Common Knowledge: The Math

We are going to formalize the role of common knowledge in games with multiple equilibria

Our model will be stylized:

We'll consider a simple coordination game with 2 players, 2 actions, symmetric payoffs

Before players play this game, some event happens

Players get some private signal about this event

Then players simultaneously choose their action in the coordination game

We'll ask the question:

When can players condition their action on their signal?

We will show:

Players can only condition their behavior on signals when those signals induce certain higher order beliefs

We will relate this to our puzzles:

By formalizing the signals generated in each of these puzzles and therefore on behavior in coordination games

Let's start by describing this step formally

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Ω is a set whose elements are all the events that could happen

Jargon…

Ω is "the (finite) set of states of the world"

$\omega \in \Omega$ is "a possible state of the world"

Examples:

Ω = {hot, cold}

Ω = {rainy, sunny}

Ω = {(hot, rainy), (hot, sunny), (cold, rainy), (cold, sunny)}

Next…

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$\pi_i: \Omega \rightarrow \Sigma$, where Σ is the set of possible signals

$\pi_i(\omega)$ represents the signal that player i gets when the event ω occurs

We allow multiple events to induce the same signal for player i. This allows us to represent events that are indistinguishable from player i's perspective

i.e. $\pi_i(\omega) = \pi_i(\omega')$

Formally, π_i partitions Ω into sets of events that are indistinguishable to player i

E.g.,

Suppose player 1 is in a basement with a thermostat but no window

Actual signal is immaterial

$\pi_i((\text{hot, sunny})) = \pi_i((\text{hot, rainy})) = \text{hot}$
$\pi_i((\text{cold, sunny})) = \pi_i((\text{cold, rainy})) = \text{cold}$

This can be represented in a condensed fashion

$\pi_i = \{\{(\text{hot, rainy}), (\text{hot, sunny})\}, \{(\text{cold, rainy}), (\text{cold, sunny})\}\}$

Suppose player 2 is in a high-rise with a window but no thermostat

$\pi_i((\text{hot, sunny})) = \pi_i((\text{cold, sunny})) = \text{sunny}$

$\pi_i((\text{hot, rainy})) = \pi_i((\text{cold, rainy})) = \text{rainy}$

This can be represented in a condensed fashion

$\pi_i = \{\{(\text{hot, rainy}), (\text{cold, rainy})\}, \{(\text{hot, sunny}), (\text{cold, sunny})\}\}$

We next want to describe players' choices

But first we need to specify their beliefs

And to specify beliefs, we need to specify how likely each event is

We let μ represent the "common prior" probability distribution over Ω

I.e. $\mu: \Omega \rightarrow (0, 1]$ s.t. $\Sigma\ ^{\mu(\omega)} = 1$

We interpret μ(ω) as the probability the event ω occurs

E.g.,

$\mu((\text{hot, sunny})) = .45$
$\mu((\text{hot, rainy})) = .05$
$\mu((\text{cold, sunny})) = .05$
$\mu((\text{cold, rainy})) = .45$

We can infer players' posterior beliefs given that they get a given signal

E.g., $\mu((\text{hot, sunny}) | \text{hot}) = \frac{\mu((\text{hot, sunny}))}{\mu((\text{hot, sunny})) + \mu((\text{hot, rainy}))} = .9$

Analogously, $\mu((\text{hot, sunny}) | \text{sunny}) = .9$

To summarize:

$< \Omega, \pi = (\pi_1, \pi_2), \mu >$ is the “information structure”

Ω

is a finite set

interpreted as the set of possible events

π_i

is a partition of Ω

interpreted as sets of events that are indistinguishable to player i

μ

is a function from Ω to $(0, 1]$

interpreted as probability an event occurs

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Regardless of the event, the players will now play the following coordination game

	A	B
A	a, a	b, c
B	c, b	d, d

Assume $a > c$ and $d > b$

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In order to analyze players choices, we need to represent the combination of the information structure and the coordination game as a (meta)game

$$\Gamma = < \underbrace{S=(S_1, S_2)}_{\text{Strategies}}, \underbrace{U=(U_1, U_2)}_{\text{Payoffs}} >$$

$S_i: \pi_i \rightarrow \{A, B\}$

$U_i: S \rightarrow IR$

$$\text{s.t. } U_i(s_1, s_2) = \Sigma_\omega\, \mu(\omega)\, U_i(s_1(\pi_1(\omega), s_2(\pi_2(\omega)))$$

Probability of a state

Expected utility in that state…

Given the actions induced…

By the signals in that state

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What does it mean for “players to condition their action on their signal”?

Players don’t play the same action for every signal

I.e.

$\exists\ \omega, \omega', i$ s.t. $S_i(\pi_i(\omega)) \neq S_i(\pi_i(\omega'))$

S_i is not a constant function

Next, we want to answer this and show…

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But before we can show this, we need some definitions

First, note that we can also refer to a set of states as an event

We'll refer to a generic event as E or F

$E, F \subseteq \Omega$ and $E, F \neq \emptyset$

Now we define a *p-evident event*

E is p-evident if $\forall \omega \in E, \forall i \; \mu(E|\pi_i(\omega)) \geq p$

Whenever E has occurred, everyone believes a state in E has occurred

Example…

(hot, sunny) is evident p for all $p \geq .9$

Proof…

For the event (hot, sunny), both players believe that it is (hot, sunny) with probability .9

Therefore, formally $\mu((\text{hot, sunny})|\pi_1(\text{hot,sunny})) \geq .9$ and $\mu((\text{hot, sunny})|\pi_2(\text{hot,sunny})) \geq .9$

Example of event that is not p-evident for high values of p…

Note $\mu(\text{hot} | \pi_2(\text{hot,rainy})) = .1$

Since 2 can only see that it's rainy, and thinks that when it's rainy, it's unlikely to be hot

Therefore hot is not evident p for any p above .1

I.e. there is a state in which it's hot but there is a player who thinks that it is unlikely to be hot

Doesn't matter that when (hot, sunny), player 2 thinks it's likely to be hot

Finally, we’re ready for the theorem

Suppose $S^* = (S_1^*, S_2^*)$ is an equilibrium, then…

$\exists i$ such that S_i^* is not a constant function

Iff

$\exists$ $E, F \subseteq \Omega$ s.t. $E \cap F = \emptyset$, E is p^*-evident and F is $(1-p^*)$-evident, where $p^* = (d-b)/(d-b+a-c)$

Sketch of proof

Suppose $\exists$ E, F $\subseteq \Omega$ s.t. E$\cap$F=$\emptyset$, E is p*-evident and F is (1-p*)-evident, where $p^* = (d-b)/(d-b+a-c)$

Then we claim there exists an i s.t. S_i^* is non-constant

We'll partially construct that strategy for you

$\forall i \qquad S_i^*(\pi_i(\omega)) = \begin{cases} A \ \forall \omega \in E \\ B \ \forall \omega \in F \end{cases}$

Neither player can benefit from deviating

E occurs➔player 1 p*-believes E has occurred➔1 p*- believes 2 plays A➔1 best responds by playing A

p* is minimal belief that other plays A such that best response is to play A

Likewise for F

For all ω not it E or F, we can deal with them (trust us)

Let’s do the other direction

Suppose $S^* = (S_1^*, S_2^*)$ is an equilibrium in which $\exists i$ such that S_i^* is not a constant function

Let E be the set of states in which both players play A, and F be the set of states in which both play B

Notice that E and F are non-empty and $E \cap F = \emptyset$

Claim: E is p^*-evident and F is $(1-p^*)$-evident

Suppose not

Then $\exists i, \omega$ s.t. $\mu(E|\pi_i(\omega)) < p^*$

Since E is the set of states in which both play A

μ(both play A$|\pi_i(\omega)) < p^*$)

Some (hard) stuff and we can show that

μ(-i plays A$|\pi_i(\omega)) < p^*$)

i best responds by playing B at $\pi_i(\omega)$, since p^* is minimal belief that other plays A such that best response is to play A. This is a contradiction

Next (time) we’ll show how the theorem maps to our applications

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