16.346 Astrodynamics
Fall 2008

Exercises 01

1. Given

$$\mathbf{a} = 2\mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$
$$\mathbf{b} = \mathbf{i}_x + 2\mathbf{i}_y - \mathbf{i}_z$$
$$\mathbf{c} = \mathbf{i}_x + \mathbf{i}_y - 2\mathbf{i}_z$$

find a vector parallel to the plane of $\mathbf{b}$ and $\mathbf{c}$ and perpendicular to $\mathbf{a}$.

2. Find the angle between the vectors

$$\mathbf{a} = \mathbf{i}_x + \mathbf{i}_y + 2\mathbf{i}_z$$
$$\mathbf{b} = 2\mathbf{i}_x - \mathbf{i}_y + \mathbf{i}_z$$

3. The vectors from the origin to the points A, B, C are

$$\mathbf{a} = \mathbf{i}_x + \mathbf{i}_y - \mathbf{i}_z$$
$$\mathbf{b} = 3\mathbf{i}_x + 3\mathbf{i}_y + 2\mathbf{i}_z$$
$$\mathbf{c} = 3\mathbf{i}_x - \mathbf{i}_y - 2\mathbf{i}_z$$

Find the distance from the origin to the plane ABC.

4. By means of products express the condition that three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be parallel to a plane.

5. **Prob G–4** Consider a triangle with sides a, b and c. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors representing the sides of the triangle, use Vector Algebra to derive the Law of Cosines

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

6. **Prob 3–13** From the pericenter and apocenter radii r_p and r_a, the semimajor and semiminor axes and the parameter of an orbit can be conveniently obtained. Show that

$$\boxed{a = \tfrac{1}{2}(r_p + r_a)} \qquad \boxed{b = \sqrt{r_p r_a}} \qquad \boxed{\frac{1}{p} = \frac{1}{2}\left(\frac{1}{r_p} + \frac{1}{r_a}\right)} \quad \text{or} \quad p = \frac{2 r_p r_a}{r_p + r_a}$$

7. Prob 3–1 To derive the equations of motion in polar coordinates we differentiate the vector representing the velocity in polar coordinates. We have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{i}_r + \left[2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}\right]\mathbf{i}_\theta = -\frac{\mu}{r^2}\,\mathbf{i}_r$$

so that

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 + \frac{\mu}{r^2} = 0 \qquad \text{and} \qquad 2\frac{dr}{dt}\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2} = 0$$

8. Prob. 3–6 The second of these equations of motion can be integrated to produce Kepler's second law

$$r^2\frac{d\theta}{dt} = h$$

which, in this form, also provides a transformation of independent variable from t to θ as given by

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta}$$

Similarly,

$$\frac{d^2r}{dt^2} = \frac{h}{r^2}\frac{d}{d\theta}\left(\frac{h}{r^2}\frac{dr}{d\theta}\right) = \frac{h^2}{r^4}\frac{d^2r}{d\theta^2} - 2\frac{h^2}{r^5}\left(\frac{dr}{d\theta}\right)^2$$

Substituting in the first of the equations of motion gives

$$\frac{1}{r^2}\frac{d^2r}{d\theta^2} - \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r} = -\frac{\mu}{h^2} = -\frac{1}{p}$$

Next, we replace the dependent variable r by its reciprocal $1/r$ to obtain

$$\frac{d}{d\theta}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) = -\frac{1}{r^2}\frac{d^2r}{d\theta^2} + \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{p} - \frac{1}{r}$$

so that

$$\boxed{\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{1}{r} = \frac{1}{p}}$$

is obtained as a linear, constant-coefficient, second-order differential equation for $1/r$.

The solution provides an independent derivation of the equation of orbit and is readily obtained as

$$\frac{1}{r} = \frac{1}{p} + c_1\cos\theta + c_2\sin\theta$$

First $\theta = \frac{1}{2}\pi \Longrightarrow c_2 = 0$ Then $\theta = 0 \Longrightarrow c_1 p = e$ Hence

$$\frac{1}{r} = \frac{1}{p} + \frac{e}{p}\cos\theta \qquad \text{or} \qquad r = \frac{p}{1 + e\cos\theta}$$

9. Prob. 3–14 Suppose the force of attraction is proportional to the distance separating m_1 and m_2 rather than inversely proportional to the square of the distance. The equations of motion would be

$$\begin{aligned} m_1 \frac{d^2\mathbf{r}_1}{dt^2} &= Gm_1m_2(\mathbf{r}_2 - \mathbf{r}_1) \\ m_2 \frac{d^2\mathbf{r}_2}{dt^2} &= Gm_2m_1(\mathbf{r}_1 - \mathbf{r}_2) \end{aligned} \qquad \Longrightarrow \qquad \frac{d\mathbf{v}}{dt} + G(m_1 + m_2)\mathbf{r} = \mathbf{0}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mu = G(m_1 + m_2)$.

Then from

$$\mathbf{r} \times \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{v}) = -\mu\mathbf{r} \times \mathbf{r} = \mathbf{0}$$

we have

$$\mathbf{h} = \mathbf{r} \times \mathbf{v}$$

so that angular momentum is preserved.

The equations of motion are linear with constant coefficients and can be solved directly. We have

$$\mathbf{r} = \cos\sqrt{\mu}\,t\,\mathbf{c}_1 + \sin\sqrt{\mu}\,t\,\mathbf{c}_2$$

where $\mathbf{c}_1$ and $\mathbf{c}_2$ are constant vectors. The motion is planar so, for convenience, we can assume that the orbit is confined to the x, y plane. Hence, the equations of motion are

$$\begin{aligned} x &= \cos\sqrt{\mu}\,t\,c_{11} + \sin\sqrt{\mu}\,t\,c_{21} \\ y &= \cos\sqrt{\mu}\,t\,c_{12} + \sin\sqrt{\mu}\,t\,c_{22} \end{aligned}$$

Next, solve for $\cos\sqrt{\mu}\,t$ and $\sin\sqrt{\mu}\,t$

$$\cos\sqrt{\mu}\,t = \frac{\begin{vmatrix} x & c_{21} \\ y & c_{22} \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{vmatrix}} \qquad \sin\sqrt{\mu}\,t = \frac{\begin{vmatrix} c_{11} & x \\ c_{12} & y \end{vmatrix}}{\begin{vmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{vmatrix}}$$

and then eliminate the trigonometric functions by squaring and adding

$$\begin{vmatrix} x & c_{21} \\ y & c_{22} \end{vmatrix}^2 + \begin{vmatrix} c_{11} & x \\ c_{12} & y \end{vmatrix}^2 = \begin{vmatrix} c_{11} & c_{21} \\ c_{12} & c_{22} \end{vmatrix}^2$$

The result is an equation for the ellipse

$$(c_{22}x - c_{21}y)^2 + (c_{11}y - c_{12}x)^2 = (c_{11}c_{22} - c_{12}c_{21})^2$$

or

$$(c_{22}^2 + c_{12}^2)x^2 - 2(c_{22}c_{21} + c_{11}c_{12})xy + (c_{21}^2 + c_{11}^2)y^2 = (c_{11}c_{22} - c_{12}c_{21})^2$$

with the mass m_1 at the center of the ellipse.