THE Z-TRANSFORM

Solution 5.1

For convergence of the Fourier transform, the sequence must be absolutely summable or square summable, (i.e. have finite energy) depending on the type of convergence to be considered.

In this problem, sequences (i) and (iv) are neither absolutely summable nor square summable, and thus their Fourier transforms do not converge. Sequences (ii) and (iii) are both absolutely summable and square summable.

Solution 5.2

(a)

(i) $$X(z) = \sum_{n=-\infty}^{+\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \delta(n) z^{-n} + \sum_{n=-\infty}^{+\infty} \left(\frac{1}{2}\right)^n u(n) z^{-n}$$

$$= 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{2} z^{-1}\right)^n = \frac{1}{1 - \frac{1}{2} z^{-1}} \quad \text{for } \left|\frac{1}{2} z^{-1}\right| < 1 .$$

$$\therefore \quad X(z) = 1 + \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{2 - \frac{1}{2} z^{-1}}{1 - \frac{1}{2} z^{-1}}; \quad |z| > \frac{1}{2}$$

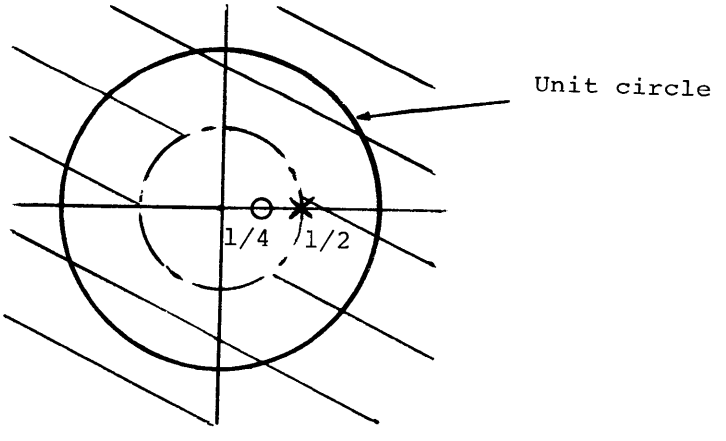


Figure S5.2-1

We obtain the z-transforms of the remaining sequences in a similar manner:

(ii) $X(z) = \dfrac{1}{1 - 3z^{-1}}; \quad |z| > 3$

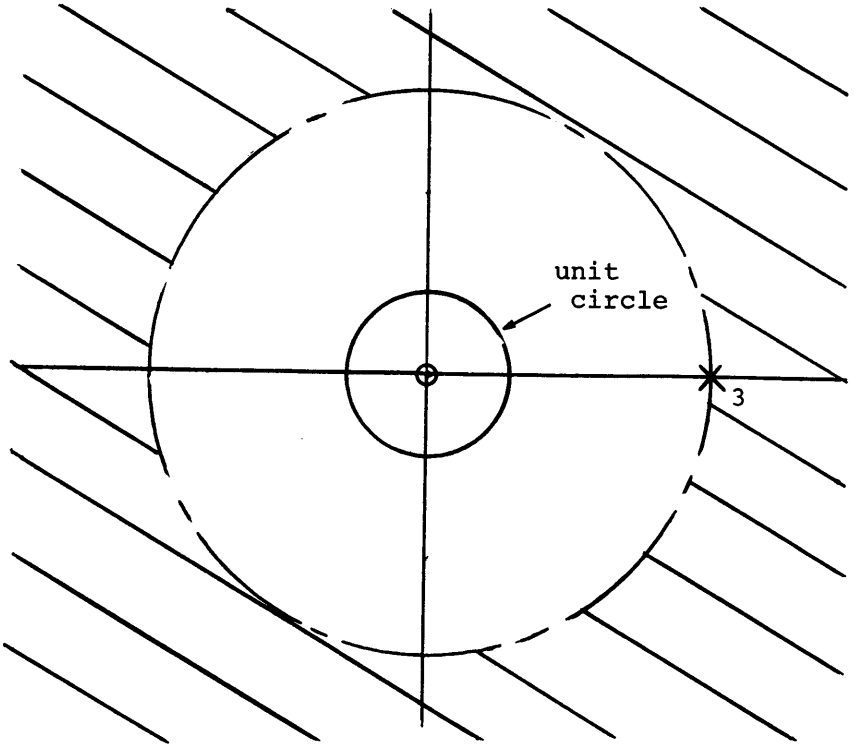


Figure S5.2-2

(iii) $X(z) = \dfrac{1}{1 - \frac{1}{2}z^{-1}} + \dfrac{1}{1 - \frac{1}{3}z^{-1}}; \qquad |z| > \frac{1}{2}$

$$= \frac{(2 - \frac{5}{6}z^{-1})}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{3}z^{-1})}; \qquad |z| > \frac{1}{2}$$

(iv) $X(z) = 1 - \frac{1}{8}z^{-3}; \; z \neq 0$

There is a third order pole at z = 0. The zeros occur at the three cube roots of (1/8), i.e. at

$$z = \frac{1}{2}, \; \frac{1}{2}\, e^{j\frac{2\pi}{3}}, \; \frac{1}{2}\, e^{-j\frac{2\pi}{3}}$$

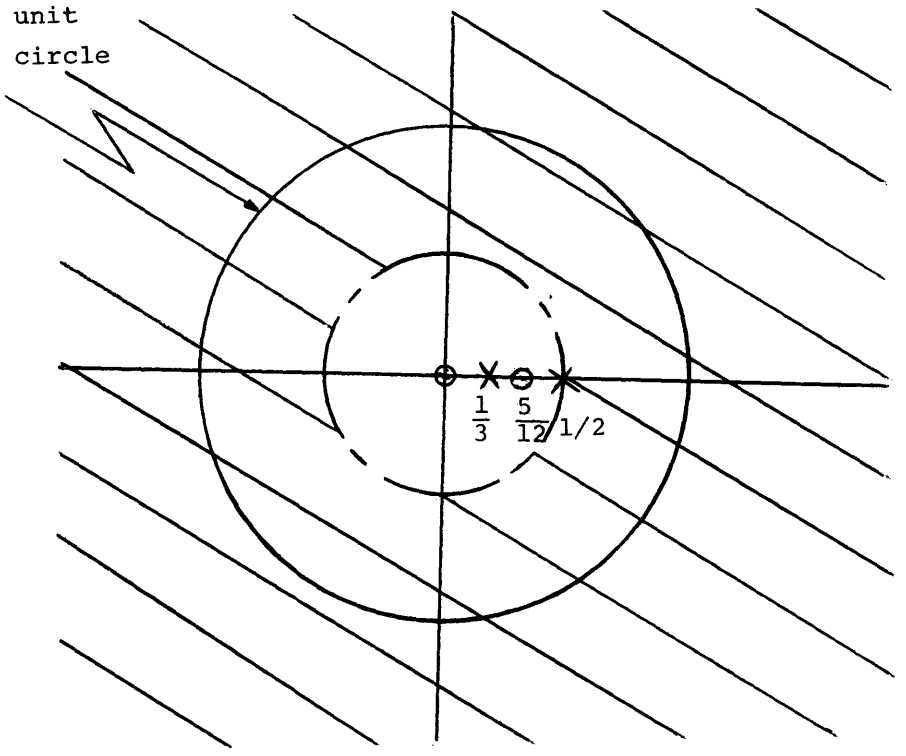
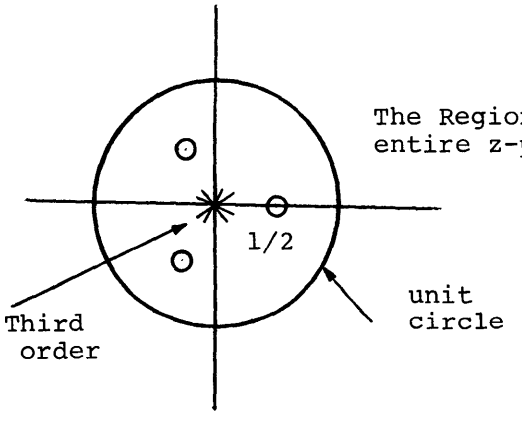


Figure S5.2-3



The Region of convergence is entire z-plane except z = 0

Figure S5.2-4

$$(v)\quad X(z) = \sum_{-\infty}^{-1} \left(\frac{1}{2}\right)^{-n} z^{-n} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

$$= \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{n} + \sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n} - 1$$

$$\sum_{0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{n} = \frac{1}{1 - \frac{1}{2}z} \; ; \quad |z| < 2$$

$$\sum_{0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{1}{1 - \frac{1}{2}z^{-1}}; \quad |z| > \frac{1}{2}$$

$$\therefore X(z) = \frac{1}{1 - \frac{1}{2}z} + \frac{1}{1 - \frac{1}{2}z^{-1}} - 1; \quad \frac{1}{2} < |z| < 2$$

$$= \frac{-\frac{3}{2}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - 2z^{-1})}; \quad \frac{1}{2} < |z| < 2$$

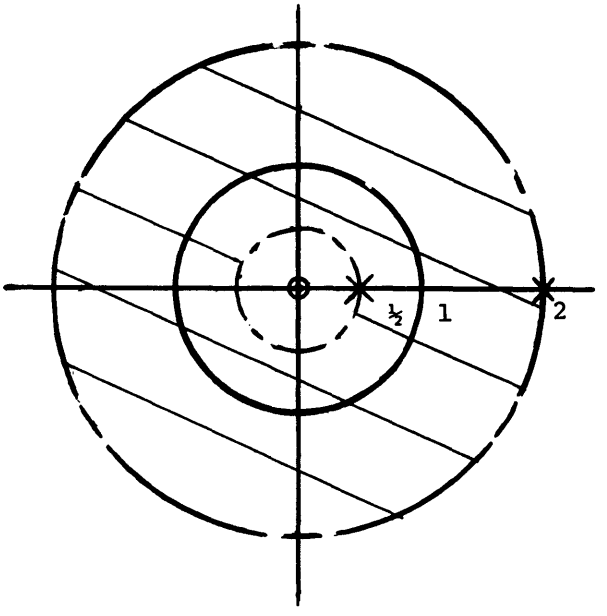


Figure S5.2-5

(b) In order that a sequence correspond to the unit sample response of a stable system the region of convergence must include the unit circle. Thus only sequence (ii) would not correspond to a stable system.

Solution 5.3

(a) If the Fourier transform converges then the region of convergence includes the unit circle. It is bounded by two circles extending outward and inward to the nearest poles. Thus the region of convergence is $\frac{1}{3} < |z| < 2$. Since it does not extend to zero or to infinity, the sequence is two-sided.

(b) If the sequence is two-sided then the region of convergence must be bounded by poles. Thus, the given pole-zero pattern with the region of convergence either as $\frac{1}{3} < |z| < 2$ or as $2 < |z| < 3$ will correspond to two-sided sequences.

Solution 5.4

Let $X_0(z)$ denote the z-transform of $x(n + n_0)$

then $X_0(z) = \sum_{n=-\infty}^{+\infty} x(n + n_0) z^{-n}$

let $m = n + n_0$. Then

$$X_0(z) = \sum_{m=-\infty}^{+\infty} x(m) z^{n_0} z^{-m}$$

$$= z^{n_0} \sum_{m=-\infty}^{+\infty} x(m) z^{-m}$$

$$= z^{n_0} X(z)$$

Solution 5.5*

(a) $$X(z) = \frac{z(1-\alpha^2)}{(1-\alpha z)(z-\alpha)} \; ; \; |\alpha| < |z| < \frac{1}{|\alpha|}$$

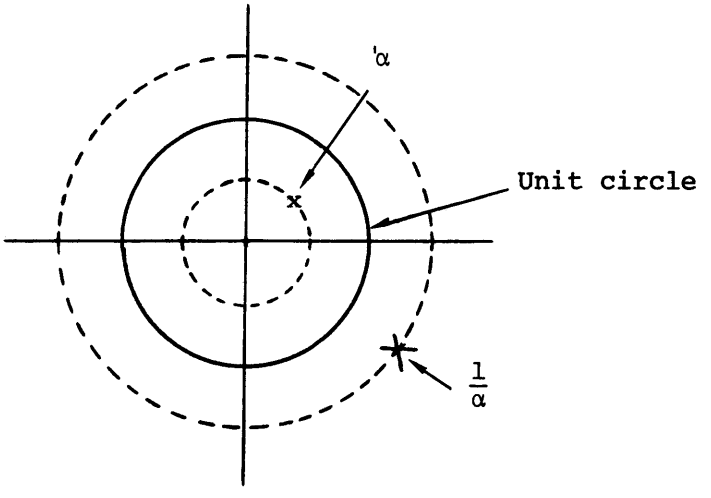


Figure S5.5-1

(b) $$X(z) = \frac{Az}{2} \left[\frac{2z \cos \phi - 2r \cos(\omega_0 - \phi)}{(z - r e^{j\omega_0})(z - r e^{-j\omega_0})} \right]; \quad |z| > r$$

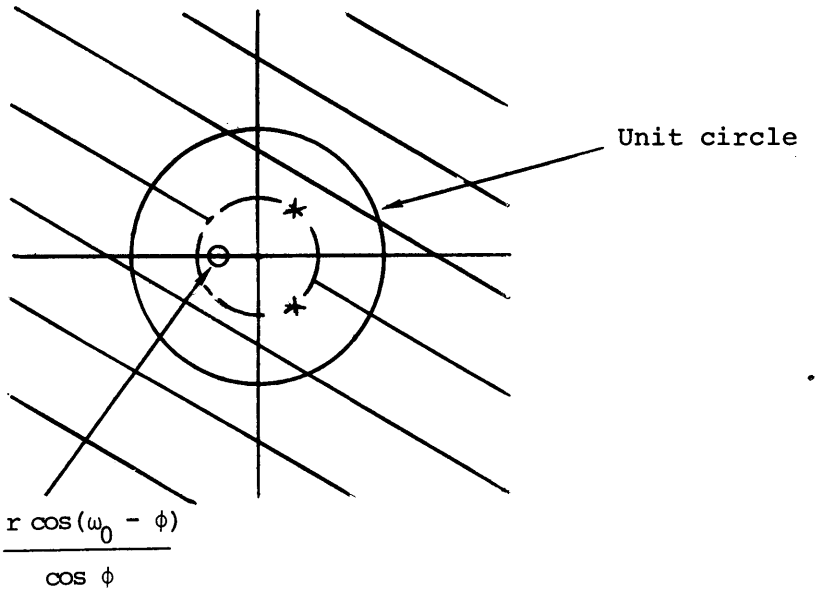


Figure S5.5-2

(c) $X(z) = \dfrac{z^N - 1}{z^{N-1}(z-1)} \qquad z \neq 0$

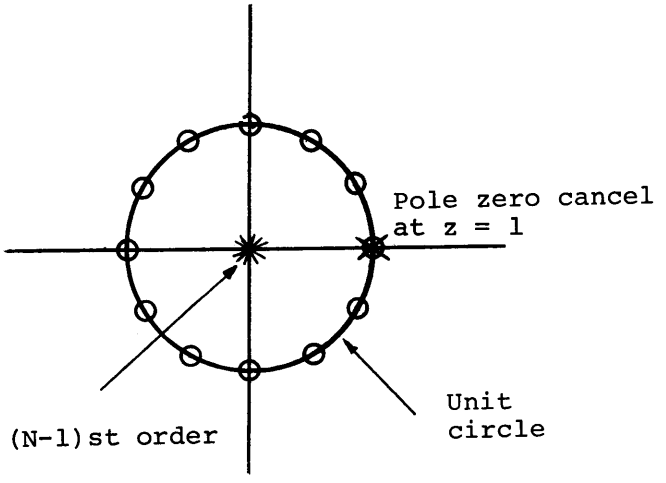


Figure S5.5-3

(d) $x_d(n+1) = x_c(n) * x_c(n)$ where $x_d(n)$ and $x_c(n)$ are the sequences in part (d) and (c) respectively. Therefore

$$X_d(z) = z^{-1} X_c(z) X_c(z)$$

Therefore the zeros in the pole-zero plot of part (c) become double zeros and there is now a (2N - 1)st order pole at z = 0.

Solution 5.6*

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$
$$= X(0) + x(1) z^{-1} + \ldots .$$

For $z \to \infty$ all of the terms in this sum approach zero with the exception of the first term. Thus

$$\lim_{z \to \infty} X(z) = x(0) \quad \text{for } x(n) \text{ causal.}$$

If $x(n) = 0$ $n > 0$ then the corresponding relation is

$$\lim_{z \to 0} X(z) = x(0) \quad .$$

Solution 5.7*

(a) From problem 5.7 we know that $\lim_{z \to \infty} X(z)$ is $x(0)$. Assuming that $x(0)$ is finite, then, $\lim_{z \to \infty} X(z)$ is finite and consequently there are no poles of $X(z)$ at $z = \infty$. Furthermore, since $x(0) \neq 0$ there are no zeros of $X(z)$ at $z = \infty$.

(b) Since $x(n) = 0$; $n < 0$ $X(z)$ is a rational function of z^{-1}, i.e. it is of the form

$$X(z) = \frac{\sum_{m=0}^{M} b_m z^{-m}}{\sum_{n=0}^{N} a_n z^{-n}}$$

The numerator factor introduces M zeros and also an $M^{\underline{th}}$ order pole at $z = 0$. The denominator factor introduces N poles and an $N^{\underline{th}}$ order zero at $z = 0$. Thus there are a total of $(M + N)$ zeros and $(M + N)$ poles i.e. the total number of poles and zeros in the entire z-plane (including $z = \infty$) is the same.

From part (a) of this problem we know that there are no poles or zeros at $z = \infty$. Thus the number of poles and zeros in the finite z-plane is the same.

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