14.30 PROBLEM SET 7

TA: Tonja Bowen Bishop

Due: Tuesday, April 25, 2006

Note: The first three problems are required, and the remaining three are practice problems. If you choose to do the practice problems now, you will receive feedback from the grader. Alternatively, you may use them later in the course to study for exams. Credit may not be awarded for solutions that do not use methods discussed in class.

Problem 1

Let $X_1, ..., X_n$ be a random sample (i.i.d.) of size n from a population with distribution $f(x)$ with mean μ and variance σ^2 (both finite). Consider a statistic formed by taking a linear combination of the X_is, $c_1X_1 + ... + c_nX_n$, where $c_i \geq 0$. For example, the sample mean, $\overline{X}$, is the linear combination with $c_i = \frac{1}{n}$ for all i.

a. Under what condition(s) is $c_1X_1 + ... + c_nX_n$ an unbiased estimator of μ?

b. What is the variance of $c_1X_1 + ... + c_nX_n$?

c. Find the linear combination (derive the c's) that minimizes the variance without violating the condition(s) you derived in part a. (i.e. find the most efficient linear unbiased estimator of μ).

Problem 2

Suppose that you have a sample of size n, $X_1, ..., X_n$, from a population with mean μ and variance σ^2. You know that your draws from the population, the X_i, are identically distributed, however, they are not independent and have $Cov(X_i, X_j) = \rho\sigma^2 > 0$, for all $i \neq j$.

a. What is $E\left(\overline{X}\right)$? Is the sample mean an unbiased estimator of μ?

b. What is the MSE of $\overline{X}$? You can use the fact that $Var\left(\sum_i X_i\right) = \sum_i Var(X_i) + \sum_i \sum_{j \neq i} Cov(X_i, X_j)$.

c. Is the sample mean a consistent estimator of μ? Would it be consistent if $\rho = 0$?

Problem 3

Suppose that $Z_1, Z_2, ..., Z_n$ is a random sample from an exponential distribution with parameter λ, so $f(x) = \begin{Bmatrix} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & elsewhere \end{Bmatrix}$

a. Find the Method of Moments estimator for λ.

b. Find the Maximum Likelihood estimator for λ.

c. Find the Maximum Likelihood estimator for $\sqrt{\lambda}$.

d. Is the estimator in part b unbiased? Is it consistent?

Problem 4

Assume a sample of continuous random variables: $X_1, X_2, ..., X_n$, where $E[X_i] = \mu$, $Var[X_i] = \sigma^2 > 0$. Consider the following estimators: $\widehat{\mu}_{1,n} = X_n, \widehat{\mu}_{2,n} = \frac{1}{n+1}\sum_{i=1}^{n} X_i$.

a. Are $\widehat{\mu}_{1,n}$ and $\widehat{\mu}_{2,n}$ unbiased?

b. Are $\widehat{\mu}_{1,n}$ and $\widehat{\mu}_{2,n}$ consistent?

c. What do you conclude about the relation between unbiased and consistent estimators?

d. For what values of n does $\widehat{\mu}_{2,n}$ have a lower mean square error than $\widehat{\mu}_{1,n}$?

Problem 5

You have a sample of size n, $X_1, ..., X_n$ from a $U[\theta_l, \theta_u]$ distribution.

a. Assume that it is known that $\theta_l = 0$. Find the MM and MLE estimators of θ_u.

b. Now assume that both θ_l and θ_u are unknown. Find the MM estimators of θ_l and θ_u.

c. Determine whether the estimators in parts a and b are unbiased and consistent.

Problem 6

Let $X_1, ..., X_n$ be a random sample of size n from a $U[0, \theta]$ population. Consider the following estimators of θ:

$$\begin{aligned} \widehat{\theta}_1 &= k_1 X_2 \\ \widehat{\theta}_2 &= k_2 \overline{X} \\ \widehat{\theta}_3 &= k_3 X_{(n)} \end{aligned}$$

where the k_i are constants and

$$X_{(n)} = \max(X_1, ..., X_n)$$

$X_{(n)}$ is known as the n^{th} order statistic.

a. Calculate $\Pr(X_{(n)} \leq x)$, given our knowledge of the population distribution.

b. Calculate the pdf of $X_{(n)}$.

c. Choose k_1, k_2, and k_3 such that all three estimators are unbiased; call these values $\widehat{k}_1$, $\widehat{k}_2$, and $\widehat{k}_3$ (In other words, solve $E\left[\widehat{k}_1 X_1\right] = \theta$, and so forth).

d. Calculate the following variances and compare them:

$$Var\left(\widehat{k}_1 X_2\right)$$
$$Var\left(\widehat{k}_2 \overline{X}\right)$$
$$Var\left(\widehat{k_3 X_{(n)}}\right)$$