2) V is differentiable w.r.t. to both arguments. Then we obtain

$$\max_{y} \left\{ f(t, x(t), y(t)) + \dot{V}(t, x(t)) + V_x(t, x(t))\, \dot{x}(t) \right\} = 0$$

and

$$\max_{y} \left\{ f(t, x(t), y(t)) + V_x(t, x(t))\, g(t, x(t), y(t)) \right\} + \dot{V}(t, x(t)) = 0 \tag{8}$$

This is the Hamilton-Jacobi-Bellman equation.

We can use it to derive the Maximum Principle conditions once more. Given an optimal solution x^*, y^* just identify

$$\lambda(t) = V_x(t, x^*(t)) \tag{9}$$

and we immediately have that $y^*(t)$ must maximize the Hamiltonian:

$$\max_{y} \left\{ f(t, x^*(t), y) + \lambda(t)\, g(t, x^*(t), y) \right\}.$$

We can also get the condition for $\lambda(t)$ (4) assuming V is twice differentiable. Differentiating both sides of (9) yields

$$\dot{\lambda}(t) = V_{x,t}(t, x^*(t)) + V_{x,x}(t, x^*(t))\, \dot{x}^*(t)$$

Differentiating (8) with respect to $x(t)$ (which can be done because the condition holds for all possible $x(t)$ not just the optimal one, we are using a value function!) and using an envelope argument we get

$$f_x(t,x(t),y(t))+V_x(t,x(t))\,g_x(t,x(t),y(t))+V_{x,x}(t,x(t))\,g(t,x(t),y(t))+V_{t,x}(t,x(t))=0$$

Combining the two (at the optimal path) yields

$$f_x(t,x^*(t),y^*(t))+\lambda(t)\,g_x(t,x^*(t),y^*(t))+\dot{\lambda}(t)=0$$

which implies (4).

2 Infinite Horizon

The problem now is to maximize

$$\int_0^\infty f(t,x(t),y(t))$$

subject to

$$\dot{x}(t)=g(t,x(t),y(t))$$

an initial condition $x(0)$ and a terminal condition

$$\lim_{t\to\infty} b(t)\,x(t)=0$$

where $b(t)$ is some given function $b: R_+ \to R_+$.

Going to infinite horizon the Maximum Principle still works, but we need to add some conditions at ∞ (to replace the condition $\lambda(T)=0$ we were using above).

2.1 Sufficiency argument

Let us go in reverse now and look first for sufficient conditions for an optimum. Suppose we have a candidate path $x^*(t), y^*(t)$ that satisfies conditions (3)-(5) for some continuous function λ and the function M defined in (6) is concave. We can compare the candidate path with any other feasible path as in the proof of Theorem 2, however at the moment of integrating by parts (7) we are left with the following expression:

$$\int_0^\infty f(t,x(t),y(t))\,dt \le \int_0^\infty f(t,x^*(t),y^*(t))\,dt+\lim_{t\to\infty}\lambda(t)\,(x^*(t)-x(t)).$$

If we assume that $\lim_{t\to\infty}\lambda(t)\,x^*(t)=0$ and $\lim_{t\to\infty}\lambda(t)\,x(t)\ge 0$ for all feasible paths $x(t)$ we then have

$$\int_0^\infty f(t,x(t),y(t))\,dt \le \int_0^\infty f(t,x^*(t),y^*(t))\,dt.$$

This proves the following.

Theorem 3 *Suppose x^* and y^* are two continuous functions that satisfy (3)-(5) for some continuous function $\lambda(.)$, X is a convex set and $M(t, x, \lambda(t))$ is concave in x for all t and the following two conditions are satisfied*

$$\lim_{t \to \infty} \lambda(t) x^*(t) = 0 \text{ (transversality condition)} \tag{10}$$

and

$$\lim_{t \to \infty} \lambda(t) x(t) \geq 0 \text{ for all feasible paths } x(t). \tag{11}$$

Then x^* and y^* are optimal. If $M(t, x, \lambda(t))$ is strictly concave in x then x^* and y^* are the unique optimal solution.

We have thus established that the transversality condition (10) (together with concavity and our usual optimality conditions) is sufficient.

2.2 Necessity

Proving that some transversality condition is necessary is more cumbersome and requires additional assumptions (although, clearly, when we deal with necessary conditions we don't need concavity). The general idea is the following. Use the HBJ equation (8) to show that

$$\dot{V}(t, x^*(t)) = -H(t, x^*(t), y^*(t), \lambda(t)).$$

If you can argue that $\lim_{t \to \infty} \dot{V}(t, x^*(t)) = 0$ then you have the transversality condition

$$\lim_{t \to \infty} H(t, x^*(t), y^*(t), \lambda(t)) = 0.$$

Under additional conditions (e.g. if $x^*(t)$ converges to a steady state level) then this condition implies (10). (See Theorems 7.12 and 7.13 in Acemoglu (2009)). However, in applications it is often more useful to use the transversality condition as a sufficient condition (as in Theorem 3 and in Theorem 7.14 in Acemoglu (2009)).

3 Discounting

The problem now is to maximize

$$\int_0^\infty e^{-\rho t} f(x(t), y(t))$$

subject to

$$\dot{x}(t) = g(t, x(t), y(t))$$

an initial condition $x(0)$ and a terminal condition

$$\lim_{t \to \infty} b(t) x(t) = 0$$

where $b(t)$ is some given function $b : R_+ \rightarrow R_+$.

First, notice that the Hamiltonian would be

$$H(t, x(t), y(t), \lambda(t)) = e^{-\rho t} f(x(t), y(t)) + \lambda(t) g(t, x(t), y(t))$$

but defining $\mu(t) = e^{\rho t} \lambda(t)$ we can define the *current-value Hamiltonian*

$$\tilde{H}(x(t), y(t), \mu(t)) = f(x(t), y(t)) + \mu(t) g(t, x(t), y(t))$$

(so that $H = e^{-\rho t} \tilde{H}$, so H is also called *present-value Hamiltonian*).

Therefore, conditions (3)-(5) become

$$\tilde{H}_y(t, x^*(t), y^*(t), \mu(t)) = 0$$

$$\begin{aligned} \dot{\mu}(t) &= \rho \mu(t) - \tilde{H}_x(t, x^*(t), y^*(t), \mu(t)) \\ \dot{x}^*(t) &= \tilde{H}_\mu(t, x^*(t), y^*(t), \mu(t)). \end{aligned}$$

The limiting conditions (10) and (11) become

$$\lim_{t \rightarrow \infty} e^{-\rho t} \mu(t) x^*(t) = 0 \text{ (transversality condition)}$$

and

$$\lim_{t \rightarrow \infty} e^{-\rho t} \mu(t) x(t) \geq 0 \text{ for all feasible paths } x(t).$$

All the results above go through and we just express them using different variables.

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