16.90 Numerical Methods for Aerospace Engineering
Spring 2014, Lecture 2

- Discretizing a function of time
- Approximating Derivative
- Truncation Error Analysis (Taylor series)
- Local order of accuracy
- The Most-Accurate-Scheme contest

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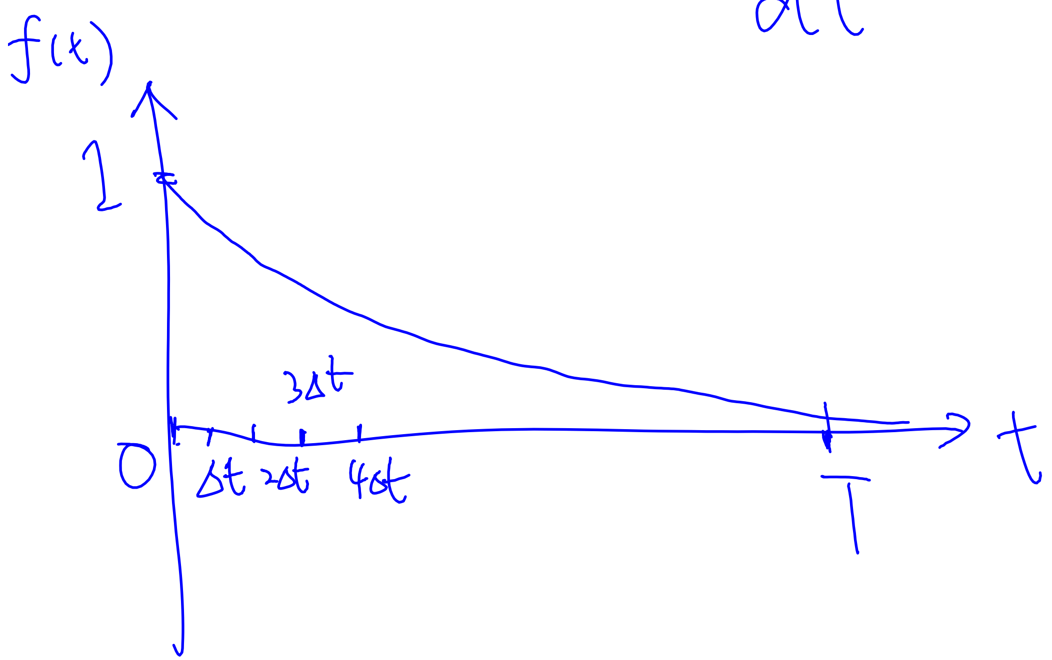
- **Discretizing a function of time**

$$f(t) = e^{-\lambda t} \qquad \frac{df}{dt} = -\lambda \cdot f(t)$$



$$\frac{df}{dt} = -f(t)^2$$

$$\frac{df}{f^2} = -dt$$

$$-\frac{1}{f} + C = -t$$

$$f(t) = \frac{1}{t+C}$$

if $f(0) = 1$ then

$$f(t) = \frac{1}{t+1}$$

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- Discretizing a function of time
- **Approximating Derivative**

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \approx \underline{\underline{\frac{f(t+\Delta t) - f(t)}{\Delta t}}}$$

$$f(t) = e^{-t} \qquad \frac{df}{dt} = -e^{-t}$$

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- Discretizing a function of time
- **Approximating Derivative**
- Truncation Error Analysis (Taylor series)

$$\tau = \left.\frac{df}{dt}\right|_t - \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Easy:

$$f(t+\Delta t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t)}{k!} \Delta t^k$$

$$= f(t) + \frac{df}{dt}\Delta t + O(\Delta t^2)$$

$$\tau = \left.\frac{df}{dt}\right|_t - \frac{\left(\cancel{f(t)} + \frac{df}{dt}\Delta t + O(\Delta t^2)\right) - \cancel{f(t)}}{\Delta t}$$

$$= \cancel{\left.\frac{df}{dt}\right|_t} - \frac{\cancel{\frac{df}{dt}\cdot\Delta t} + O(\Delta t^2)}{\cancel{\Delta t}}$$

$$= \frac{O(\Delta t^2)}{\Delta t} = O(\Delta t)$$

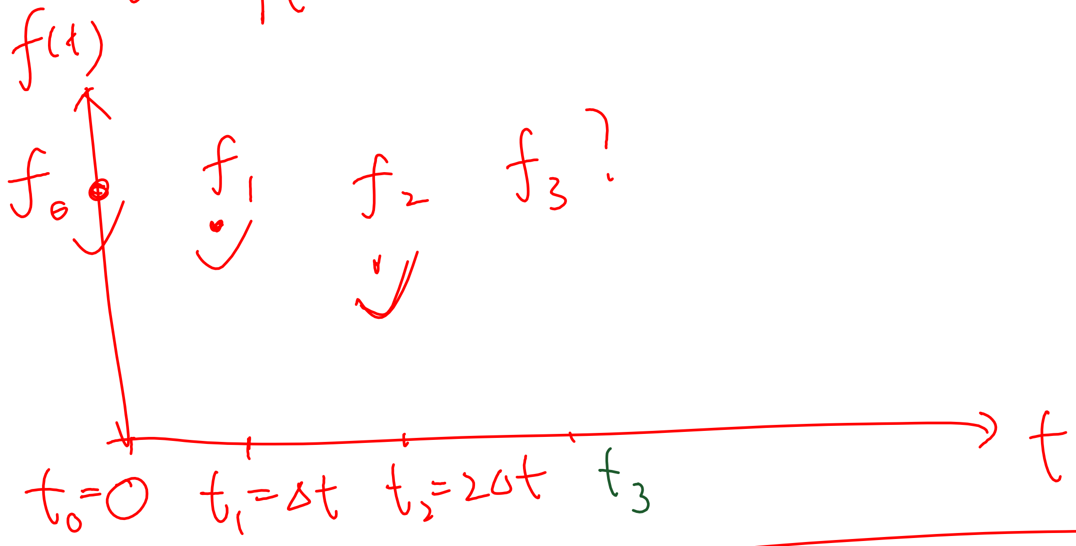
$$\tau = \left.\frac{df}{dt}\right|_t - \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(t+\Delta t)}{k!}(-\Delta t)^k$$

$$\boxed{f(y) = \sum_{k=0}^{\infty} \left(\frac{f^{(k)}(x)}{k!}\right)(y-x)^k}$$

$$\boxed{\frac{df}{dy} = \sum_{k=0}^{\infty} k\frac{f^{(k)}(x)}{k!}(y-x)^{k-1}}$$

$$\left.\frac{df}{dt}\right|_t = \frac{f(t+\Delta t) - f(t)}{\Delta t} \quad (*) \quad ✓$$



$$\frac{df}{dt} = -\lambda \cdot f(t) \quad (✓)$$

$$\left.\frac{df}{dt}\right|_{t_2} = \underbrace{\frac{f(t_3) - f(t_2)}{\Delta t} = -\lambda f(t_2)}$$

$$\boxed{f_3 = f_2 - \Delta t \cdot \lambda f_2}$$

$$f_{i+1} = f_i - \Delta t \lambda f_i$$

Forward Euler

$$f_4 = f_3 - \Delta t \lambda f_3$$

Different scheme.

$$\frac{du}{dt} = -\lambda u(t)$$

$$\left.\frac{du}{dt}\right|_t = \frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t}$$

① Why it's a good approximation

Taylor: $$u(t+\Delta t) = u(t) + \left.\frac{du}{dt}\right|_t \Delta t + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2} + O(\Delta t^3)$$

$$u(t-\Delta t) = u(t) + \left.\frac{du}{dt}\right|_t (-\Delta t) + \frac{d^2u}{dt^2}\frac{\Delta t^2}{2} + O((-\Delta t)^3)$$

$$\frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = \frac{2\Delta t \left.\frac{du}{dt}\right|_t + O(\Delta t^3)}{2\Delta t}$$

$$= \left.\frac{du}{dt}\right|_t + O(\Delta t^2)$$

$$O(1) \supset O(\Delta t) \supset O(\Delta t^2) \supset O(\Delta t^3) \cdots$$

② How to make scheme

$$\left.\frac{du}{dt}\right|_t = \frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t}$$

$$\downarrow$$

$$\frac{du}{dt} = -\lambda u$$

$$\frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = -\lambda u(t)$$

$$u(t+\Delta t) = u(t-\Delta t) - 2\Delta t \lambda u(t)$$

$$u_{i+1} = u_{i-1} - 2\Delta t \lambda u_i$$ <u>Midpoint</u>

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(1) $\tau = \left.\frac{du}{dt}\right|_t - \frac{u(t+\Delta t) - u(t)}{\Delta t} = O(\Delta t)$

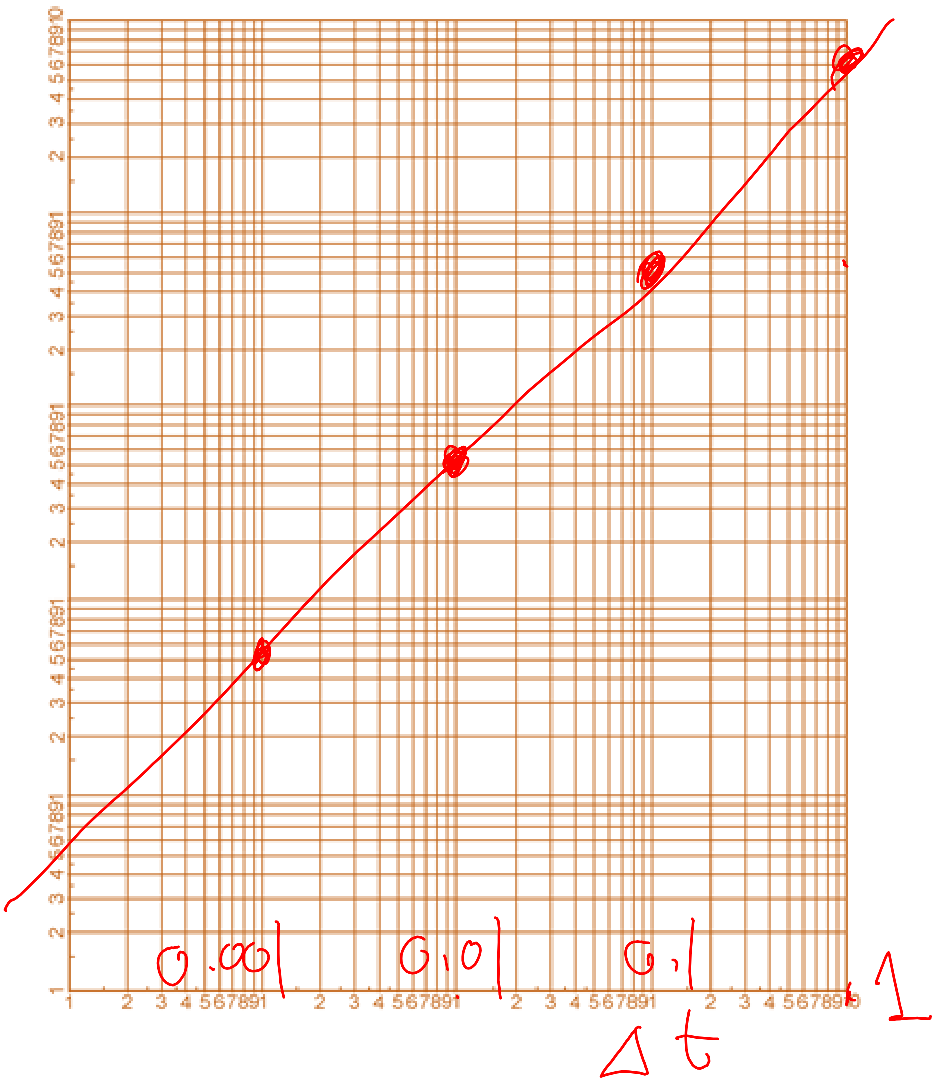
F.W.

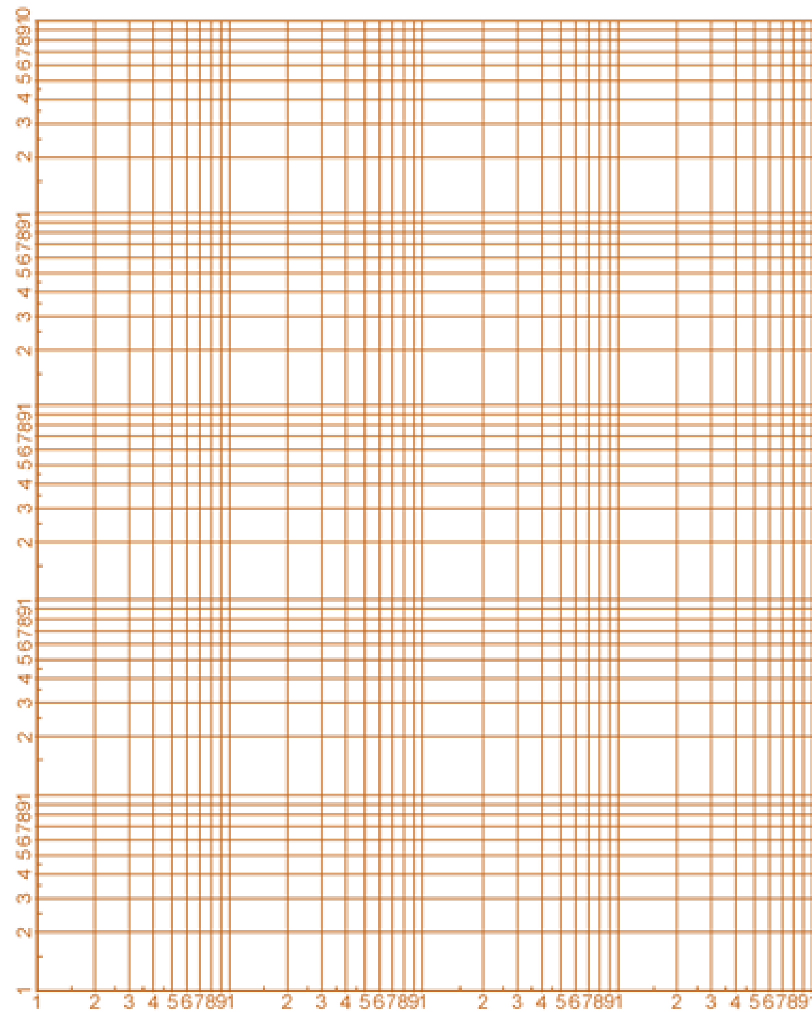
(2) $\tau = \left.\frac{du}{dt}\right|_t - \frac{u(t+\Delta t) - u(t-\Delta t)}{2\Delta t} = O(\Delta t^2)$

Midpoint

Local order of accuracy: convergence rate of truncation error (log-log plot)

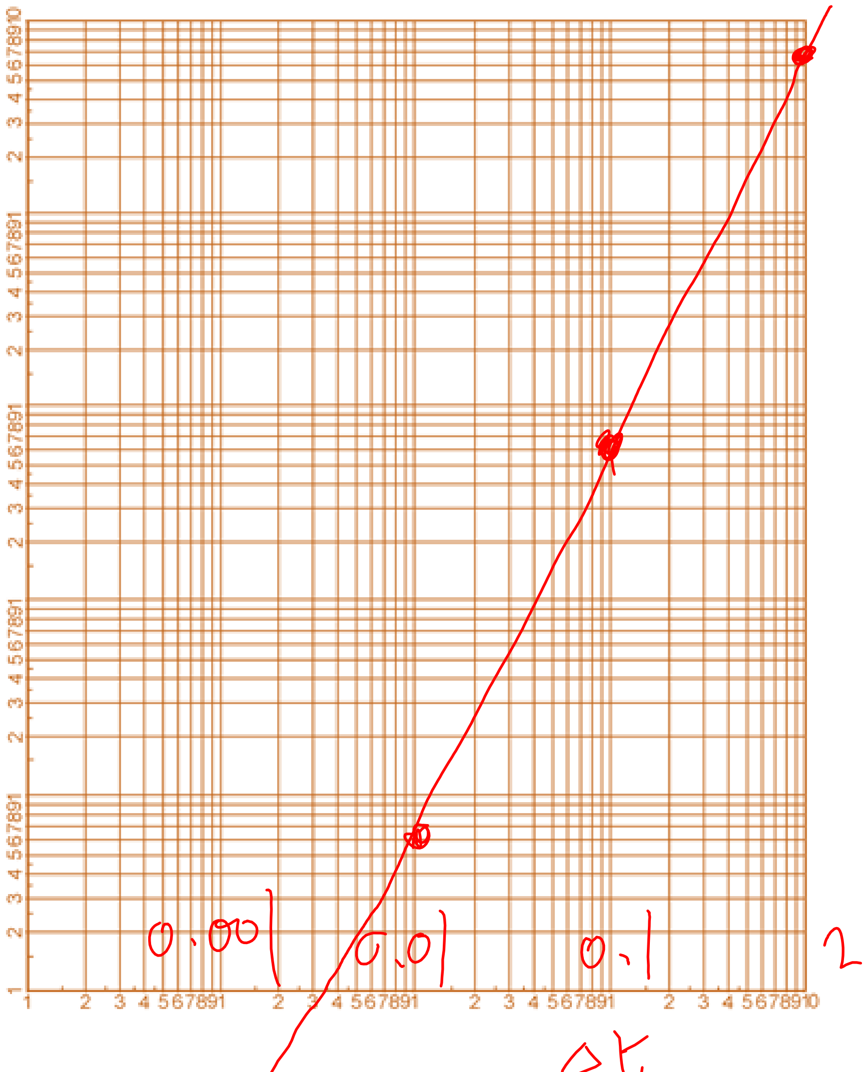
- **First** order accuracy: truncation error decreases as

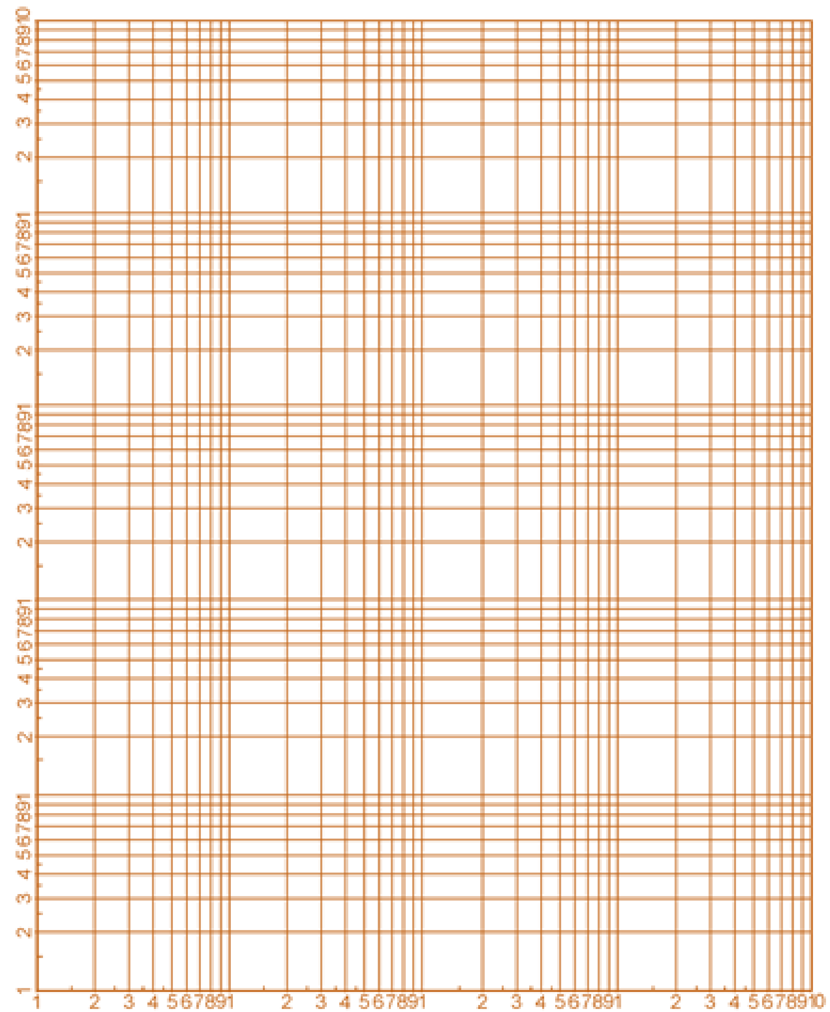




Local order of accuracy: convergence rate of truncation error (log-log plot)

- **Second** order accuracy: truncation error decreases as





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Local order: the best X scheme

1. Best implicit, one step scheme

$$u(t+\Delta t) = \boxed{?\,\frac{du}{dt}\Big|_{t+\Delta t}} + ?\,u(t) + ?\,\frac{du}{dt}\Big|_{t}$$

2. Best explicit, two step scheme

$$u(t+\Delta t) = ?\,u(t) + ?\,\frac{du}{dt}\Big|_{t} + ?\,u(t-\Delta t) + ?\,\frac{du}{dt}\Big|_{t-\Delta t}$$

3. Best implicit, two step scheme

$$u(t+\Delta t) = ?\,\frac{du}{dt}\Big|_{t+\Delta t} + ?\,u(t) + ?\,\frac{du}{dt}\Big|_{t} + ?\,u(t-\Delta t) + ?\,\frac{du}{dt}\Big|_{t-\Delta t}$$

4. Best explicit, three step scheme

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