Answers to Selected Problems from Past Final Exams

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1 Problem 1, Final 2004

Denote the type with probability 0.4 as strong, and the type with probability 0.6 as weak.

It is clear that if player 1 is strong, he will play A over D at node 1.

Suppose that player 2 believes that P1 is strong with probability μ and weak with probability $1-\mu$. Since player 1 plays A whenever he is strong, it follows that

$$\mu = \frac{P(S)}{P(S) + P(A|W)\,P(W)} \geq 0.4$$

Then, P2 chooses α iff $2(1-\mu) \geq 1$, or $1-\mu \geq 0.5$, or $\mu \leq 0.5$. Otherwise, P2 chooses δ. Clear that neither separating nor pooling works here. Why?

Separating: must involve weak P1 playing d, and P2 playing δ. But then, weak P1 wants to play a.

Pooling: P2 wants to play α, and weak P2 then wants to play d.

Consider the following: Weak P1 plays a mixed strategy such that $\mu = 0.5$, so $P(a|W) = \frac{1}{P(W)}\left[\frac{P(S)}{0.5} - P(S)\right] = \frac{P(S)}{P(W)} = \frac{2}{3}$, and P2 plays a mixed strategy such that weak P1 is indifferent. Specifically, P2 plays α with probability α, so weak P1 gets $\alpha + 3(1-\alpha)$ from a, and 2 from d. Therefore, we have $3 - 2\alpha = 2$, or $\alpha = 0.5$.

Check that this is an SE. The profile specified is $\left(\left\{A, \frac{2}{3}a + \frac{1}{3}d, \frac{1}{2}\alpha + \frac{1}{2}\delta\right\}, (0.5, 0.5)\right)$. We implicitly did this in the derivation.

Sequential Rationality: At 1(strong), A is a strictly dominant strategy. At 1(weak), P1 is indifferent between a and d. At 2, P2 is indifferent between α and δ.

Consistency: Since strong P1 always plays A, (so $P(A|S) = 1$), and weak P1 plays a with probability $\frac{2}{3}$, we have that $\mu = 0.5$.

2 Problem 4, Final 2004

The timing of the game is as follows:

1) Professor chooses cutoff score $\gamma \in [0, 100]$

2) Student observes type $t \in \{H, L\}$ and decides whether to take class. If does not take class, professor gets 0, and student gets W_t, where $0 < W_L < W_H < 100$

3) If student takes class, he exerts effort e and gets grade $s = e \cdot (t = L) + 2e \cdot (t = H)$. The professor's payoff is s. The student's payoff is $100 \cdot (s \geq \gamma) - e/2$.

I will let $t = 0$ stand for the low-ability student and $t = 1$ stand for the high-ability student. Hence, the grade production function of a student with ability t is $s(e) = 2^{t-1}e$.

Consider the last stage of the game. Then, the student solves $\max_e \{100 \cdot (2^t e \geq \gamma) - e/2\} = \max((100 - e/2)(2^t e \geq \gamma), -e$

If student decides to fail, sets $e = 0$, and gets 0.

If student decides to pass, sets $e = \frac{\gamma}{2^t}$, and gets $100 - \frac{\gamma}{2^{t+1}}$. In particular, gets $U_L = 100 - \frac{\gamma}{2}$, and $U_H = 100 - \frac{\gamma}{4}$.

Consider the second stage. Then, the student takes the class iff $100 - \frac{\gamma}{2^{t+1}} \geq W_t$. In particular, low types take the class iff $100 - \frac{\gamma}{2} > W_L$ and high types take the class iff $100 - \frac{\gamma}{4} > W_H$.

Now, consider the first stage. The professor's payoff is $\gamma \cdot E\left(100 - \frac{\gamma}{2^{t+1}} \geq W_t\right) = \gamma \cdot \left(\pi \cdot \left[100 - \frac{\gamma}{4} \geq W_H\right] + (1-\pi) \cdot \left[100 -\right.\right.$

If everyone takes the course, the professor should set $\gamma = \min\left(4\left(100 - W_H\right), 2\left(100 - W_L\right)\right)$, and would get $\min\left(2\left(100 - W_H\right), \left(100 - W_L\right)\right)$

If only high types take the course, the professor should set $\gamma = 4\left(100 - W_H\right)$, and would get $2\left(100 - W_H\right)\pi$

If only low types take the course, the professor should set $\gamma = 2\left(100 - W_L\right)$,and would get $\left(100 - W_L\right)(1-\pi)$

Suppose that $\pi > \frac{1}{2}\frac{100-W_L}{100-W_H}$ and $W_H < \frac{100+W_L}{2}$. Then, $2\left(100 - W_H\right) > 2\left(100 - \frac{100+W_L}{2}\right) = 100 - W_L$, so making the course for all would get the professor $100 - W_L > \left(100 - W_L\right)(1-\pi)$.

Now, $2\left(100 - W_H\right)\pi > 100 - W_L$, so he will make the course for the high types only.

Suppose that $\pi > 1 - 2\frac{100-W_H}{100-W_L}$ and $W_H > \frac{100+W_L}{2}$. Then, $2\left(100 - W_H\right) < \left(100 - W_L\right)$, so $P_U = 2\left(100 - W_H\right) > 2\left(100 - W_H\right)\pi$. Moreover,

$\left(100 - W_L\right)(1-\pi) < \left(100 - W_L\right)\left(1 - \left(1 - 2\frac{100-W_H}{100-W_L}\right)\right) = 2\left(100 - W_H\right)$, so P_U is maximum and everyone takes the course.

Suppose that $\pi < \frac{1}{2}\frac{100-W_L}{100-W_H}$ and $W_H < \frac{100+W_L}{2}$. Then, $2\left(100 - W_H\right) > 100 - W_L$, so $P_U = 100 - W_L > \left(100 - W_L\right)(1-\pi)$. Now, $2\left(100 - W_H\right)\pi < \frac{1}{2}\frac{100-W_L}{100-W_H} = 100 - W_L$, so everyone takes the course (since P_U is maximum).

The grades are $100 - W_L$, $2\left(100 - W_H\right) < 100 - W_L$, and $100 - W_L$. Therefore, the grades at the first and third institutions are the same. The grade at the second institution is too low.

3 Problem 2, Final 2007 (Regular Exam)

First note that we have one type for player 2, and three types for player 1: A, B, and C. Player 2 knows whether player 1 played L1,L2,L3 or R1,R2,R3, but doesn't know player 1's type. Also note how I labeled all the nodes in this diagram.

It is clear that 1A has R1 as a dominant strategy, and 1B has L1 as a dominant strategy. Therefore, at information set 2L, player 2 knows that player 1 is either type A or B and played Left, so player 2 must conclude that player 1 has type B. Therefore, player 2 will play b at information set 2L.

Consider player 2's play at information set 2R. Then, player 2 knows that player 1's type is not B. Let μ be player 2's belief that he is at 2RA, and $1-\mu$ be his belief that he is at 2RC. Note that $\mu = \frac{P(2R|A)P(A)}{P(2R|A)P(A)+P(2R|C)P(C)} = \frac{(1/3)}{(1/3)+(1/3)P(2R|C)} \geq \frac{1}{2}$.

What is player 2's best response given a belief μ? Player 2 will play l iff $\mu \geq 2(1-\mu)$ or $\mu \geq \frac{2}{3}$.

Suppose that type 1C always plays L3.Then, $\mu = 1$, and player 2 should play l, which he wants to do. However, type 1C would do better by playing R3 and getting 2, so this is not an SE.

Suppose that type 1C always plays R3. Then, $\mu = \frac{1}{2}$ and player 2 should play r. But then, type 1C would do better by playing L3 and getting 1, so this is not an SE.

Hence, it is clear that type 1C has to mix, and therefore, has to be indifferent between L3 and R3. Moreover, player 2 has to mix at information set 2R, so as to make type 1C indifferent. Then, player 2 herself must be indifferent between l and r, so it must be the case that $\mu = \frac{2}{3}$. The only way we can have $\mu = \frac{2}{3}$ is if $P(2R|C) = \left(\frac{1}{2} - \frac{1}{3}\right)3 = \frac{1}{2}$, so type 1C must mix between L3 and R3 with probability $\frac{1}{2}$. Now, suppose that player 2 mixes with probability p of going left. We have that for type 1C, $U(L3) = 1$, and $U(R3) = 2p$, so must have $p = \frac{1}{2}$. We have now pinned down the strategies and beliefs of all the players, and the sequential equilibrium is

$$\left\{\text{R1}, \text{L1}, \left\{\frac{1}{2}\text{R3} + \frac{1}{2}\text{L3}\right\}, \text{b}, \left\{\frac{1}{2}l + \frac{1}{2}r\right\}; \upsilon = 0, \mu = \frac{2}{3}\right\}$$

How to check that this is an SE: Check sequential rationality and consistency. We did all the calculations in the work, so just write what you would have needed to check:

1) SR at node 1A: 1A has a dominant strategy R1

2) SR at node 1B: 1B has a dominant strategy L2

3) SR at node 1C: 1C is indifferent between L3 and R3, so he can mix

4) SR at information set 2L: P2 believes he is at 2LB, so plays b

5) SR at information set 2R: P2 believes he is at 2RA with probability $\frac{2}{3}$, so is indifferent between l and r, and can mix

1) Consistency at information set 2L: Only 1B ever plays L2, so the probability that we are at 2LA, $\nu = 0$.

2) Consistency at information set 2R: 1A always plays R1, and 1C plays R3 one-half of the time, so the probability we are at 2RA, $\mu = \frac{2}{3}$.

4 Problem 2, Final 2010

Three bidders, two objects. Values independent and uniform; each $v_i \in [0,1]$. Perform a first-price auction.

a) Set of players: $N = \{1,2,3\}$

b) Set of types: $T_i = [0,1] \ \forall i \in N$

c) Set of actions: $b_i \in R$

d) Set of expectations: $p_i\left(t_{-i}|t_i\right) = 1 \left(t_{-i} \in [0,1]^2\right)$

e) Set of payoffs: $u_i\left(b,v\right) = \left\{ \begin{array}{c} v_i - b_i, b_i > \min\left(b_i\right) \\ 0, b_i = \min\left(b_i\right) \end{array} \right.$

We are to find a symmetric BNE in strictly increasing and differentiable strategies.

Step 0: Define what you are looking for:

We have $b_i\left(v_i\right) = b\left(v_i\right)$, where $b'\left(v_i\right)$ exists and $b'\left(v_i\right) > 0$ for $v_i \in (0,1)$.

Step 1: Compute expected utility

$$\begin{aligned}
U\left(b_i|v_i\right) &= E\left(u_i\left(b_i, b\left(v_j\right)_{j\neq i}, v\right)|v_i\right) \\
&= \int_{v_{\square i}\in[0,1]^2} u_i\left(b_i, b\left(v_j\right)_{j\neq i}, v\right) dv_{-i} \\
&= \int_{b_i > \min\left(b_j\right)} \left(v_i - b_i\right) dv_{-i} \\
&= \left(v_i - b_i\right) P\left(b_i > \min\left(b\left(v_j\right)\right)\right) \\
&= \left(v_i - b_i\right)\left(1 - \prod_{j\neq i} P\left(b_i < b\left(v_j\right)\right)\right) \\
&= \left(v_i - b_i\right)\left(1 - \prod_{j\neq i} P\left(b^{-1}\left(b_i\right) < v_j\right)\right) \\
&= \left(v_i - b_i\right)\left(1 - \left[1 - b^{-1}\left(b_i\right)\right]^2\right)
\end{aligned}$$

Step 2: Take the FOC

$$0 = \frac{dU}{db_i} = -\left(1 - \left[1 - b^{-1}\left(b_i\right)\right]^2\right) + 2\left[1 - b^{-1}\left(b_i\right)\right]\left(v_i - b_i\right)\left(\frac{1}{b'\left(b^{-1}\left(b_i\right)\right)}\right)$$

Now, make the substitution $b_i = b\left(v_i\right)$, so $b^{-1}\left(b\left(v_i\right)\right) = v_i$. Hence,

$$\begin{aligned}0 &= -\left(1-[1-v_i]^2\right)+2[1-v_i](v_i-b(v_i))\left(\frac{1}{b'(v_i)}\right)\\ &= -\left(1-[1-v_i]^2\right)b'(v_i)+2[1-v_i](v_i-b(v_i))\end{aligned}$$

Step 3: Check the solution satisfies our hypotheses
We have $b_i(v_i)=b(v_i)$, and we have

$$\left(1-[1-v_i]^2\right)b'(v_i)=2[1-v_i](v_i-b(v_i))>0 \text{ for } v_i\in(0,1)$$

Step 4: Solve if you can (lots of partial credit if stop here)
Notice that

$$\frac{d}{dv_i}\left(\left(1-[1-v_i]^2\right)(-b(v_i))\right)=-\left(1-[1-v_i]^2\right)b'(v_i)+2[1-v_i](-b(v_i))$$

Therefore,

$$\frac{d}{dv_i}\left(\left(1-(1-v_i)^2\right)b(v_i)\right)=2v_i(1-v_i)$$

and

$$\left(1-(1-v_i)^2\right)b(v_i)=\int 2v_i(1-v_i)\,dv_i=2\left[\frac{1}{2}v_i^2-\frac{1}{3}v_i^3\right]+C$$

so

$$b(v)=\frac{v^2\left[1-\frac{2}{3}v\right]+C}{v(2-v)}$$

To have $\lim_{v\to 0}b(v)=0$, we need $0=\lim_{v\to 0}\frac{v^2\left[1-\frac{2}{3}v\right]+C}{v(2-v)}=\lim_{v\to 0}\frac{v\left[1-\frac{2}{3}v\right]}{(2-v)}+\lim_{v\to 0}\frac{C}{v(2-v)}=0+\lim_{v\to 0}\frac{C}{v(2-v)}$, which implies that $C=0$.

Therefore,

$$b(v)=v\frac{2}{3}\frac{(3-2v)}{(4-2v)}\leq\frac{2}{3}v$$

and the bidders shade their bids.

5 Problem 4, Final 2010

Stage game: $a\in[0,1]$, $b\in[0,1]$, $u_A(a,b)=2b-a$, $u_B(a,b)=2a-b$.

a) Clear that the best response functions are $a_A(b)=0$, $b_B(a)=0$, since u_A and u_B are decreasing in own strategy. Moreover, clear that any strategy $\hat{a}>0$ is strictly dominated by 0, since

$$u_A(0,b)-u_A(\hat{a},b)=\hat{a}>0$$

Hence, all strategies besides 0 are strictly dominated and deleted in the first round. Therefore, the only rationalizable strategies are $\{0\}$ for each player. The only NE of the stage game is therefore $(a,b)=(0,0)$

b) The worst punishment is the minmax payoff, which is also the Nash equilibrium in this stage game. Consider the following strategy profile: play (a^*,b^*) until someone deviates, and play $(0,0)$ if anyone ever deviates. Then, the single deviation test requires that for every history without a deviation, we have

for A: $2b^*\leq\frac{2b^*-a^*}{1-\delta}\iff 2b^*\left(\frac{\delta}{1-\delta}\right)-\alpha^*\frac{1}{1-\delta}\geq 0$, and for B: $2a^*\leq\frac{2a^*-b^*}{1-\delta}\iff 2a^*\left(\frac{\delta}{1-\delta}\right)-b^*\frac{1}{1-\delta}\geq 0$.

Suppose that $a^* \geq b^*$. Then, if $2b^*\left(\frac{\delta}{1-\delta}\right) - \alpha^* \frac{1}{1-\delta} \geq 0$, it must be the case that $2a^*\left(\frac{\delta}{1-\delta}\right) - b^* \frac{1}{1-\delta} > 2b^*\left(\frac{\delta}{1-\delta}\right) - \alpha^* \frac{1}{1-\delta} \geq 0$. So therefore, only one of the conditions can be binding.

In particular, the frontier of (a^*, b^*) that is sustainable with discount factor δ is given by

$$\min(a^*, b^*) \geq \frac{1}{2\delta}\max(a^*, b^*)$$

It is clear that if $\delta < 0.5$, this inequality cannot hold for $\max(a^*, b^*) > 0$ because $\frac{1}{2\delta}\max(a^*, b^*) > \max(a^*, b^*) \geq \min(a^*, b^*)$, since $2\delta < 1$. Hence, for $\delta < 0.5$, the only sustainable candidate profile is $(a, b) = (0, 0)$. For $\delta \geq 0.5$, the sustainable cooperation contributions (a, b) must satisfy

$$\begin{cases} b \geq \frac{1}{2\delta}a, a > b \\ 2\delta a \geq b, \ b > a \end{cases}$$

The region of sustainable contributions looks like a "kite" with vertices at $(0,0)$, $\left(1, \frac{1}{2\delta}\right)$, $\left(\frac{1}{2\delta}, 1\right)$ and $(1, 1)$. Therefore, the maximum contribution for $\delta \geq 0.5$ from any given player is equal to unity. As δ grows, the contribution pattern can become more and more unequal, with the limit being $\left(1, \frac{1}{2}\right)$, with the payoff for the higher contributor approaching zero.

c) Since play in the stage game has no impact on the continuation value, it must be an NE, so we must have $\left(\hat{a}, \hat{b}\right) = (0, 0)$.

We now look for values of (a_A, b_A) and (a_B, b_B) such that the following is an equilibrium: A accepts any proposal (a, b) such that $\frac{2b-a}{1-\delta} \geq \frac{2b_B - a_B}{1-\delta}$ and B accepts any proposal (a, b) such that $\frac{2a-b}{1-\delta} \geq \frac{2a_A - b_A}{1-\delta}$, A proposes (a_A, b_A) and B proposes (a_B, b_B). Suppose there exists an SPE of this form, and Alice has been made an offer (a, b). Then, next period, Alice will offer (a_A, b_A) and Bob will (barely) accept, so Alice will accept (a, b) iff $\frac{2b-a}{1-\delta} \geq \delta \frac{2b_A - a_A}{1-\delta}$. Hence, it must be the case that

$$(2b_B - a_B) = \delta(2b_A - a_A)$$

Similarly, for Bob, it must be the case that

$$(2a_A - b_A) = \delta(2a_B - b_B)$$

If Alice is proposing, it is obvious that she will not propose (a, b) such that $\frac{2a-b}{1-\delta} < \frac{2a_A - b_A}{1-\delta}$, because her payoff from rejection is $\delta \frac{2b_B - a_B}{1-\delta} < \frac{2b_B - a_B}{1-\delta} < \frac{2b_A - a_A}{1-\delta}$. However, she would want to propose (a, b) such that $\frac{2a-b}{1-\delta} = \frac{2a_A - b_A}{1-\delta}$ and to maximize $\frac{2b-a}{1-\delta}$. Hence, Alice would solve

$$\max_{a,b}(2b - a) \text{ st. } 2a - b = 2a_A - b_A, \ b \in [0, 1], \ a \in [0, 1]$$

Now, we then have that $b = 2a - (2a_A - b_A)$, so the problem becomes

$$\max_{a,b}(3a - 2(2a_A - b_A)) \text{ st. } 2a - (2a_A - b_A) \in [0, 1], \ a \in [0, 1]$$

It is clear that Alice will have a corner solution. Setting $a = 1$ cannot be optimal for Alice. Neither can Alice set $a = 0$, since then, b would have to be negative. Finally, it is obvious that we cannot have $2a - (2a_A - b_A) = 0$, since then we would require $b_A = 0$, which implies $a_A = 0$, which is clearly inefficient. Therefore, Alice will set $b = 2a - (2a_A - b_A) = 1$, so

$$b_A = 1$$

By symmetry, it must be the case that $a_B = 1$. Hence, the acceptance conditions reduce to:

$$(2b_B - 1) = \delta(2 - a_A)$$

and

$$(2a_A - 1) = \delta (2 - b_B)$$

By symmetry, it must be the case that $a_A = b_B =: x$, so we have the single equation

$$2x - 1 = \delta (2 - x) \Rightarrow x (\delta) = \frac{2\delta + 1}{2 + \delta}$$

Note that $x(\delta)$ is increasing in δ from $x(0) = \frac{1}{2}$ to $x(1) = 1$.

Therefore, the profile in which A offers $(a_A, b_A) = (x(\delta), 1)$, B offers $(a_B, b_B) = (1, x(\delta))$, and they accept only offers yielding them at least $U(a_B, b_B)$ and $U(a_A, b_A)$ respectively is an SPE. We can check single deviation at acceptance and proposal nodes (the checks are identical for A and B), but we constructed the SPE assuming these checks hold.