14.124(Solutions for Homework 1)

1 Question 1

Definition 1 *MLRP:* $\frac{f(x)}{g(x)}$ *is increasing in* x.

Definition 2 *FOSD:* $F(x) \leq G(x) \forall x$

Show that $MLRP \Rightarrow FOSD$

Proof: $MLRP$ implies that there exists x_0 such that $\frac{f(x_0)}{g(x_0} = 1$. If not then $f(x) > g(x) \forall x$ or $f(x) < g(x) \forall x$. But since $\int f(x)dx = \int g(x)dx = 1$, neither of these cases are possible.

We have two cases:

C1: $x < x_0 : f(x) \leq g(x) \rightarrow {}_{-\infty}^{x} f(x) \leq {}_{-\infty}^{x} g(x) \rightarrow F(x) \leq G(x)$

C2: $x > x_0 : f(x) > g(x) \rightarrow {}_{x}^{\infty} f(x) \geq {}_{x}^{\infty} g(x) \rightarrow 1 - F(x) \leq 1 - G(x) \rightarrow F(x) \leq G(x)$

2 Question 2

You want to design an experiment, that is a random variable Y (that takes value in [0;1], for simplicity, and is characterized by the joint distribution $p(\theta,y)$) such that the distribution of posteriors generated by this experiment is given by $f(p)$. In the "experiment" the posterior will be given by $Pr(\theta_1/y)$ and the probability that this posterior arises is simply $Pr(y)$. In the statement, the probability that posterior p arises would be given by $f(p)$. Therefore, we'd like to take $Pr(\theta_1/y)=p$ and $Pr(y)=f(p)$ (for y=p) which directly defines $p(\theta_1,y)=pf(p)$;$Pr(\theta_2/y)=1-p$; $p(\theta_2,y)=(1-p)f(p)$. Does such a random variable exists ?

We have $p(\theta,y) \geq 0$ for all y,θ and

$\int p(\theta,y)dyd\theta = \int p(\theta_1,y)dy + \int p(\theta_2,y)dy$

$= \int pf(p)dp + \int (1-p)f(p)dp$ (by construction, since $p(\theta_1,y)=pf(p)$ for y=p)

$= p_0 + (1-p_0) = 1$ (by hypothesis)

Therefore such an experiment exists (we can construct a random variable Y such that the joint distribution of (Y,θ) is $p(\theta,y)$ since p is non negative and sums up to 1) and the prior is given by $Pr(\theta_1) = \int p(\theta_1,y)dy = p_0$ as wanted

We can then define the likelihood functions using Bayes rule and we have $Pr(y/\theta_1) = Pr(\theta_1/y)Pr(y)/Pr(\theta_1) = pf(p)/p_0$ and similarly *for* $Pr(y/\theta_2)$. You can then directly verify that the experiment defined by the outcome y, these likelihood functions and the prior p_0 generate posteriors distributed according to f.

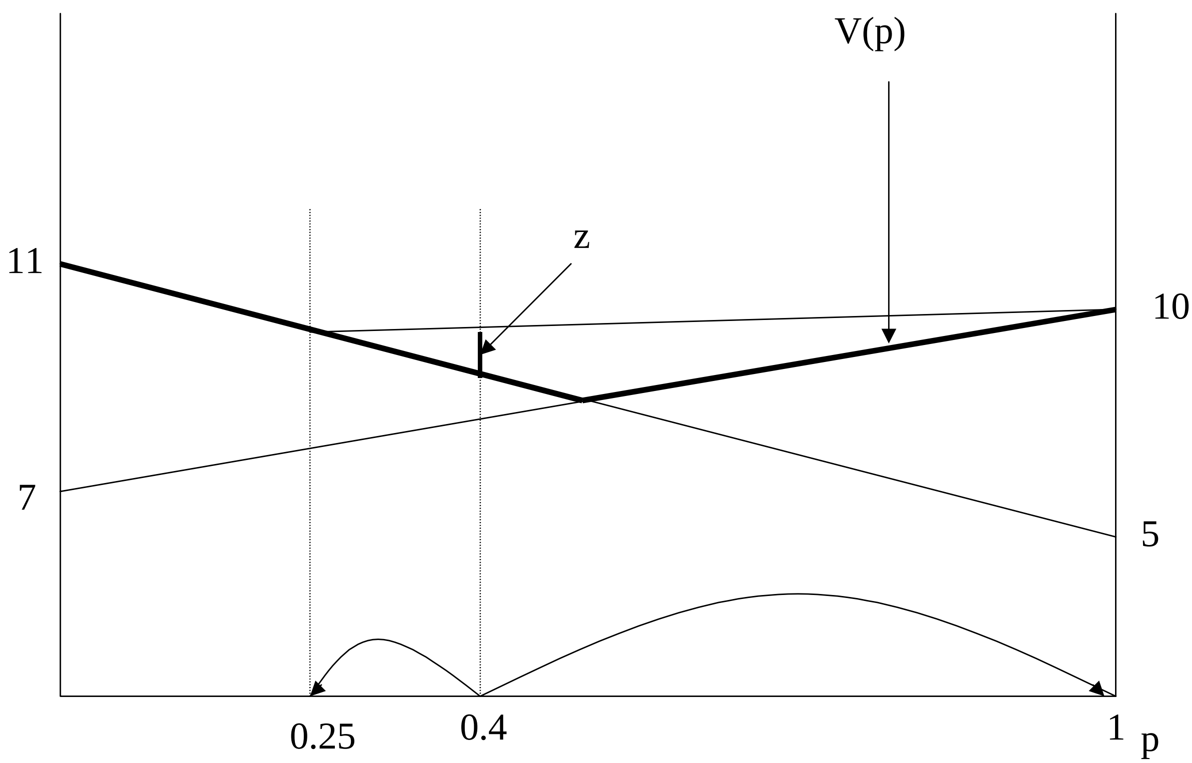
3 Question 3

1. We are given $u(a, s)$ therefore we can derive $v(a, p)$:

$$\begin{aligned} v(a_1, p) &= pu(a_1, s_1) + (1-p)\, u(a_1, s_2) = 7 + 3p \\ v(a_2, p) &= pu(a_2, s_1) + (1-p)\, u(a_2, s_2) = 11 - 6p \end{aligned}$$

The upper envelope of $v(a, p)$ will be the $V(p)$ (it indicates the maximum expected utility that the agent can reach if faced with probability of s_1 equal to p):

$$V(p) = \max_a v(a, p)$$



At $p = 0.4$ the optimal decision is a_2 since:

$$
\begin{aligned}
v(a_1, p) &= 10(0.4) + 7(0.6) = 8.2 \\
v(a_2, p) &= 5(0.4) + 11(0.6) = 8.6
\end{aligned}
$$

2. We are given the likelihood matrix:

$$
L = \begin{bmatrix} \Pr(y_1|s_1) & \Pr(y_2|s_1) \\ \Pr(y_1|s_2) & \Pr(y_2|s_2) \end{bmatrix} = \begin{bmatrix} \lambda_1 & (1-\lambda_1) \\ \lambda_2 & (1-\lambda_2) \end{bmatrix}
$$

The probability of observing the two signals is:

$$
\begin{aligned}
\Pr(y_1) &= \Pr(y_1|s_1)\Pr(s_1) + \Pr(y_1|s_2)\Pr(s_2) = 0.4\lambda_1 + 0.6\lambda_2 \\
\Pr(y_2) &= \Pr(y_2|s_1)\Pr(s_1) + \Pr(y_2|s_2)\Pr(s_2) = 1 - 0.4\lambda_1 - 0.6\lambda_2
\end{aligned}
$$

Let's calculate the posterior probabilities:

$$
\begin{aligned}
\Pr(s_1|y_1) &= \frac{\Pr(y_1|s_1)\Pr(s_1)}{\Pr(y_1)} = \frac{0.4\lambda_1}{0.4\lambda_1 + 0.6\lambda_2} \\
\Pr(s_1|y_2) &= \frac{\Pr(y_2|s_1)\Pr(s_1)}{\Pr(y_2)} = \frac{0.4(1-\lambda_1)}{1 - 0.4\lambda_1 - 0.6\lambda_2}
\end{aligned}
$$

- If $\lambda_1 = \lambda_2 = \frac{1}{2}$ the information system has no value because the same signal y_1 is as likely to appear in the two states of nature. Whenever $\lambda_1 = \lambda_2$ the information system has no value.
- If $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = 0$ the the likelihood matrix is the following:

$$
L = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix}
$$

  Posteriors in this case are:

$$
\begin{aligned}
\Pr(s_1|y_1) &= 1 \\
\Pr(s_1|y_2) &= 0.25
\end{aligned}
$$

And probabilities of the two signals are:

$$\Pr(y_1) = 0.2$$
$$\Pr(y_2) = 0.8$$

- When observe y_1:

$$v(a_1, 1) = 10$$
$$v(a_2, 1) = 5$$

therefore the optimal choice as a function of the signal is:

$$a(y_1) = \arg\max_a v(a, p = 1) = a_1$$

Hence:

$$V(1) = 10$$

- When observe y_2:

$$v(a_1, 0.25) = 7.75$$
$$v(a_2, 0.25) = 9.5$$

therefore the optimal choice as a function of the signal is:

$$a(y_2) = \arg\max_a v(a, p = 0.25) = a_2$$

hence:

$$V(0.25) = 9.5$$

We can know calculate V_Y as:

$$V_Y = (9.5)\,0.8 + (10)\,0.2 = 9.6$$

We can now calculate the value of information as:

$$Z = V_Y - V(0.4) = 9.6 - 8.6 = 1$$

3. We know that the Blackwell theorem gives general conditions under which one information system is preferred to another. So we just have to prove that the information system $\left(\lambda_1 = \frac{1}{2}\alpha + \frac{1}{2}\beta,\ \lambda_2 = \beta\right)$ is a garbling of the information system $\left(\lambda_1 = \frac{1}{2},\ \lambda_2 = 0\right)$.

   We have to find a Markov matrix M:

$$\begin{aligned} M &= \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \\ m_{11} + m_{12} &= 1 \\ m_{21} + m_{22} &= 1 \\ m_{ij} &\geq 0 \end{aligned}$$

   such that the following relationship between the likelihood matrices holds:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} \frac{\alpha+\beta}{2} & 1 - \frac{\alpha+\beta}{2} \\ \beta & 1-\beta \end{bmatrix}$$

   You can verify that such matrix M exists and is equal to the following:

$$M = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

4 Question 4

4.1 a)

The Pareto problem is

$$\max_{e,w} f_e u(s_2) + (1 - f_e) u(s_1) - e$$

subject to

$$\text{IR)} \quad f_e (x_2 - s_2) + (1 - f_e)(x_1 - s_1) \geq 0$$

The individual rationality constraint for the principal will bind (otherwise, can just pay the agent more). Since we only have the Pareto problem here, the agent will get a constant wage (in effect, the principal is like a competitive insurance company here)

$$s_1 = s_2 = f_e x_2 + (1 - f_e) x_1$$

The effort level should be chosen to solve

$$\max_{e} \{u(f_e x_2 + (1 - f_e) x_1) - e\}$$

Effort level e_H is optimal iff

$$u(f_H x_2 + (1 - f_H) x_1) - u(f_L x_2 + (1 - f_L) x_1) \geq e_H - e_L$$

The second-best problem is the same, but also adds an incentive constraint: for implementing e_H

$$\text{IC)} \quad (f_H - f_L)(u(s_2) - u(s_1)) \geq e_H - e_L$$

Then, it is clear that a constant wage cannot implement e_H, since it will set the LHS to zero. Hence, the IC constraint will bind if e_H is optimal to implement. To implement e_H, we will have

$$\begin{aligned} (f_H - f_L)(u(s_2) - u(s_1)) &= e_H - e_L \\ f_H (x_2 - s_2) + (1 - f_H)(x_1 - s_1) &= 0 \end{aligned} \qquad (2)$$

To implement e_L, we will have a constant wage

$$s = f_L x_2 + (1 - f_L) x_1$$

It is efficient to implement e_H iff

$$f_H u(s_2) + (1 - f_H) u(s_1) - e_H \geq u(s) - e_L$$

and using 2,

$$\begin{aligned} &\Rightarrow \quad u(s_1) + f_H (u(s_2) - u(s_1)) \geq u(s) + e_H - e_L \\ &\Rightarrow \quad u(s_1) + \frac{f_L}{f_H \quad f_L}(e_H - e_L) \geq u(s) \end{aligned}$$

4.2 b)

Suppose there is now a third effort level $e_M > \frac{1}{2}(e_H + e_L)$ with $f_M = \frac{1}{2}(f_H + f_L)$. Then, to implement e_M we need the following incentive constraints:

$$\begin{aligned} \text{ICH} \quad (f_H - f_M)(u(s_2) - u(s_1)) &\le e_H - e_M \\ \text{ICL} \quad (f_M - f_L)(u(s_2) - u(s_1)) &\ge e_M - e_L \end{aligned}$$

Substituting in for f_M and e_M, we get

$$\begin{aligned} \text{ICH} \quad \frac{1}{2}(f_H - f_L)(u(s_2) - u(s_1)) &< \frac{1}{2}(e_H - e_L) \\ \text{ICL} \quad \frac{1}{2}(f_H - f_L)(u(s_2) - u(s_1)) &> \frac{1}{2}(e_H - e_L) \end{aligned}$$

which cannot both hold simultaneously. Therefore, e_M cannot be implemented.

4.3 c)

Now suppose $e_M < \frac{1}{2}(e_H + e_L)$ and e_H was optimal to implement in Part (a). Hence,

$$s_1^* \ge f_L x_2 + (1 - f_L) x_1$$

where

$$u(s_2^*) - u(s_1^*) = \frac{e_H - e_L}{f_H - f_L}$$

and (s_1^*, s_2^*) denote the optimal contract in Part (a).

Our constraints are now:

$$\begin{aligned} \text{IR} \quad f_H(x_2 - s_2) + (1 - f_H)(x_1 - s_1) &\ge 0 \\ \text{IC} \quad (f_H - f_L)(u(s_2) - u(s_1)) &\ge e_H - e_L \\ \text{ICM} \quad \frac{1}{2}(f_H - f_L)(u(s_2) - u(s_1)) &\ge e_H - e_M \end{aligned}$$

Since we have that

$$\frac{1}{2}(f_H - f_L)(u(s_2^*) - u(s_1^*)) = \frac{1}{2}(e_H - e_L) < e_H - e_M$$

constraint (ICM) fails the contract we derived in Part (a) no longer implements e_H once e_M is available.

We can still implement e_H under certain conditions. Constraint (ICM) will bind, and therefore, constraint IC) will not bind. The sharing rule will satisfy

$$\begin{aligned} f_H(x_2 - s_2) + (1 - f_H)(x_1 - s_1) &= 0 \\ \frac{1}{2}(f_H - f_L)(u(s_2) - u(s_1)) &= e_H - e_M \end{aligned}$$