Decision Making Under Risk

14.123 Microeconomic Theory III
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Road map

1. Expected Utility Maximization
   1. Representation
   2. Characterization
2. Indifference Sets under Expected Utility Maximization

Choice Theory – Summary

1. X = set of alternatives
2. Ordinal Representation: $U : X \to R$ is an ordinal representation of $\succcurlyeq$ iff:

   $$x \succcurlyeq y \Leftrightarrow U(x) \geq U(y)\ \forall x,y \in X.$$
3. If $\succcurlyeq$ has an ordinal representation, then $\succcurlyeq$ is complete and transitive.
4. Assume X is a compact, convex subset of a separable metric space. A preference relation has a continuous ordinal representation if and only if it is continuous.
5. Let $\succcurlyeq$ be continuous and $x' \succ x \succ x''$. For any continuous $\varphi:[0,1] \to X$ with $\varphi(1)=x'$ and $\varphi(0)=x''$, there exists t such that $\phi(t) \sim x$.

Model

- DM = Decision Maker
- DM cares only about consequences
  - C = Finite set of consequences
- Risk = DM has to choose from alternatives
  - whose consequences are unknown
  - But the probability of each consequence is known
- Lottery: a probability distribution on C
- P = set of all lotteries p,q,r
- $X = P$
- Compounding lotteries are reduced to simple lotteries!

Expected Utility Maximization
Von Neumann-Morgenstern representation

A lottery (in P)

Expected value of u under p

$$p \succeq q \Leftrightarrow \underbrace{\sum_{c \in C} u(c)p(c)}_{U(p)} \geq \underbrace{\sum_{c \in C} u(c)q(c)}_{U(q)}$$

- $U : P \to \mathbb{R}$ is an ordinal representation of $\succcurlyeq$.
- $U(p)$ is the expected value of u under p.
- U is linear and hence continuous.

Expected Utility Maximization
Characterization (VNM Axioms)

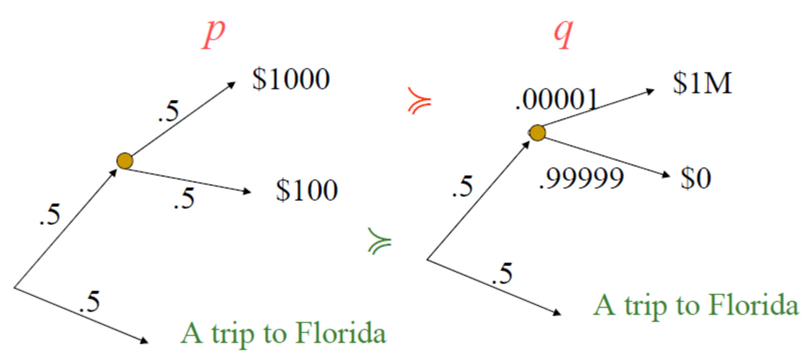
Axiom A1: $\succcurlyeq$ is complete and transitive.

Axiom A2 (Continuity): $\succcurlyeq$ is continuous.

Independence Axiom

Axiom A3: For any $p,q,r \in P$, $a \in (0,1]$,

$$ap+(1-a)r \succcurlyeq aq+(1-a)r \Leftrightarrow p \succcurlyeq q.$$



Expected Utility Maximization Characterization Theorem

- $\succcurlyeq$ has a von Neumann – Morgenstern representation iff $\succcurlyeq$ satisfies Axioms A1-A3;
- i.e. $\succcurlyeq$ is a continuous preference relation with Independence Axiom.
- *u* and *v* represent $\succcurlyeq$ iff $v = au + b$ for some $a > 0$ and any *b*.

Exercise

- Consider a relation $\succcurlyeq$ among positive real numbers represented by VNM utility function u with $u(x) = x^2$.
- Can this relation be represented by VNM utility function $u^*(x) = x^{1/2}$?
- What about $u^{**}(x) = 1/x$?

Implications of Independence Axiom (Exercise)

- For any p,q,r,r' with $r \sim r'$ and any a in $(0,1]$,
  $$ap+(1-a)r \succcurlyeq aq+(1-a)r' \Leftrightarrow p \succcurlyeq q.$$
- **Betweenness:** For any p,q,r and any a,
  $$p \sim q \Rightarrow ap+(1-a)r \sim aq+(1-a)r.$$
- **Monotonicity:** If $p \succ q$ and $a > b$, then
  $$ap + (1-a)q \succ bp + (1-b)q.$$
- **Extreme Consequences:** $\exists c^B, c^W \in C: \forall p \in P,$
  $$c^B \succcurlyeq p \succcurlyeq c^W.$$

Proof of Characterization Theorem

- $c^B \sim c^W$ trivial. Assume $c^B \succ c^W$.
- Define $\phi : [0,1] \to P$ by $\phi(t) = tc^B+(1-t)c^W$.
- Monotonicity: $\phi(t) \succcurlyeq \phi(t') \Leftrightarrow t \geq t'$.
- Continuity: $\forall p \in P$, $\exists$ unique $U(p) \in [0,1]$ s.t. $p \sim \phi(U(p))$.
- Check Ordinal Representation:

$$p \succcurlyeq q \quad \Leftrightarrow \quad \phi(U(p)) \succcurlyeq \phi(U(q)) \quad \Leftrightarrow \quad U(p) \geq U(q)$$

- U is linear:

$$U(ap+(1-a)q) = aU(p)+(1-a)U(q)$$

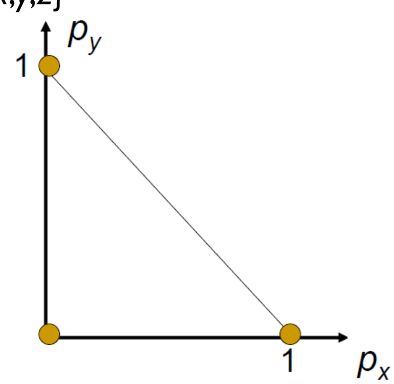
- Because $ap+(1-a)q \sim a\phi(U(p))+(1-a)\phi(U(q))$

$$= \phi(aU(p)+(1-a)U(q)),$$

Indifference Sets under Independence Axiom

1. Indifference sets are straight lines
2. … and parallel to each other.

Example: $C = \{x,y,z\}$



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