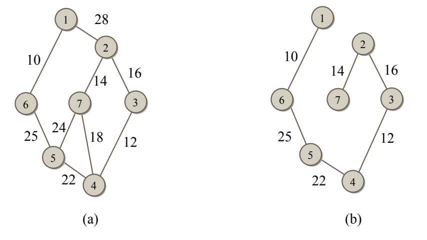
1.204 Lecture 11

Greedy algorithms:
Minimum spanning trees

Minimum spanning tree

- If G is an undirected, connected graph, a subgraph T of G is a spanning tree iff T is a tree with n nodes (or, equivalently, n-1 arcs)
  - A minimum spanning tree is the spanning tree T of G with minimum arc costs



A graph and its minimum cost spanning tree

Figure by MIT OpenCourseWare.

Applications of minimum spanning trees

- **Building wiring, mechanicals**
- **Water, power, gas, CATV, phone, road distribution networks**
- **Copper (conventional) phone networks**
  - MST algorithms not needed, done heuristically
- **Wireless telecom networks**
  - Cell tower connectivity with microwave 'circuits'
  - Cost is not a function of distance, but reliability is
  - East-west links preferred to north-south (ice, sun,…)
  - Topography matters: DEM data
  - Move to fiber optics as better technology
  - Problem is to have a cost-effective, reliable network
    - Not to find the minimum spanning tree
- **System engineer looks at entire issue**
  - MST is one component of a broader solution

Prim's algorithm

- **Greedy method to build minimum spanning tree**
  - Start at an arbitrary node (root)
  - The set of arcs selected always form a tree T
    - Initially the tree T is just the root. No arcs added to it yet.
  - The next arc (u,v) to be included in T is:
    - Minimum cost arc such that
    - Both nodes u and v are not in T already
  - Add arc (u,v) and node v to T
    - Mark node v as being in T, or <u>visited</u> (u is already in the tree)
    - $T \cup \{(u,v)\}$ is now the new tree T
  - End when all nodes in tree have been visited, or
    - Equivalently, when (n-1) arcs have been put in the spanning tree

Prim’s algorithm example

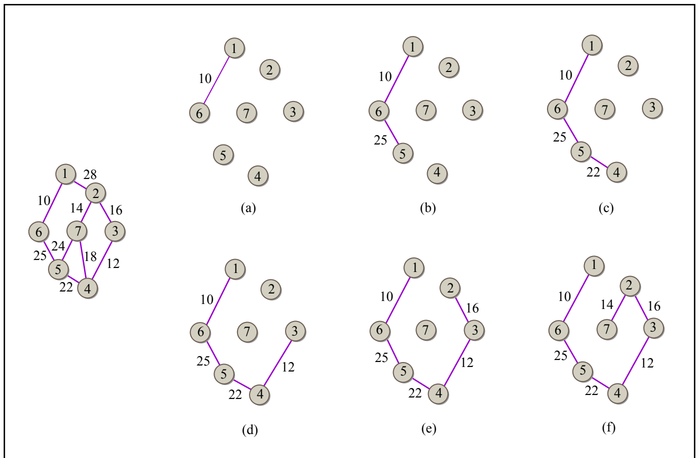


Figure by MIT OpenCourseWare.

Standard Prim: data members, constructor

```
public class Prim {    // Assumes connected graph; not checked
   private int nodes;  // Assumes consecutive node numbers
   private int[] head;
   private int[] to;
   private int[] dist;
   private int[] P;           // Predecessor node back to root
   private boolean[] visited;         // Has node been visited
   private int MSTcost;

   Prim(int n, int[] h, int[] t, int[] d) {
        nodes = n;            // Or set nodes= head.length-1
        head = h;
        to = t;
        dist = d;
   }
```

Standard Prim: prim(), p.1

```
public int prim(int root) {
    P = new int[nodes];           // Predecessor node in MST
    visited = new boolean[nodes]; // Has node been visited
    for (int i = 0; i < nodes; i++) {     // Initialize
           P[i] = -1;     // No predecessor on path
    }
    visited[root] = true;         // Initialize root node

    // Continued on next slide
```

Standard Prim: prim(), p.2

```
for (int i = 0; i < nodes-1; i++) {   // Add nodes-1 arcs
  int minDist = Integer.MAX_VALUE;
  int nextNode = -1;   // Next node to be added to MST
  int pred = -1; // Predecessor of next node added to MST
  // Find node w/ min distance via arc from already visited set
  for (int node = 0; node < nodes; node++) {
    if (visited[node])
      for (int arc = head[node]; arc < head[node + 1]; arc++) {
        int dest = to[arc];
        if (!visited[dest] && dist[arc] < minDist) {
          minDist = dist[arc];
          nextNode = dest;
          pred = node;
        }
      }
  }
  visited[nextNode] = true;
  P[nextNode] = pred;
  MSTcost += minDist; }
return MSTcost;}
```

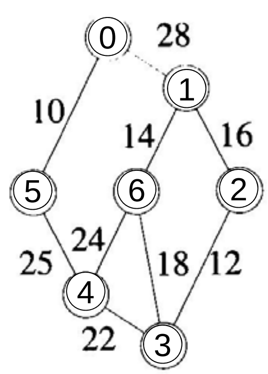
Standard Prim: print(), main()

```
public void print() {
     System.out.println("i \tP");
     for (int i = 0; i < nodes; i++) {
             if (P[i] == -1)
                     System.out.println(i + "\t-");
             else
                     System.out.println(i + "\t" + P[i]);
     }
     System.out.println("MST cost: " + MSTcost);
}

public static void main(String[] args) {
     // Create test data (H&S p. 237—see download)
     Prim p = new Prim(nodes, hh, tt, dd);
     p.prim(root);
     p.print();
}
```

Prim's algorithm code, standard version, output

```
i  P
0  -
1  2
2  3
3  4
4  5
5  0
6  1
MST cost: 99
```



Node numbers start at 0, not 1, compared to first example

Better Prim algorithm

- **In each node iteration in the standard version:**
  - **We go through the arcs out of every visited node each time a node is added to the tree, looking for the shortest arc from any node**
  - **This is a lot of repetitive work: We look at each arc about n/2 times to see if it's the shortest, and it almost never is**
  - **Standard Prim is O(na), for number of nodes n and arcs a**
- **If we keep the arcs out of visited nodes in a heap, we can just add arcs from a newly visited node to the heap, an O(lg n) operation, rather than the O(n) standard scan**
  - **In each iteration we then delete the shortest arc from the heap:**
    - **If its destination has been visited, ignore it and delete the next arc from the heap**
    - **Otherwise, add the arc to the MST**
- **This is O(a lg n), where a is the number of arcs**
  - **Complexity proof easy except whether to use 'lg n' or 'lg a'**
  - **Since 'n' and 'a' usually proportional, it's not a major issue**
  - **Also, sorting to create the network takes O(a lg a) steps**

PrimHeap: arc class

```
public class MSTArc implements Comparable {
   int from;          // Package access
   int to;            // Package access
   int dist;          // Package access

   public MSTArc(int f, int t, int d) {
        from= f;
        to= t;
        dist= d;
   }
   public String toString() {
        return (" from: "+ from+ " to: "+ to + " dist: "+ dist);
   }
   public int compareTo(Object o) {
        MSTArc other = (MSTArc) o;
        if (dist > other.dist)        // Ascending sort with
               return -1;             // max heap to get min arc
        else if (dist < other.dist)
               return 1;
        else
               return 0;
}  }
```

PrimHeap: data members, constructor

```
public class PrimHeap {   // Assumes connected graph; not checked
   private int nodes;     // Assumes consecutive node numbers
   private int arcs;
   private int[] head;
   private int[] to;
   private int[] dist;
   private boolean[] visited; // Has node been visited in Prim
   private int MSTcost;
   private Heap g;
   private MSTArc[] inMST;    // Arcs in MST

   PrimHeap(int n, int a, int[] h, int[] t, int[] d) {
        nodes = n;
        arcs= a;
        head = h;
        to = t;
        dist = d;
        g= new Heap(arcs);
        inMST= new MSTArc[nodes];
  }
```

PrimHeap: prim()

```
public int prim(int root) {
     visited = new boolean[nodes];
     MSTArc inArc= null;
     int k= 0;                      // Index of arcs in MST
     visited[root] = true;          // Initialize root node
     for (int arc= head[root]; arc< head[root+1]; arc++)
       g.insert(new MSTArc(root, to[arc], dist[arc]));

     for (int i = 0; i < nodes-1; i++) {   // Add (nodes-1) arcs
       do {  // Find shortest arc to node not yet visited
         inArc= (MSTArc) g.delete();
       } while (visited[inArc.to]);
       inMST[k++]= inArc;
       int inNode= inArc.to;
       visited[inNode] = true;
       MSTcost += inArc.dist;
       for (int arc= head[inNode]; arc< head[inNode+1]; arc++)
         g.insert(new MSTArc(inNode, to[arc], dist[arc]));
     }                              // O(a lg n)
     return MSTcost;
}
```

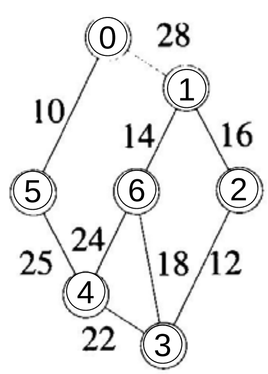
PrimHeap: print(), main()

```
public void print() {
     System.out.println("Arcs in MST");
     for (int i = 0; i < nodes-1; i++) {
             System.out.println(inMST[i]);
     }
     System.out.println("MST cost: " + MSTcost);
}

public static void main(String[] args) {
     // Create test data (H&S p. 237)—see download
     PrimHeap p = new PrimHeap(nodes, arcs, hh, tt, dd);
     p.prim(root);
     p.print();
}
```

Prim's algorithm code, heap version, output

```
Arcs in MST
 from: 0 to: 5 dist: 10
 from: 5 to: 4 dist: 25
 from: 4 to: 3 dist: 22
 from: 3 to: 2 dist: 12
 from: 2 to: 1 dist: 16
 from: 1 to: 6 dist: 14
MST cost: 99
```



Node numbers start at 0, not 1, compared to first example

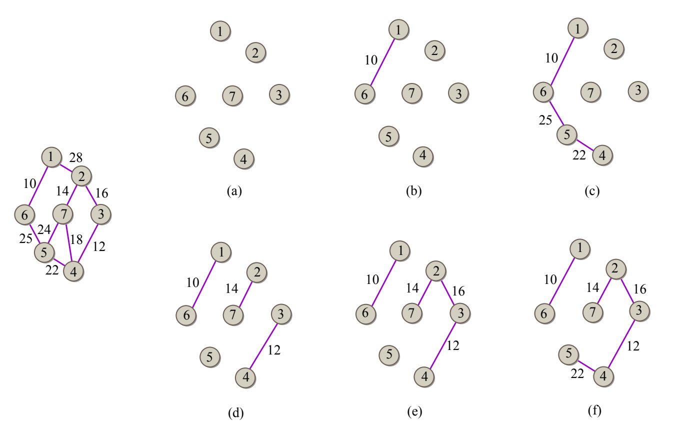
Kruskal's algorithm

- **A different greedy method to build minimum spanning tree:**

```
Tree T is empty;
Heap A contains all arcs, from lowest to highest cost
While (T has fewer than n-1 arcs) && (A has more arcs) {
  Delete arc (v,w) from A
  If arc (v,w) does not create a cycle in T
   Add arc (v,w) to T
  Else
   Discard arc (v,w)
```

- **To detect cycles, we need to know if the origin and destination nodes of the candidate entering arc are already connected**
  - **Doing this efficiently is key to Kruskal's algorithm**
  - **We place connected nodes in the same Set**
  - **The arcs will be a forest (set of disconnected subtrees) until the end**
- **We place the arcs in a Heap**
  - **We only need the minimum arc in each iteration, not a complete sort**

Kruskal's algorithm example



Stages in Kruskal's algorithm

Figure by MIT OpenCourseWare.

Kruskal: data members, constructor

```
public class Kruskal {    // Assumes connected graph; not checked
   private int nodes;     // Assumes consecutive node numbers
   private int arcs;
   private MSTArc[] inMST; // Arcs in MST
   private int MSTcost;
   private Heap g;
   private Set s;
   Kruskal(int n, int a, MSTArc[] arcList) {
        nodes = n;
        arcs = a;
        inMST= new MSTArc[nodes];
        s= new Set(nodes);
        g= new Heap(arcList);
   }
```

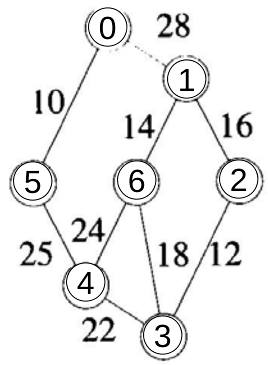
Kruskal: kruskal()

```
   public void kruskal() {
        int i= 0;  // Index in inMST array where arcs are placed
        for (int arc= 0; arc < arcs; arc++) {
                MSTArc d= (MSTArc) g.delete();
                int j= s.collapsingFind(d.from);
                int k= s.collapsingFind(d.to);
                if (j != k) {
                        inMST[i++]= d;
                        MSTcost += d.dist;
                        s.weightedUnion(j, k);
                }
                if (i == nodes - 1)
                        break;
        }
   }

// print() and main() same as PrimHeap (except call kruskal() in main)
// MSTArc class same as in PrimHeap
// Once you're comfortable with the MST codes, move them to Graph class
// KruskalAdjArray class in download uses adjacency array
```

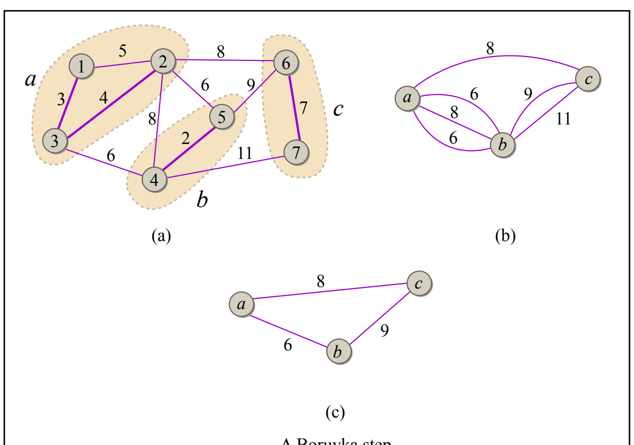
Kruskal's algorithm code, output

```
Arcs in MST
 from: 0 to: 5 dist: 10
 from: 3 to: 2 dist: 12
 from: 1 to: 6 dist: 14
 from: 1 to: 2 dist: 16
 from: 3 to: 4 dist: 22
 from: 4 to: 5 dist: 25
MST cost: 99
```



Node numbers start at 0, not 1, compared to first example

Improving Kruskal: Boruvka steps



A Boruvka step

Figure by MIT OpenCourseWare.

Summary: Minimum spanning trees

- Prim:
  - Without heap: O(na), where n is number of nodes
  - With heap: O(a lg n), where a is number of arcs
- Kruskal
  - Standard: O(a lg n), where a is number of arcs
  - Randomized: O'(n + a), where O' is 'high probability' running time of random element
    - See text, p. 53-54
- Prim with heap and standard Kruskal are usual implementation choices
  - Fast, straightforward
- Add these to your Graph class if you wish
  - Use symmetric directed graph in implementation
  - Minor changes to constructor for add'l data members