

# chapter 6

*electromagnetic  
induction*

In our development thus far, we have found the electric and magnetic fields to be uncoupled. A net charge generates an electric field while a current is the source of a magnetic field. In 1831 Michael Faraday experimentally discovered that a time varying magnetic flux through a conducting loop also generated a voltage and thus an electric field, proving that electric and magnetic fields are coupled.

## 6-1 FARADAY'S LAW OF INDUCTION

### 6-1-1 The Electromotive Force (EMF)

Faraday's original experiments consisted of a conducting loop through which he could impose a dc current via a switch. Another short circuited loop with no source attached was nearby, as shown in Figure 6-1. When a dc current flowed in loop 1, no current flowed in loop 2. However, when the voltage was first applied to loop 1 by closing the switch, a transient current flowed in the opposite direction in loop 2.

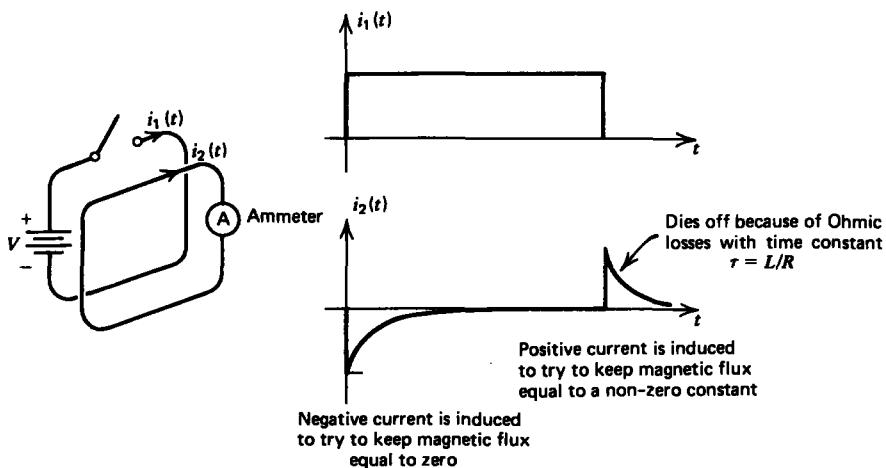


Figure 6-1 Faraday's experiments showed that a time varying magnetic flux through a closed conducting loop induced a current in the direction so as to keep the flux through the loop constant.

When the switch was later opened, another transient current flowed in loop 2, this time in the same direction as the original current in loop 1. Currents are induced in loop 2 whenever a time varying magnetic flux due to loop 1 passes through it.

In general, a time varying magnetic flux can pass through a circuit due to its own or nearby time varying current or by the motion of the circuit through a magnetic field. For any loop, as in Figure 6-2, Faraday's law is

$$\text{EMF} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad (1)$$

where EMF is the electromotive force defined as the line integral of the electric field. The minus sign is introduced on the right-hand side of (1) as we take the convention that positive flux flows in the direction perpendicular to the direction of the contour by the right-hand rule.

### 6-1-2 Lenz's Law

The direction of induced currents is always such as to oppose any changes in the magnetic flux already present. Thus in Faraday's experiment, illustrated in Figure 6-1, when the switch in loop 1 is first closed there is no magnetic flux in loop 2 so that the induced current flows in the opposite direction with its self-magnetic field opposite to the imposed field. The induced current tries to keep a zero flux through

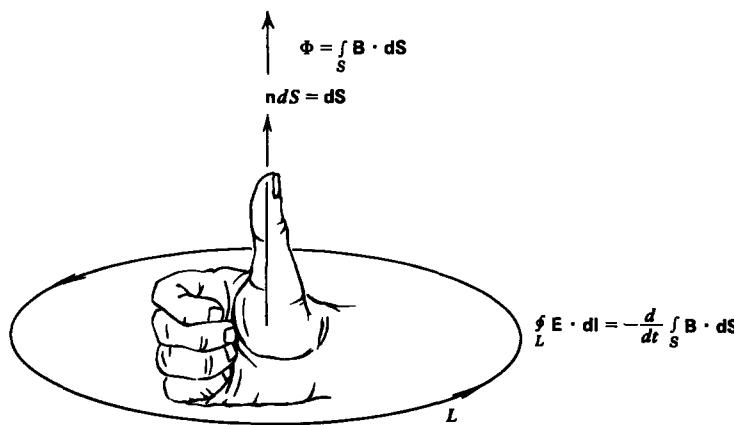


Figure 6-2 Faraday's law states that the line integral of the electric field around a closed loop equals the time rate of change of magnetic flux through the loop. The positive convention for flux is determined by the right-hand rule of curling the fingers on the right hand in the direction of traversal around the loop. The thumb then points in the direction of positive magnetic flux.

loop 2. If the loop is perfectly conducting, the induced current flows as long as current flows in loop 1, with zero net flux through the loop. However, in a real loop, resistive losses cause the current to exponentially decay with an  $L/R$  time constant, where  $L$  is the self-inductance of the loop and  $R$  is its resistance. Thus, in the dc steady state the induced current has decayed to zero so that a constant magnetic flux passes through loop 2 due to the current in loop 1.

When the switch is later opened so that the current in loop 1 goes to zero, the second loop tries to maintain the constant flux already present by inducing a current flow in the same direction as the original current in loop 1. Ohmic losses again make this induced current die off with time.

If a circuit or any part of a circuit is made to move through a magnetic field, currents will be induced in the direction such as to try to keep the magnetic flux through the loop constant. The force on the moving current will always be opposite to the direction of motion.

Lenz's law is clearly demonstrated by the experiments shown in Figure 6-3. When a conducting ax is moved into a magnetic field, eddy currents are induced in the direction where their self-flux is opposite to the applied magnetic field. The Lorentz force is then in the direction opposite to the motion of the ax. This force decreases with time as the currents decay with time due to Ohmic dissipation. If the ax was slotted, effectively creating a very high resistance to the eddy currents, the reaction force becomes very small as the induced current is small.

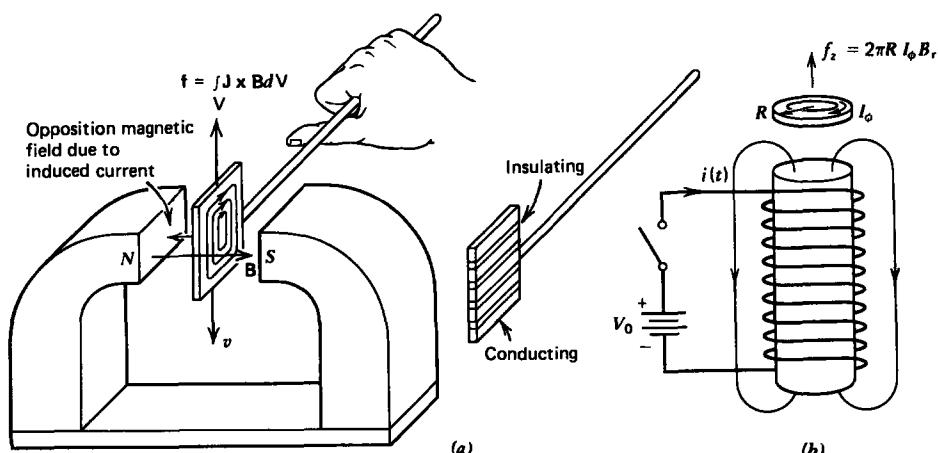


Figure 6-3 Lenz's law. (a) Currents induced in a conductor moving into a magnetic field exert a force opposite to the motion. The induced currents can be made small by slotting the ax. (b) A conducting ring on top of a coil is flipped off when a current is suddenly applied, as the induced currents try to keep a zero flux through the ring.

When the current is first turned on in the coil in Figure 6-3b, the conducting ring that sits on top has zero flux through it. Lenz's law requires that a current be induced opposite to that in the coil. Instantaneously there is no  $z$  component of magnetic field through the ring so the flux must return radially. This creates an upwards force:

$$\mathbf{f} = 2\pi RI \times \mathbf{B} = 2\pi RI_B \mathbf{B}_r \mathbf{i}_z \quad (2)$$

which flips the ring off the coil. If the ring is cut radially so that no circulating current can flow, the force is zero and the ring does not move.

### (a) Short Circuited Loop

To be quantitative, consider the infinitely long time varying line current  $I(t)$  in Figure 6-4, a distance  $r$  from a rectangular loop of wire with Ohmic conductivity  $\sigma$ , cross-sectional area  $A$ , and total length  $l = 2(D+d)$ . The magnetic flux through the loop due to  $I(t)$  is

$$\begin{aligned}\Phi_m &= \int_{z=-D/2}^{D/2} \int_r^{r+d} \mu_0 H_\phi(r') dr' dz \\ &= \frac{\mu_0 I D}{2\pi} \int_r^{r+d} \frac{dr'}{r'} = \frac{\mu_0 I D}{2\pi} \ln \frac{r+d}{r}\end{aligned} \quad (3)$$

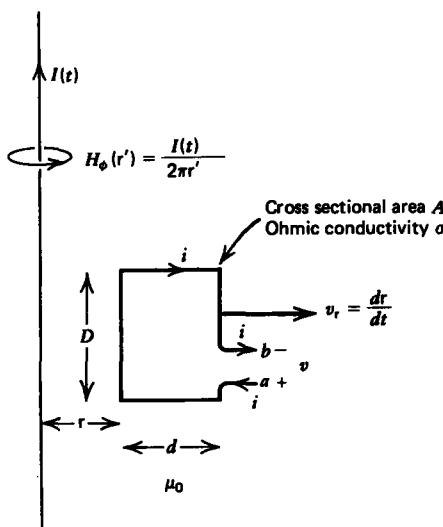


Figure 6-4 A rectangular loop near a time varying line current. When the terminals are short circuited the electromotive force induces a current due to the time varying mutual flux and/or because of the motion of the circuit through the imposed nonuniform magnetic field of the line current. If the loop terminals are open circuited there is no induced current but a voltage develops.

The mutual inductance  $M$  is defined as the flux to current ratio where the flux through the loop is due to an external current. Then (3) becomes

$$\Phi_m = M(r)I, \quad M(r) = \frac{\mu_0 D}{2\pi} \ln \frac{r+d}{r} \quad (4)$$

When the loop is short circuited ( $v = 0$ ), the induced Ohmic current  $i$  gives rise to an electric field [ $E = J/\sigma = i/(A\sigma)$ ] so that Faraday's law applied to a contour within the wire yields an electromotive force just equal to the Ohmic voltage drop:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \frac{il}{\sigma A} = iR = -\frac{d\Phi}{dt} \quad (5)$$

where  $R = l/(\sigma A)$  is the resistance of the loop. By convention, the current is taken as positive in the direction of the line integral.

The flux in (5) has contributions both from the imposed current as given in (3) and from the induced current proportional to the loop's self-inductance  $L$ , which for example is given in Section 5-4-3c for a square loop ( $D = d$ ):

$$\Phi = M(r)I + Li \quad (6)$$

If the loop is also moving radially outward with velocity  $v_r = dr/dt$ , the electromotively induced Ohmic voltage is

$$\begin{aligned} -iR &= \frac{d\Phi}{dt} \\ &= M(r) \frac{dI}{dt} + I \frac{dM(r)}{dt} + L \frac{di}{dt} \\ &= M(r) \frac{dI}{dt} + I \frac{dM}{dr} \frac{dr}{dt} + L \frac{di}{dt} \end{aligned} \quad (7)$$

where  $L$  is not a function of the loop's radial position.

If the loop is stationary, only the first and third terms on the right-hand side contribute. They are nonzero only if the currents change with time. The second term is due to the motion and it has a contribution even for dc currents.

**Turn-on Transient.** If the loop is stationary ( $dr/dt = 0$ ) at  $r = r_0$ , (7) reduces to

$$L \frac{di}{dt} + iR = -M(r_0) \frac{dI}{dt} \quad (8)$$

If the applied current  $I$  is a dc step turned on at  $t = 0$ , the solution to (8) is

$$i(t) = -\frac{M(r_0)I}{L} e^{-(R/L)t}, \quad t > 0 \quad (9)$$

where the impulse term on the right-hand side of (8) imposes the initial condition  $i(t=0) = -M(r_0)I/L$ . The current is negative, as Lenz's law requires the self-flux to oppose the applied flux.

**Turn-off Transient.** If after a long time  $T$  the current  $I$  is instantaneously turned off, the solution is

$$i(t) = \frac{M(r_0)I}{L} e^{-(R/L)(t-T)}, \quad t > T \quad (10)$$

where now the step decrease in current  $I$  at  $t = T$  reverses the direction of the initial current.

**Motion with a dc Current.** With a dc current, the first term on the right-hand side in (7) is zero yielding

$$L \frac{di}{dt} + iR = \frac{\mu_0 IDd}{2\pi r(r+d)} \frac{dr}{dt} \quad (11)$$

To continue, we must specify the motion so that we know how  $r$  changes with time. Let's consider the simplest case when the loop has no resistance ( $R = 0$ ). Then (11) can be directly integrated as

$$Li = -\frac{\mu_0 ID}{2\pi} \ln \frac{1+d/r}{1+d/r_0} \quad (12)$$

where we specify that the current is zero when  $r = r_0$ . This solution for a lossless loop only requires that the total flux of (6) remain constant. The current is positive when  $r > r_0$  as the self-flux must aid the decreasing imposed flux. The current is similarly negative when  $r < r_0$  as the self-flux must cancel the increasing imposed flux.

The force on the loop for all these cases is only due to the force on the  $z$ -directed current legs at  $r$  and  $r+d$ :

$$\begin{aligned} f_r &= \frac{\mu_0 DiI}{2\pi} \left( \frac{1}{r+d} - \frac{1}{r} \right) \\ &= -\frac{\mu_0 DiId}{2\pi r(r+d)} \end{aligned} \quad (13)$$

being attractive if  $iI > 0$  and repulsive if  $iI < 0$ .

### (b) Open Circuited Loop

If the loop is open circuited, no induced current can flow and thus the electric field within the wire is zero ( $\mathbf{J} = \sigma \mathbf{E} = 0$ ). The electromotive force then only has a contribution from the gap between terminals equal to the negative of the voltage:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_b^a \mathbf{E} \cdot d\mathbf{l} = -v = -\frac{d\Phi}{dt} \Rightarrow v = \frac{d\Phi}{dt} \quad (14)$$

Note in Figure 6-4 that our convention is such that the current  $i$  is always defined positive flowing out of the positive voltage terminal into the loop. The flux  $\Phi$  in (14) is now only due to the mutual flux given by (3), as with  $i = 0$  there is no self-flux. The voltage on the moving open circuited loop is then

$$v = M(r) \frac{dI}{dt} + I \frac{dM}{dr} \frac{dr}{dt} \quad (15)$$

### (c) Reaction Force

The magnetic force on a short circuited moving loop is always in the direction opposite to its motion. Consider the short circuited loop in Figure 6-5, where one side of the loop moves with velocity  $v_x$ . With a uniform magnetic field applied normal to the loop pointing out of the page, an expansion of the loop tends to link more magnetic flux requiring the induced current to flow clockwise so that its self-flux is in the direction given by the right-hand rule, opposite to the applied field. From (1) we have

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \frac{i l}{\sigma A} = iR = -\frac{d\Phi}{dt} = B_0 D \frac{dx}{dt} = B_0 D v_x \quad (16)$$

where  $l = 2(D+x)$  also changes with time. The current is then

$$i = \frac{B_0 D v_x}{R} \quad (17)$$

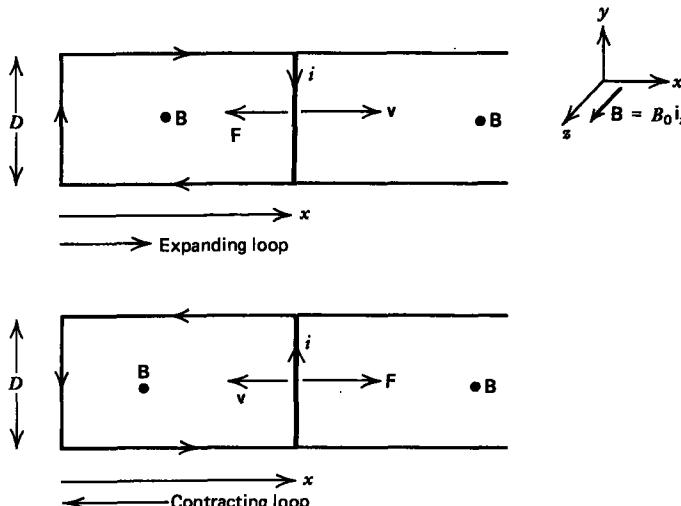


Figure 6-5 If a conductor moves perpendicular to a magnetic field a current is induced in the direction to cause the Lorentz force to be opposite to the motion. The total flux through the closed loop, due to both the imposed field and the self-field generated by the induced current, tries to remain constant.

where we neglected the self-flux generated by  $i$ , assuming it to be much smaller than the applied flux due to  $B_0$ . Note also that the applied flux is negative, as the right-hand rule applied to the direction of the current defines positive flux into the page, while the applied flux points outwards.

The force on the moving side is then to the left,

$$\mathbf{f} = -iD\mathbf{i}_y \times B_0\mathbf{i}_z = -iDB_0\mathbf{i}_x = -\frac{B_0^2 D^2 v_x}{R}\mathbf{i}_x \quad (18)$$

opposite to the velocity.

However if the side moves to the left ( $v_x < 0$ ), decreasing the loop's area thereby linking less flux, the current reverses direction as does the force.

### 6-1-3 Laminations

The induced eddy currents in Ohmic conductors results in Ohmic heating. This is useful in induction furnaces that melt metals, but is undesired in many iron core devices. To reduce this power loss, the cores are often sliced into many thin sheets electrically insulated from each other by thin oxide coatings. The current flow is then confined to lie within a thin sheet and cannot cross over between sheets. The insulating laminations serve the same purpose as the cuts in the slotted ax in Figure 6-3a.

The rectangular conductor in Figure 6-6a has a time varying magnetic field  $B(t)$  passing through it. We approximate the current path as following the rectangular shape so that

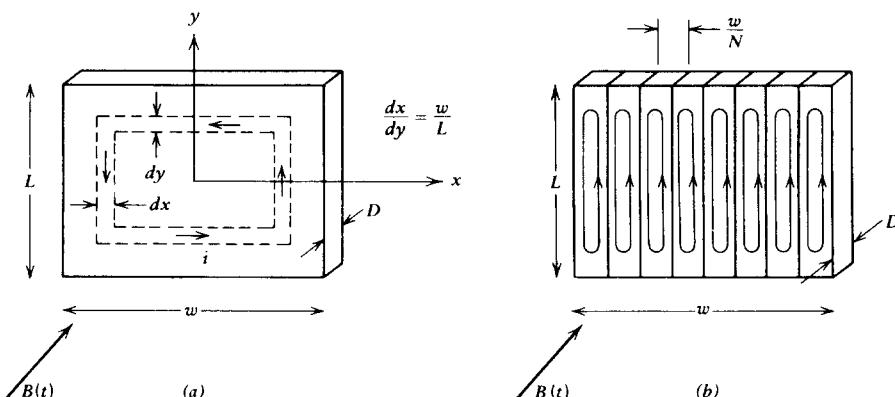


Figure 6-6 (a) A time varying magnetic field through a conductor induces eddy currents that cause Ohmic heating. (b) If the conductor is laminated so that the induced currents are confined to thin strips, the dissipated power decreases.

the flux through the loop of incremental width  $dx$  and  $dy$  of area  $4xy$  is

$$\Phi = -4xyB(t) \quad (19)$$

where we neglect the reaction field of the induced current assuming it to be much smaller than the imposed field. The minus sign arises because, by the right-hand rule illustrated in Figure 6-2, positive flux flows in the direction opposite to  $B(t)$ . The resistance of the loop is

$$R_x = \frac{4}{\sigma D} \left( \frac{y}{dx} + \frac{x}{dy} \right) = \frac{4}{\sigma D} \frac{L}{w} \frac{x}{dx} \left[ 1 + \left( \frac{w}{L} \right)^2 \right] \quad (20)$$

The electromotive force around the loop then just results in an Ohmic current:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = i R_x = \frac{-d\Phi}{dt} = 4xy \frac{dB}{dt} = \frac{4L}{w} x^2 \frac{dB}{dt} \quad (21)$$

with dissipated power

$$dp = i^2 R_x = \frac{4Dx^3 \sigma L (dB/dt)^2 dx}{w [1 + (w/L)^2]} \quad (22)$$

The total power dissipated over the whole sheet is then found by adding the powers dissipated in each incremental loop:

$$\begin{aligned} P &= \int_0^{w/2} dp \\ &= \frac{4D (dB/dt)^2 \sigma L}{w [1 + (w/L)^2]} \int_0^{w/2} x^3 dx \\ &= \frac{LDw^3 \sigma (dB/dt)^2}{16 [1 + (w/L)^2]} \end{aligned} \quad (23)$$

If the sheet is laminated into  $N$  smaller ones, as in Figure 6-6b, each section has the same solution as (23) if we replace  $w$  by  $w/N$ . The total power dissipated is then  $N$  times the power dissipated in a single section:

$$P = \frac{LD(w/N)^3 \sigma (dB/dt)^2 N}{16 [1 + (w/NL)^2]} = \frac{\sigma LDw^3 (dB/dt)^2}{16N^2 [1 + (w/NL)^2]} \quad (24)$$

As  $N$  becomes large so that  $w/NL \ll 1$ , the dissipated power decreases as  $1/N^2$ .

#### 6-1-4 Betatron

The cyclotron, discussed in Section 5-1-4, is not used to accelerate electrons because their small mass allows them to

reach relativistic speeds, thereby increasing their mass and decreasing their angular speed. This puts them out of phase with the applied voltage. The betatron in Figure 6-7 uses the transformer principle where the electrons circulating about the evacuated toroid act like a secondary winding. The imposed time varying magnetic flux generates an electric field that accelerates the electrons.

Faraday's law applied to a contour following the charge's trajectory at radius  $R$  yields

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = E_\phi 2\pi R = - \frac{d\Phi}{dt} \quad (25)$$

which accelerates the electrons as

$$m \frac{dv_\phi}{dt} = -eE_\phi = \frac{e}{2\pi R} \frac{d\Phi}{dt} \Rightarrow v_\phi = \frac{e}{2\pi m R} \Phi \quad (26)$$

The electrons move in the direction so that their self-magnetic flux is opposite to the applied flux. The resulting Lorentz force is radially inward. A stable orbit of constant radius  $R$  is achieved if this force balances the centrifugal force:

$$m \frac{dv_r}{dt} = \frac{mv_\phi^2}{R} - ev_\phi B_z(R) = 0 \quad (27)$$

which from (26) requires the flux and magnetic field to be related as

$$\Phi = 2\pi R^2 B_z(R) \quad (28)$$

This condition cannot be met by a uniform field (as then  $\Phi = \pi R^2 B_z$ ) so in practice the imposed field is made to approximately vary with radial position as

$$B_z(r) = B_0 \left( \frac{R}{r} \right) \Rightarrow \Phi = 2\pi \int_{r=0}^R B_z(r) r dr = 2\pi R^2 B_0 \quad (29)$$

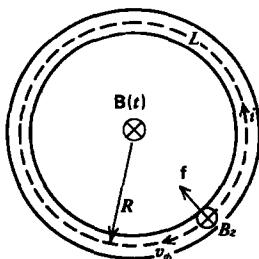


Figure 6-7 The betatron accelerates electrons to high speeds using the electric field generated by a time varying magnetic field.

where  $R$  is the equilibrium orbit radius, so that (28) is satisfied.

The magnetic field must remain curl free where there is no current so that the spatial variation in (29) requires a radial magnetic field component:

$$\nabla \times \mathbf{B} = \left( \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r} \right) \mathbf{i}_\phi = 0 \Rightarrow B_r = -\frac{B_0 R}{r^2} z \quad (30)$$

Then any  $z$ -directed perturbation displacements

$$\begin{aligned} \frac{d^2 z}{dt^2} &= \frac{ev_\phi}{m} B_r(R) = -\left(\frac{eB_0}{m}\right)^2 z \\ \Rightarrow z &= A_1 \sin \omega_0 t + A_2 \cos \omega_0 t, \quad \omega_0 = \frac{eB_0}{m} \end{aligned} \quad (31)$$

have sinusoidal solutions at the cyclotron frequency  $\omega_0 = eB_0/m$ , known as betatron oscillations.

### 6-1-5 Faraday's Law and Stokes' Theorem

The integral form of Faraday's law in (1) shows that with magnetic induction the electric field is no longer conservative as its line integral around a closed path is non-zero. We may convert (1) to its equivalent differential form by considering a stationary contour whose shape does not vary with time. Because the area for the surface integral does not change with time, the time derivative on the right-hand side in (1) may be brought inside the integral but becomes a partial derivative because  $\mathbf{B}$  is also a function of position:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad (32)$$

Using Stokes' theorem, the left-hand side of (32) can be converted to a surface integral,

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_S \nabla \times \mathbf{E} \cdot d\mathbf{S} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad (33)$$

which is equivalent to

$$\int_S \left( \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right) \cdot d\mathbf{S} = 0 \quad (34)$$

Since this last relation is true for any surface, the integrand itself must be zero, which yields Faraday's law of induction in differential form as

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (35)$$

## 6-2 MAGNETIC CIRCUITS

Various alloys of iron having very high values of relative permeability are typically used in relays and machines to constrain the magnetic flux to mostly lie within the permeable material.

### 6-2-1 Self-Inductance

The simple magnetic circuit in Figure 6-8 has an  $N$  turn coil wrapped around a core with very high relative permeability idealized to be infinite. There is a small air gap of length  $s$  in the core. In the core, the magnetic flux density  $\mathbf{B}$  is proportional to the magnetic field intensity  $\mathbf{H}$  by an infinite permeability  $\mu$ . The  $\mathbf{B}$  field must remain finite to keep the flux and coil voltage finite so that the  $\mathbf{H}$  field in the core must be zero:

$$\lim_{\mu \rightarrow \infty} \mathbf{B} = \mu \mathbf{H} \Rightarrow \begin{cases} \mathbf{H} = 0 \\ \mathbf{B} \text{ finite} \end{cases} \quad (1)$$

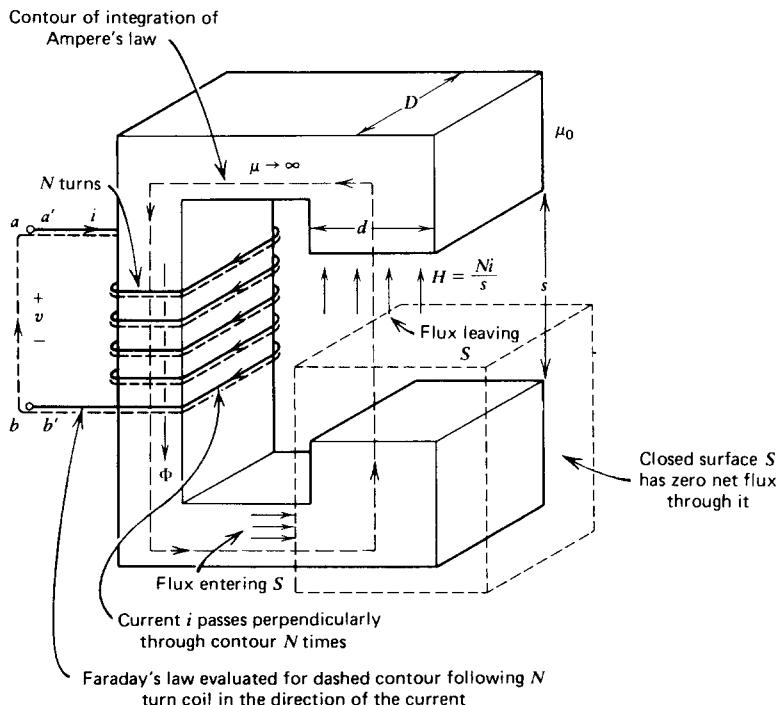


Figure 6-8 The magnetic field is zero within an infinitely permeable magnetic core and is constant in the air gap if we neglect fringing. The flux through the air gap is constant at every cross section of the magnetic circuit and links the  $N$  turn coil  $N$  times.

The  $\mathbf{H}$  field can then only be nonzero in the air gap. This field emanates perpendicularly from the pole faces as no surface currents are present so that the tangential component of  $\mathbf{H}$  is continuous and thus zero. If we neglect fringing field effects, assuming the gap  $s$  to be much smaller than the width  $d$  or depth  $D$ , the  $\mathbf{H}$  field is uniform throughout the gap. Using Ampere's circuital law with the contour shown, the only nonzero contribution is in the air gap,

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = Hs = I_{\text{total enclosed}} = Ni \quad (2)$$

where we realize that the coil current crosses perpendicularly through our contour  $N$  times. The total flux in the air gap is then

$$\Phi = \mu_0 H D d = \frac{\mu_0 N D d}{s} i \quad (3)$$

Because the total flux through any closed surface is zero,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \quad (4)$$

all the flux leaving  $S$  in Figure 6-8 on the air gap side enters the surface through the iron core, as we neglect leakage flux in the fringing field. The flux at any cross section in the iron core is thus constant, given by (3).

If the coil current  $i$  varies with time, the flux in (3) also varies with time so that a voltage is induced across the coil. We use the integral form of Faraday's law for a contour that lies within the winding with Ohmic conductivity  $\sigma$ , cross sectional area  $A$ , and total length  $l$ . Then the current density and electric field within the wire is

$$J = \frac{i}{A}, \quad E = \frac{J}{\sigma} = \frac{i}{\sigma A} \quad (5)$$

so that the electromotive force has an Ohmic part as well as a contribution due to the voltage across the terminals:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \underbrace{\int_{a'}^{b'} \frac{i}{\sigma A} dt}_{iR \text{ in wire}} + \underbrace{\int_b^a \mathbf{E} \cdot d\mathbf{l}}_{-v \text{ across terminals}} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (6)$$

The surface  $S$  on the right-hand side is quite complicated because of the spiral nature of the contour. If the coil only had one turn, the right-hand side of (6) would just be the time derivative of the flux of (3). For two turns, as in Figure 6-9, the flux links the coil twice, while for  $N$  turns the total flux

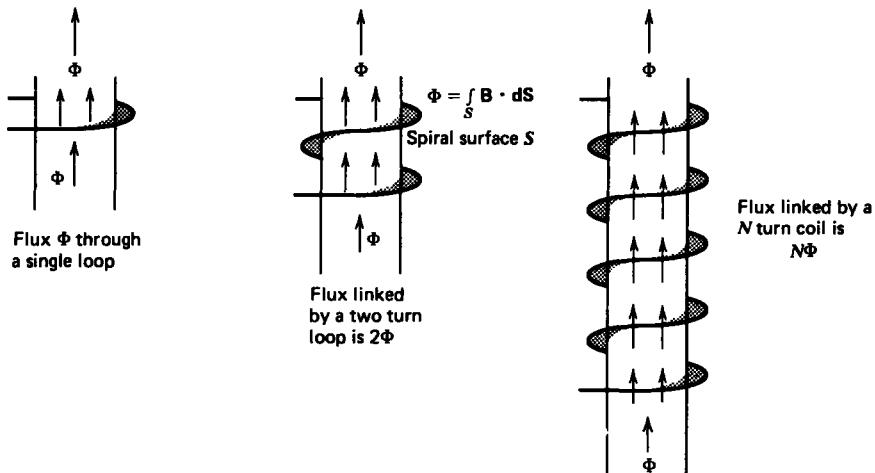


Figure 6-9 The complicated spiral surface for computation of the linked flux by an  $N$  turn coil can be considered as  $N$  single loops each linking the same flux  $\Phi$ .

linked by the coil is  $N\Phi$ . Then (6) reduces to

$$v = iR + L \frac{di}{dt} \quad (7)$$

where the self-inductance is defined as

$$L = \frac{N\Phi}{i} = \frac{N \int_S \mathbf{B} \cdot d\mathbf{S}}{\oint_L \mathbf{H} \cdot d\mathbf{l}} = \frac{\mu_0 N^2 Dd}{s} \text{ henry } [\text{kg}\cdot\text{m}^2\cdot\text{A}^{-2}\cdot\text{s}^{-2}] \quad (8)$$

For linearly permeable materials, the inductance is always independent of the excitations and only depends on the geometry. Because of the fixed geometry, the inductance is a constant and thus was taken outside the time derivative in (7). In geometries that change with time, the inductance will also be a function of time and must remain under the derivative. The inductance is always proportional to the square of the number of coil turns. This is because the flux  $\Phi$  in the air gap is itself proportional to  $N$  and it links the coil  $N$  times.

### EXAMPLE 6-1 SELF-INDUCTANCES

Find the self-inductances for the coils shown in Figure 6-10.

#### (a) Solenoid

An  $N$  turn coil is tightly wound upon a cylindrical core of radius  $a$ , length  $l$ , and permeability  $\mu$ .

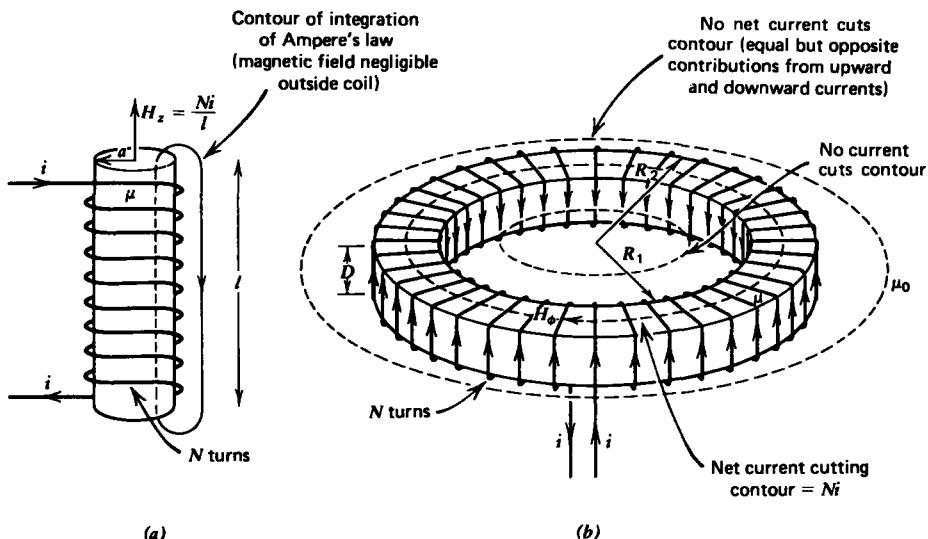


Figure 6-10 Inductances. (a) Solenoidal coil; (b) toroidal coil.

## SOLUTION

A current  $i$  flowing in the wire approximates a surface current

$$K_\phi = Ni/l$$

If the length  $l$  is much larger than the radius  $a$ , we can neglect fringing field effects at the ends and the internal magnetic field is approximately uniform and equal to the surface current,

$$H_z = K_\phi = \frac{Ni}{l}$$

as we assume the exterior magnetic field is negligible. The same result is obtained using Ampere's circuital law for the contour shown in Figure 6-10a. The flux links the coil  $N$  times:

$$L = \frac{N\Phi}{i} = \frac{N\mu H_z \pi a^2}{i} = \frac{N^2 \mu \pi a^2}{l}$$

### (b) Toroid

An  $N$  turn coil is tightly wound around a donut-shaped core of permeability  $\mu$  with a rectangular cross section and inner and outer radii  $R_1$  and  $R_2$ .

## SOLUTION

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Applying Ampere's circuital law to the three contours shown in Figure 6-10b, only the contour within the core has a net current passing through it:

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = H_\phi 2\pi r = \begin{cases} 0, & r < R_1 \\ Ni, & R_1 < r < R_2 \\ 0, & r > R_2 \end{cases}$$

The inner contour has no current through it while the outer contour enclosing the whole toroid has equal but opposite contributions from upward and downward currents.

The flux through any single loop is

$$\begin{aligned} \Phi &= \mu D \int_{R_1}^{R_2} H_\phi dr \\ &= \frac{\mu D N i}{2\pi} \int_{R_1}^{R_2} \frac{dr}{r} \\ &= \frac{\mu D N i}{2\pi} \ln \frac{R_2}{R_1} \end{aligned}$$

so that the self-inductance is

$$L = \frac{N\Phi}{i} = \frac{\mu D N^2}{2\pi} \ln \frac{R_2}{R_1}$$

### 6-2-2 Reluctance

Magnetic circuits are analogous to resistive electronic circuits if we define the magnetomotive force (MMF)  $\mathcal{F}$  analogous to the voltage (EMF) as

$$\mathcal{F} = Ni \quad (9)$$

The flux then plays the same role as the current in electronic circuits so that we define the magnetic analog to resistance as the reluctance:

$$\mathcal{R} = \frac{\mathcal{F}}{\Phi} = \frac{N^2}{L} = \frac{(\text{length})}{(\text{permeability})(\text{cross-sectional area})} \quad (10)$$

which is proportional to the reciprocal of the inductance.

The advantage to this analogy is that the rules of adding reluctances in series and parallel obey the same rules as resistances.

## (a) Reluctances in Series

For the iron core of infinite permeability in Figure 6-11a, with two finitely permeable gaps the reluctance of each gap is found from (8) and (10) as

$$\mathcal{R}_1 = \frac{s_1}{\mu_1 a_1 D}, \quad \mathcal{R}_2 = \frac{s_2}{\mu_2 a_2 D} \quad (11)$$

so that the flux is

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}_1 + \mathcal{R}_2} = \frac{Ni}{\mathcal{R}_1 + \mathcal{R}_2} \Rightarrow L = \frac{N\Phi}{i} = \frac{N^2}{\mathcal{R}_1 + \mathcal{R}_2} \quad (12)$$

The iron core with infinite permeability has zero reluctance. If the permeable gaps were also iron with infinite permeability, the reluctances of (11) would also be zero so that the flux

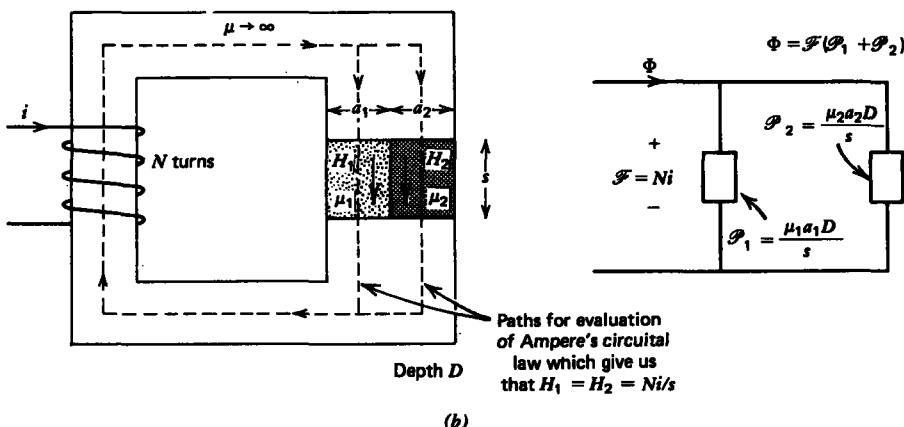
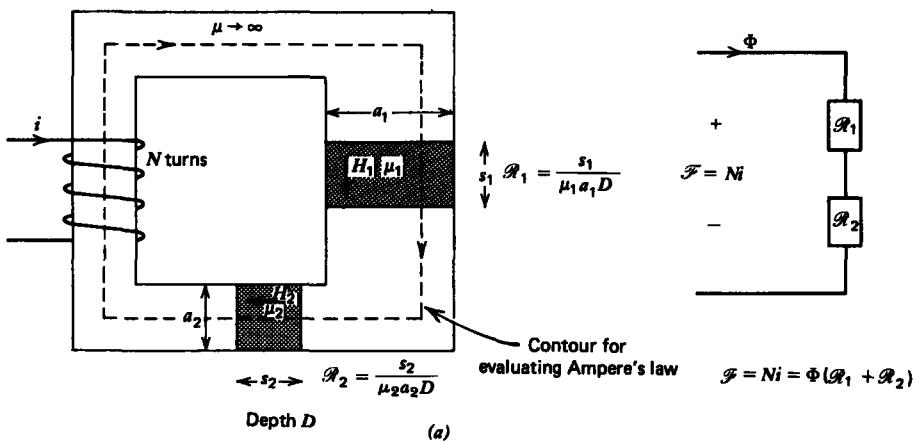


Figure 6-11 Magnetic circuits are most easily analyzed from a circuit approach where (a) reluctances in series add and (b) permeances in parallel add.

in (12) becomes infinite. This is analogous to applying a voltage across a short circuit resulting in an infinite current. Then the small resistance in the wires determines the large but finite current. Similarly, in magnetic circuits the small reluctance of a closed iron core of high permeability with no gaps limits the large but finite flux determined by the saturation value of magnetization.

The  $\mathbf{H}$  field is nonzero only in the permeable gaps so that Ampere's law yields

$$H_1s_1 + H_2s_2 = Ni \quad (13)$$

Since the flux must be continuous at every cross section,

$$\Phi = \mu_1 H_1 a_1 D = \mu_2 H_2 a_2 D \quad (14)$$

we solve for the  $\mathbf{H}$  fields as

$$H_1 = \frac{\mu_2 a_2 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1}, \quad H_2 = \frac{\mu_1 a_1 Ni}{\mu_1 a_1 s_2 + \mu_2 a_2 s_1} \quad (15)$$

### (b) Reluctances in Parallel

If a gap in the iron core is filled with two permeable materials, as in Figure 6-11b, the reluctance of each material is still given by (11). Since each material sees the same magnetomotive force, as shown by applying Ampere's circuital law to contours passing through each material,

$$H_1s = H_2s = Ni \Rightarrow H_1 = H_2 = \frac{Ni}{s} \quad (16)$$

the  $H$  fields in each material are equal. The flux is then

$$\Phi = (\mu_1 H_1 a_1 + \mu_2 H_2 a_2) D = \frac{Ni(\mathcal{R}_1 + \mathcal{R}_2)}{\mathcal{R}_1 \mathcal{R}_2} = Ni(\mathcal{P}_1 + \mathcal{P}_2) \quad (17)$$

where the permeances  $\mathcal{P}_1$  and  $\mathcal{P}_2$  are just the reciprocal reluctances analogous to conductance.

### 6-2-3 Transformer Action

#### (a) Voltages are not Unique

Consider two small resistors  $R_1$  and  $R_2$  forming a loop enclosing one leg of a closed magnetic circuit with permeability  $\mu$ , as in Figure 6-12. An  $N$  turn coil excited on one leg with a time varying current generates a time varying flux that is approximately

$$\Phi(t) = \mu N A i_1 / l \quad (18)$$

where  $l$  is the average length around the core.

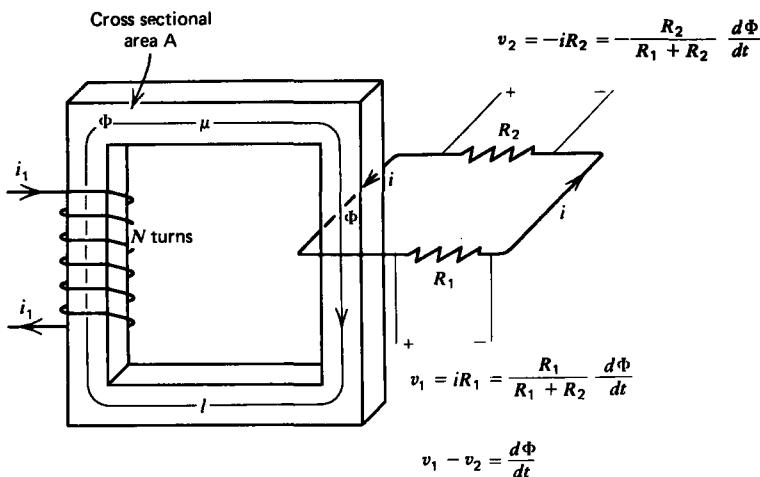


Figure 6-12 Voltages are not unique in the presence of a time varying magnetic field. A resistive loop encircling a magnetic circuit has different measured voltages across the same node pair. The voltage difference is equal to the time rate of magnetic flux through the loop.

Applying Faraday's law to the resistive loop we have

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = i(R_1 + R_2) = +\frac{d\Phi(t)}{dt} \Rightarrow i = \frac{1}{R_1 + R_2} \frac{d\Phi}{dt} \quad (19)$$

where we neglect the self-flux produced by the induced current  $i$  and reverse the sign on the magnetic flux term because  $\Phi$  penetrates the loop in Figure 6-12 in the direction opposite to the positive convention given by the right-hand rule illustrated in Figure 6-2.

If we now measured the voltage across each resistor, we would find different values and opposite polarities even though our voltmeter was connected to the same nodes:

$$v_1 = iR_1 = +\frac{R_1}{R_1 + R_2} \frac{d\Phi}{dt}$$

$$v_2 = -iR_2 = -\frac{R_2}{R_1 + R_2} \frac{d\Phi}{dt} \quad (20)$$

This nonuniqueness of the voltage arises because the electric field is no longer curl free. The voltage difference between two points depends on the path of the connecting wires. If any time varying magnetic flux passes through the contour defined by the measurement, an additional contribution results.

### (b) Ideal Transformers

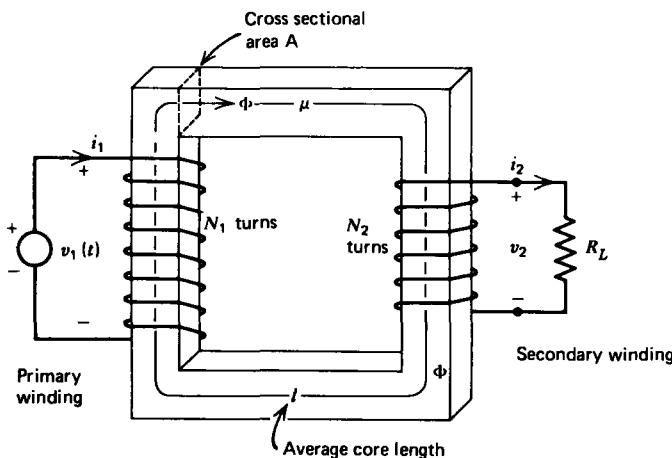
Two coils tightly wound on a highly permeable core, so that all the flux of one coil links the other, forms an ideal transformer, as in Figure 6-13. Because the iron core has an infinite permeability, all the flux is confined within the core. The currents flowing in each coil,  $i_1$  and  $i_2$ , are defined so that when they are positive the fluxes generated by each coil are in the opposite direction. The total flux in the core is then

$$\Phi = \frac{N_1 i_1 - N_2 i_2}{\mathcal{R}}, \quad \mathcal{R} = \frac{l}{\mu A} \quad (21)$$

where  $\mathcal{R}$  is the reluctance of the core and  $l$  is the average length of the core.

The flux linked by each coil is then

$$\begin{aligned} \lambda_1 &= N_1 \Phi = \frac{\mu A}{l} (N_1^2 i_1 - N_1 N_2 i_2) \\ \lambda_2 &= N_2 \Phi = \frac{\mu A}{l} (N_1 N_2 i_1 - N_2^2 i_2) \end{aligned} \quad (22)$$



(a)

Figure 6-13 (a) An ideal transformer relates primary and secondary voltages by the ratio of turns while the currents are in the inverse ratio so that the input power equals the output power. The  $H$  field is zero within the infinitely permeable core. (b) In a real transformer the nonlinear B-H hysteresis loop causes a nonlinear primary current  $i_1$  with an open circuited secondary ( $i_2 = 0$ ) even though the imposed sinusoidal voltage  $v_1 = V_0 \cos \omega t$  fixes the flux to be sinusoidal. (c) A more complete transformer equivalent circuit.

$$\left. \begin{aligned} \frac{v_1}{v_2} &= \frac{N_1}{N_2} \\ \frac{i_1}{i_2} &= \frac{N_2}{N_1} \end{aligned} \right\} \Rightarrow v_1 i_1 = v_2 i_2$$

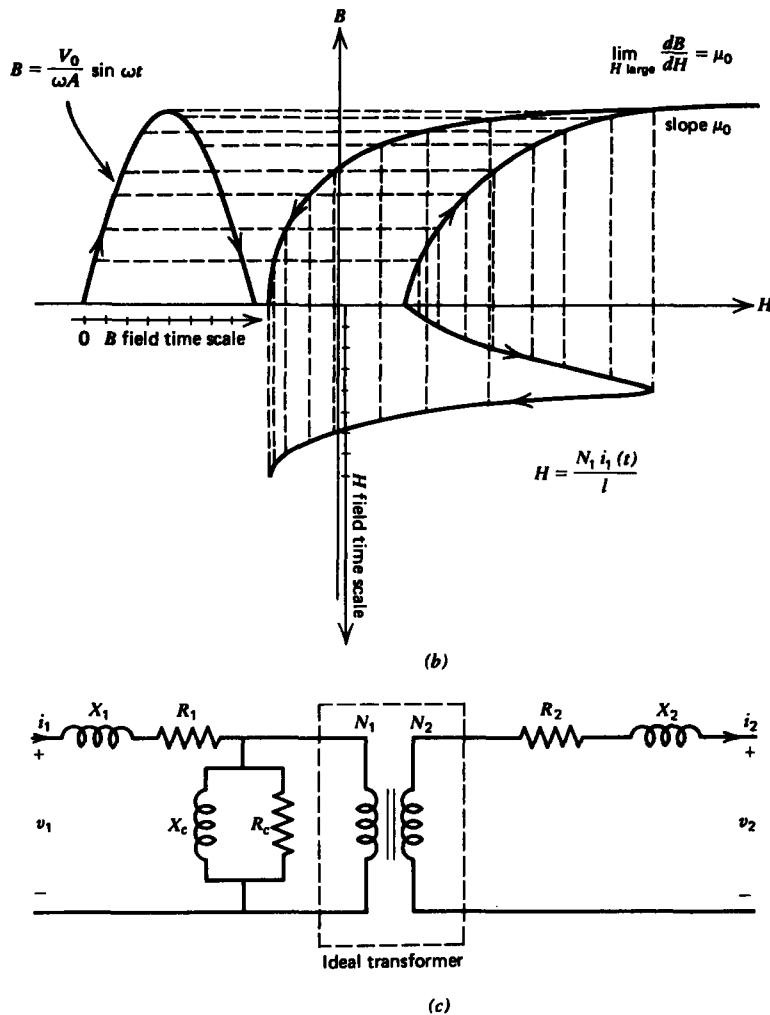


Figure 6.13.

which can be written as

$$\begin{aligned}\lambda_1 &= L_1 i_1 - M i_2 \\ \lambda_2 &= M i_1 - L_2 i_2\end{aligned}\quad (23)$$

where  $L_1$  and  $L_2$  are the self-inductances of each coil alone and  $M$  is the mutual inductance between coils:

$$L_1 = N_1^2 L_0, \quad L_2 = N_2^2 L_0, \quad M = N_1 N_2 L_0, \quad L_0 = \mu A/l \quad (24)$$

In general, the mutual inductance obeys the equality:

$$M = k(L_1 L_2)^{1/2}, \quad 0 \leq k \leq 1 \quad (25)$$

where  $k$  is called the coefficient of coupling. For a noninfinite core permeability,  $k$  is less than unity because some of the flux of each coil goes into the free space region and does not link the other coil. In an ideal transformer, where the permeability is infinite, there is no leakage flux so that  $k = 1$ .

From (23), the voltage across each coil is

$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \quad (26)$$

$$v_2 = \frac{d\lambda_2}{dt} = M \frac{di_1}{dt} - L_2 \frac{di_2}{dt}$$

Because with no leakage, the mutual inductance is related to the self-inductances as

$$M = \frac{N_2}{N_1} L_1 = \frac{N_1}{N_2} L_2 \quad (27)$$

the ratio of coil voltages is the same as the turns ratio:

$$\frac{v_1}{v_2} = \frac{d\lambda_1/dt}{d\lambda_2/dt} = \frac{N_1}{N_2} \quad (28)$$

In the ideal transformer of infinite core permeability, the inductances of (24) are also infinite. To keep the voltages and fluxes in (26) finite, the currents must be in the inverse turns ratio

$$\frac{i_1}{i_2} = \frac{N_2}{N_1} \quad (29)$$

The electrical power delivered by the source to coil 1, called the primary winding, just equals the power delivered to the load across coil 2, called the secondary winding:

$$v_1 i_1 = v_2 i_2 \quad (30)$$

If  $N_2 > N_1$ , the voltage on winding 2 is greater than the voltage on winding 1 but current  $i_2$  is less than  $i_1$  keeping the powers equal.

If primary winding 1 is excited by a time varying voltage  $v_1(t)$  with secondary winding 2 loaded by a resistor  $R_L$  so that

$$v_2 = i_2 R_L \quad (31)$$

the effective resistance seen by the primary winding is

$$R_{\text{eff}} = \frac{v_1}{i_1} = \frac{N_1}{N_2} \frac{v_2}{(N_2/N_1)i_2} = \left(\frac{N_1}{N_2}\right)^2 R_L \quad (32)$$

A transformer is used in this way as an impedance transformer where the effective resistance seen at the primary winding is increased by the square of the turns ratio.

### (c) Real Transformers

When the secondary is open circuited ( $i_2 = 0$ ), (29) shows that the primary current of an ideal transformer is also zero. In practice, applying a primary sinusoidal voltage  $V_0 \cos \omega t$  will result in a small current due to the finite self-inductance of the primary coil. Even though this self-inductance is large if the core permeability  $\mu$  is large, we must consider its effect because there is no opposing flux as a result of the open circuited secondary coil. Furthermore, the nonlinear hysteresis curve of the iron as discussed in Section 5-5-3c will result in a nonsinusoidal current even though the voltage is sinusoidal. In the magnetic circuit of Figure 6.13a with  $i_2 = 0$ , the magnetic field is

$$\mathbf{H} = \frac{N_1 i_1}{l} \quad (33)$$

while the imposed sinusoidal voltage also fixes the magnetic flux to be sinusoidal

$$v_1 = \frac{d\Phi}{dt} = V_0 \cos \omega t \Rightarrow \Phi = BA = \frac{V_0}{\omega} \sin \omega t \quad (34)$$

Thus the upper half of the nonlinear B-H magnetization characteristic in Figure 6-13b has the same shape as the flux-current characteristic with proportionality factors related to the geometry. Note that in saturation the B-H curve approaches a straight line with slope  $\mu_0$ . For a half-cycle of flux given by (34), the nonlinear open circuit magnetizing current is found graphically as a function of time in Figure 6-13b. The current is symmetric over the negative half of the flux cycle. Fourier analysis shows that this nonlinear current is composed of all the odd harmonics of the driving frequency dominated by the third and fifth harmonics. This causes problems in power systems and requires extra transformer windings to trap the higher harmonic currents, thus preventing their transmission.

A more realistic transformer equivalent circuit is shown in Figure 6-13c where the leakage reactances  $X_1$  and  $X_2$  represent the fact that all the flux produced by one coil does not link the other. Some small amount of flux is in the free space region surrounding the windings. The nonlinear inductive reactance  $X_c$  represents the nonlinear magnetization characteristic illustrated in Figure 6-13b, while  $R_c$  represents the power dissipated in traversing the hysteresis

loop over a cycle. This dissipated power per cycle equals the area enclosed by the hysteresis loop. The winding resistances are  $R_1$  and  $R_2$ .

### 6-3 FARADAY'S LAW FOR MOVING MEDIA

#### 6-3-1 The Electric Field Transformation

If a point charge  $q$  travels with a velocity  $\mathbf{v}$  through a region with electric field  $\mathbf{E}$  and magnetic field  $\mathbf{B}$ , it experiences the combined Coulomb–Lorentz force

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (1)$$

Now consider another observer who is travelling at the same velocity  $\mathbf{v}$  as the charge carrier so that their relative velocity is zero. This moving observer will then say that there is no Lorentz force, only a Coulombic force

$$\mathbf{F}' = q\mathbf{E}' \quad (2)$$

where we indicate quantities measured by the moving observer with a prime. A fundamental postulate of mechanics is that all physical laws are the same in every inertial coordinate system (systems that travel at constant relative velocity). This requires that the force measured by two inertial observers be the same so that  $\mathbf{F}' = \mathbf{F}$ :

$$\mathbf{E}' = \mathbf{E} + \mathbf{v} \times \mathbf{B} \quad (3)$$

The electric field measured by the two observers in relative motion will be different. This result is correct for material velocities much less than the speed of light and is called a Galilean field transformation. The complete relativistically correct transformation slightly modifies (3) and is called a Lorentzian transformation but will not be considered here.

In using Faraday's law of Section 6-1-1, the question remains as to which electric field should be used if the contour  $L$  and surface  $S$  are moving. One uses the electric field that is measured by an observer moving at the same velocity as the convecting contour. The time derivative of the flux term cannot be brought inside the integral if the surface  $S$  is itself a function of time.

#### 6-3-2 Ohm's Law for Moving Conductors

The electric field transformation of (3) is especially important in modifying Ohm's law for moving conductors. For nonrelativistic velocities, an observer moving along at the

same velocity as an Ohmic conductor measures the usual Ohm's law in his reference frame,

$$\mathbf{J}' = \sigma \mathbf{E}' \quad (4)$$

where we assume the conduction process is unaffected by the motion. Then in Galilean relativity for systems with no free charge, the current density in all inertial frames is the same so that (3) in (4) gives us the generalized Ohm's law as

$$\mathbf{J}' = \mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \quad (5)$$

where  $\mathbf{v}$  is the velocity of the conductor.

The effects of material motion are illustrated by the parallel plate geometry shown in Figure 6-14. A current source is applied at the left-hand side that distributes itself uniformly as a surface current  $K_x = \pm I/D$  on the planes. The electrodes are connected by a conducting slab that moves to the right with constant velocity  $U$ . The voltage across the current source can be computed using Faraday's law with the contour shown. Let us have the contour continually expanding with the 2-3 leg moving with the conductor. Applying Faraday's law we have

$$\begin{aligned} \oint_L \mathbf{E}' \cdot d\mathbf{l} &= \int_1^2 \mathbf{E}'^0 \cdot d\mathbf{l} + \int_2^3 \mathbf{E}'^0 \cdot d\mathbf{l} + \int_3^4 \mathbf{E}'^0 \cdot d\mathbf{l} + \int_4^1 \mathbf{E}'^0 \cdot d\mathbf{l} \\ &= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \end{aligned} \quad (6)$$

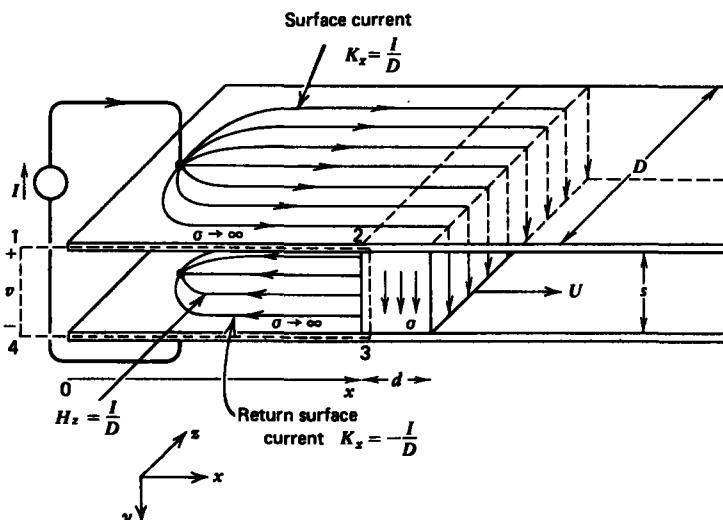


Figure 6-14 A moving, current-carrying Ohmic conductor generates a speed voltage as well as the usual resistive voltage drop.

where the electric field used along each leg is that measured by an observer in the frame of reference of the contour. Along the 1–2 and 3–4 legs, the electric field is zero within the stationary perfect conductors. The second integral within the moving Ohmic conductor uses the electric field  $\mathbf{E}'$ , as measured by a moving observer because the contour is also expanding at the same velocity, and from (4) and (5) is related to the terminal current as

$$\mathbf{E}' = \frac{\mathbf{J}'}{\sigma} = \frac{I}{\sigma Dd} \mathbf{i}, \quad (7)$$

In (6), the last line integral across the terminals defines the voltage.

$$\frac{Is}{\sigma Dd} - v = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{d}{dt} (\mu_0 H_z s) \quad (8)$$

The first term is just the resistive voltage drop across the conductor, present even if there is no motion. The term on the right-hand side in (8) only has a contribution due to the linearly increasing area ( $dx/dt = U$ ) in the free space region with constant magnetic field,

$$H_z = I/D \quad (9)$$

The terminal voltage is then

$$v = I \left( R + \frac{\mu_0 U s}{D} \right), \quad R = \frac{s}{\sigma D d} \quad (10)$$

We see that the speed voltage contribution arose from the flux term in Faraday's law. We can obtain the same solution using a contour that is stationary and does not expand with the conductor. We pick the contour to just lie within the conductor at the time of interest. Because the contour does not expand with time so that both the magnetic field and the contour area does not change with time, the right-hand side of (6) is zero. The only difference now is that along the 2–3 leg we use the electric field as measured by a stationary observer,

$$\mathbf{E} = \mathbf{E}' - \mathbf{v} \times \mathbf{B} \quad (11)$$

so that (6) becomes

$$IR + \frac{\mu_0 U Is}{D} - v = 0 \quad (12)$$

which agrees with (10) but with the speed voltage term now arising from the electric field side of Faraday's law.

This speed voltage contribution is the principle of electric generators converting mechanical work to electric power

when moving a current-carrying conductor through a magnetic field. The resistance term accounts for the electric power dissipated. Note in (10) that the speed voltage contribution just adds with the conductor's resistance so that the effective terminal resistance is  $v/I = R + (\mu_0 U s/D)$ . If the slab moves in the opposite direction such that  $U$  is negative, the terminal resistance can also become negative for sufficiently large  $U$  ( $U < -RD/\mu_0 s$ ). Such systems are unstable where the natural modes grow rather than decay with time with any small perturbation, as illustrated in Section 6-3-3b.

### 6-3-3 Faraday's Disk (Homopolar Generator)\*

#### (a) Imposed Magnetic Field

A disk of conductivity  $\sigma$  rotating at angular velocity  $\omega$  transverse to a uniform magnetic field  $B_0 \hat{i}_z$ , illustrates the basic principles of electromechanical energy conversion. In Figure 6-15a we assume that the magnetic field is generated by an  $N$  turn coil wound on the surrounding magnetic circuit,

$$B_0 = \frac{\mu_0 N i_f}{s} \quad (13)$$

The disk and shaft have a permeability of free space  $\mu_0$ , so that the applied field is not disturbed by the assembly. The shaft and outside surface at  $r = R_0$  are highly conducting and make electrical connection to the terminals via sliding contacts.

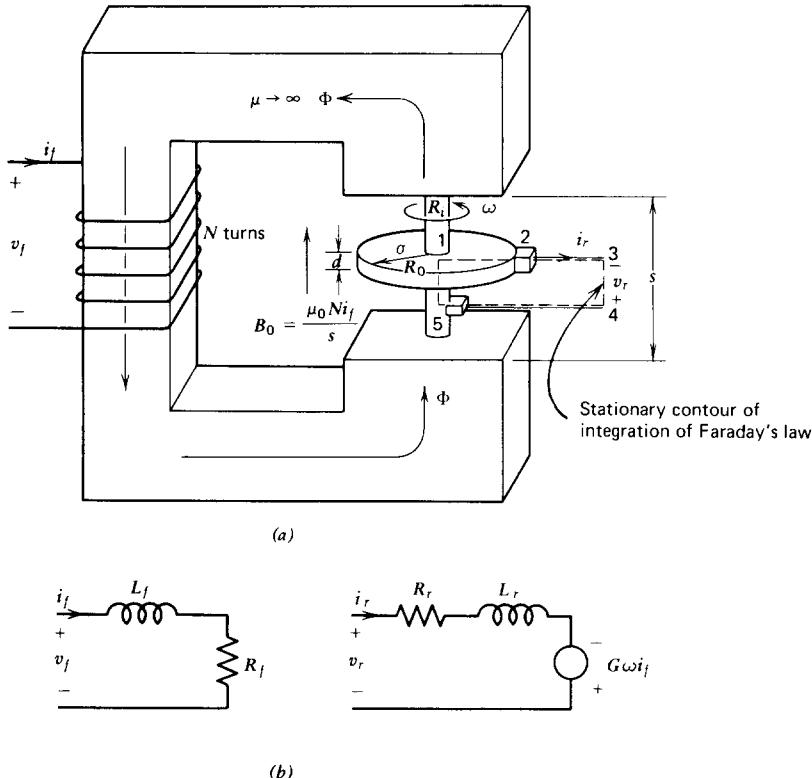
We evaluate Faraday's law using the contour shown in Figure 6-15a where the 1-2 leg within the disk is stationary so the appropriate electric field to be used is given by (11):

$$E_r = \frac{J_r}{\sigma} - \omega r B_0 = \frac{i_r}{2\pi\sigma r} - \omega r B_0 \quad (14)$$

where the electric field and current density are radial and  $i_r$  is the total rotor terminal current. For the stationary contour with a constant magnetic field, there is no time varying flux through the contour:

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_1^2 E_r dr + \int_3^4 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{-v_r} = 0 \quad (15)$$

\* Some of the treatment in this section is similar to that developed in: H. H. Woodson and J. R. Melcher, Electromechanical Dynamics, Part I, Wiley, N.Y., 1968, Ch. 6.



**Figure 6-15** (a) A conducting disk rotating in an axial magnetic field is called a homopolar generator. (b) In addition to Ohmic and inductive voltages there is a speed voltage contribution proportional to the speed of the disk and the magnetic field.

Using (14) in (15) yields the terminal voltage as

$$\begin{aligned}
 v_r &= \int_{R_i}^{R_0} \left( \frac{i_r}{2\pi r \sigma d} - \omega r B_0 \right) dr \\
 &= \frac{i_r}{2\pi \sigma d} \ln \frac{R_0}{R_i} - \frac{\omega B_0}{2} (R_0^2 - R_i^2) \\
 &= i_r R_r - G \omega i_f
 \end{aligned} \tag{16}$$

where  $R_r$  is the internal rotor resistance of the disk and  $G$  is called the speed coefficient:

$$R_r = \frac{\ln(R_0/R_i)}{2\pi \sigma d}, \quad G = \frac{\mu_0 N}{2s} (R_0^2 - R_i^2) \tag{17}$$

We neglected the self-magnetic field due to the rotor current, assuming it to be much smaller than the applied field  $B_0$ , but

it is represented in the equivalent rotor circuit in Figure 6-15b as the self-inductance  $L_r$  in series with a resistor and a speed voltage source linearly dependent on the field current. The stationary field coil is represented by its self-inductance and resistance.

For a copper disk ( $\sigma = 6 \times 10^7$  siemen/m) of thickness 1 mm rotating at 3600 rpm ( $\omega = 120\pi$  radian/sec) with outer and inner radii  $R_o = 10$  cm and  $R_i = 1$  cm in a magnetic field of  $B_0 = 1$  tesla, the open circuit voltage is

$$v_{oc} = -\frac{\omega B_0}{2} (R_o^2 - R_i^2) \approx -1.9 \text{ V} \quad (18)$$

while the short circuit current is

$$i_{sc} = \frac{v_{oc}}{2\pi\sigma d} \ln(R_o/R_i) \approx 3 \times 10^5 \text{ amp} \quad (19)$$

Homopolar generators are typically high current, low voltage devices. The electromagnetic torque on the disk due to the Lorentz force is

$$\begin{aligned} T &= \int_{\phi=0}^{2\pi} \int_{z=0}^d \int_{r=R_i}^{R_o} r \mathbf{i}_r \times (\mathbf{J} \times \mathbf{B}) r \, dr \, d\phi \, dz \\ &= -i_r B_0 \mathbf{i}_z \int_{R_i}^{R_o} r \, dr \\ &= -\frac{i_r B_0}{2} (R_o^2 - R_i^2) \mathbf{i}_z \\ &= -G i_r \mathbf{i}_z \end{aligned} \quad (20)$$

The negative sign indicates that the Lorentz force acts on the disk in the direction opposite to the motion. An external torque equal in magnitude but opposite in direction to (20) is necessary to turn the shaft.

This device can also be operated as a motor if a rotor current into the disk ( $i_r < 0$ ) is imposed. Then the electrical torque causes the disk to turn.

### (b) Self-Excited Generator

For generator operation it is necessary to turn the shaft and supply a field current to generate the magnetic field. However, if the field coil is connected to the rotor terminals, as in Figure 6-16a, the generator can supply its own field current. The equivalent circuit for self-excited operation is shown in Figure 6-16b where the series connection has  $i_r = i_f$ .

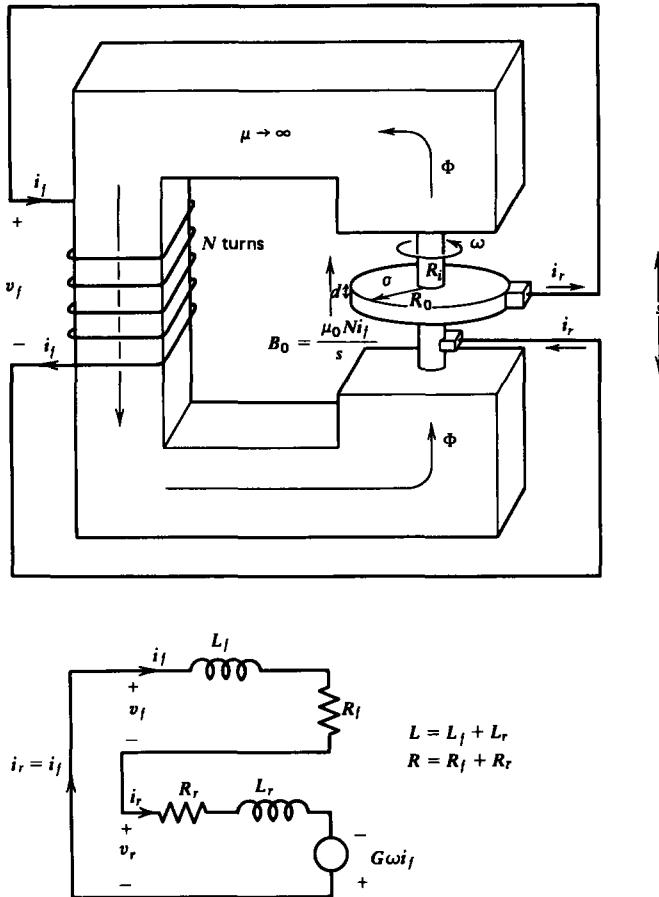


Figure 6-16 A homopolar generator can be self-excited where the generated rotor current is fed back to the field winding to generate its own magnetic field.

Kirchoff's voltage law around the loop is

$$\frac{di}{dt} + i(R - G\omega) = 0, \quad R = R_r + R_f, \quad L = L_r + L_f \quad (21)$$

where  $R$  and  $L$  are the series resistance and inductance of the coil and disk. The solution to (21) is

$$i = I_0 e^{-(R-G\omega)/L t} \quad (22)$$

where  $I_0$  is the initial current at  $t = 0$ . If the exponential factor is positive

$$G\omega > R \quad (23)$$

the current grows with time no matter how small  $I_0$  is. In practice,  $I_0$  is generated by random fluctuations (noise) due to residual magnetism in the iron core. The exponential growth is limited by magnetic core saturation so that the current reaches a steady-state value. If the disk is rotating in the opposite direction ( $\omega < 0$ ), the condition of (23) cannot be satisfied. It is then necessary for the field coil connection to be reversed so that  $i_r = -i_f$ . Such a dynamo model has been used as a model of the origin of the earth's magnetic field.

### (c) Self-Excited ac Operation

Two such coupled generators can spontaneously generate two phase ac power if two independent field windings are connected, as in Figure 6-17. The field windings are connected so that if the flux through the two windings on one machine add, they subtract on the other machine. This accounts for the sign difference in the speed voltages in the equivalent circuits,

$$L \frac{di_1}{dt} + (R - G\omega)i_1 + G\omega i_2 = 0 \quad (24)$$

$$L \frac{di_2}{dt} + (R - G\omega)i_2 - G\omega i_1 = 0$$

where  $L$  and  $R$  are the total series inductance and resistance. The disks are each turned at the same angular speed  $\omega$ .

Since (24) are linear with constant coefficients, solutions are of the form

$$i_1 = I_1 e^{st}, \quad i_2 = I_2 e^{st} \quad (25)$$

which when substituted back into (24) yields

$$\begin{aligned} (Ls + R - G\omega)I_1 + G\omega I_2 &= 0 \\ -G\omega I_1 + (Ls + R - G\omega)I_2 &= 0 \end{aligned} \quad (26)$$

For nontrivial solutions, the determinant of the coefficients of  $I_1$  and  $I_2$  must be zero,

$$(Ls + R - G\omega)^2 = -(G\omega)^2 \quad (27)$$

which when solved for  $s$  yields the complex conjugate natural frequencies,

$$\begin{aligned} s &= -\frac{(R - G\omega)}{L} \pm j \frac{G\omega}{L} \\ I_1/I_2 &= \pm j \end{aligned} \quad (28)$$

where the currents are  $90^\circ$  out of phase. If the real part of  $s$  is positive, the system is self-excited so that any perturbation

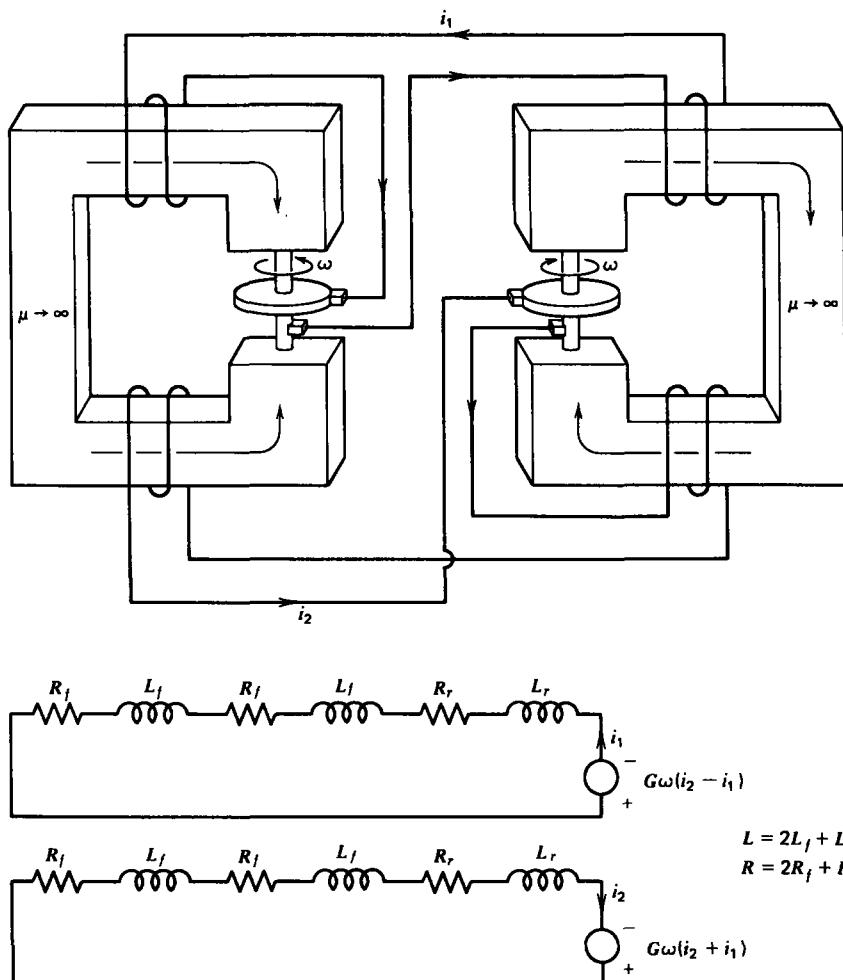


Figure 6-17 Cross-connecting two homopolar generators can result in self-excited two-phase alternating currents. Two independent field windings are required where on one machine the fluxes add while on the other they subtract.

grows at an exponential rate:

$$G\omega > R \quad (29)$$

The imaginary part of  $s$  yields the oscillation frequency

$$\omega_0 = \text{Im}(s) = G\omega/L \quad (30)$$

Again, core saturation limits the exponential growth so that two-phase power results. Such a model may help explain the periodic reversals in the earth's magnetic field every few hundred thousand years.

#### (d) Periodic Motor Speed Reversals

If the field winding of a motor is excited by a dc current, as in Figure 6-18, with the rotor terminals connected to a generator whose field and rotor terminals are in series, the circuit equation is

$$\frac{di}{dt} + \frac{(R - G_g \omega_g)}{L} i = \frac{G_m \omega_m}{L} I_f \quad (31)$$

where  $L$  and  $R$  are the total series inductances and resistances. The angular speed of the generator  $\omega_g$  is externally

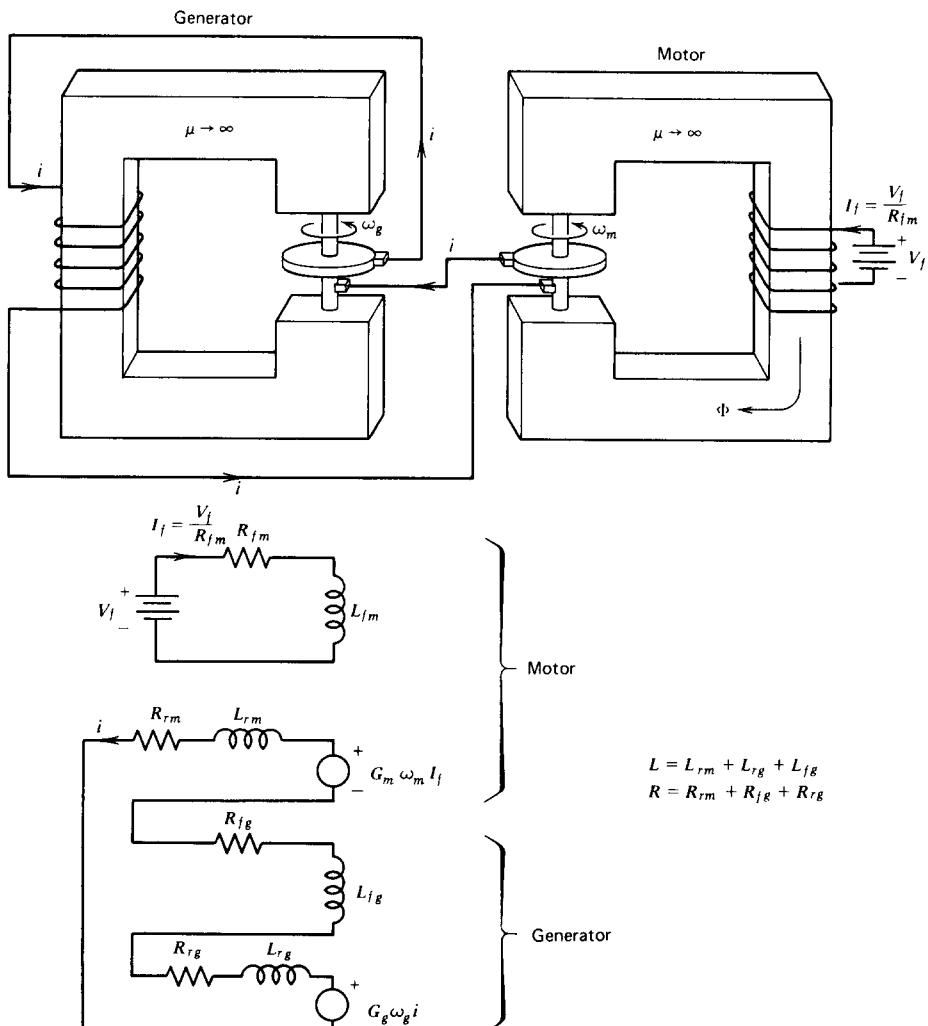


Figure 6-18 Cross connecting a homopolar generator and motor can result in spontaneous periodic speed reversals of the motor's shaft.

constrained to be a constant. The angular acceleration of the motor's shaft is equal to the torque of (20),

$$J \frac{d\omega_m}{dt} = -G_m I_f i \quad (32)$$

where  $J$  is the moment of inertia of the shaft and  $I_f = V_f/R_{fm}$  is the constant motor field current.

Solutions of these coupled, linear constant coefficient differential equations are of the form

$$\begin{aligned} i &= I e^{st} \\ \omega &= W e^{st} \end{aligned} \quad (33)$$

which when substituted back into (31) and (32) yield

$$\begin{aligned} I \left( s + \frac{R - G_m \omega_g}{L} \right) - W \left( \frac{G_m I_f}{L} \right) &= 0 \\ I \left( \frac{G_m I_f}{J} \right) + W s &= 0 \end{aligned} \quad (34)$$

Again, for nontrivial solutions the determinant of coefficients of  $I$  and  $W$  must be zero,

$$s \left( s + \frac{R - G_m \omega_g}{L} \right) + \frac{(G_m I_f)^2}{JL} = 0 \quad (35)$$

which when solved for  $s$  yields

$$s = -\frac{(R - G_m \omega_g)}{2L} \pm \left[ \left( \frac{R - G_m \omega_g}{2L} \right)^2 - \frac{(G_m I_f)^2}{JL} \right]^{1/2} \quad (36)$$

For self-excitation the real part of  $s$  must be positive,

$$G_m \omega_g > R \quad (37)$$

while oscillations will occur if  $s$  has an imaginary part,

$$\frac{(G_m I_f)^2}{JL} > \left( \frac{R - G_m \omega_g}{2L} \right)^2 \quad (38)$$

Now, both the current and shaft's angular velocity spontaneously oscillate with time.

### 6-3-4 Basic Motors and Generators

#### (a) ac Machines

Alternating voltages are generated from a dc magnetic field by rotating a coil, as in Figure 6-19. An output voltage is measured via slip rings through carbon brushes. If the loop of area  $A$  is vertical at  $t = 0$  linking zero flux, the imposed flux

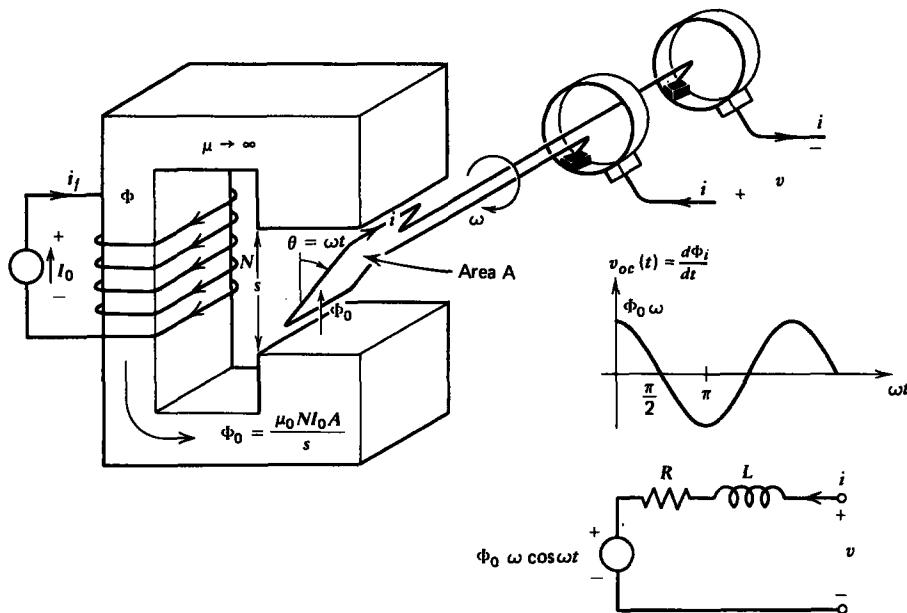


Figure 6-19 A coil rotated within a constant magnetic field generates a sinusoidal voltage.

through the loop at any time, varies sinusoidally with time due to the rotation as

$$\Phi_i = \Phi_0 \sin \omega t \quad (39)$$

Faraday's law applied to a stationary contour instantaneously passing through the wire then gives the terminal voltage as

$$v = iR + \frac{d\Phi}{dt} = iR + L \frac{di}{dt} + \Phi_0 \omega \cos \omega t \quad (40)$$

where  $R$  and  $L$  are the resistance and inductance of the wire. The total flux is equal to the imposed flux of (39) as well as self-flux (accounted for by  $L$ ) generated by the current  $i$ . The equivalent circuit is then similar to that of the homopolar generator, but the speed voltage term is now sinusoidal in time.

### (b) dc Machines

DC machines have a similar configuration except that the slip ring is split into two sections, as in Figure 6-20a. Then whenever the output voltage tends to change sign, the terminals are also reversed yielding the waveform shown, which is of one polarity with periodic variations from zero to a peak value.

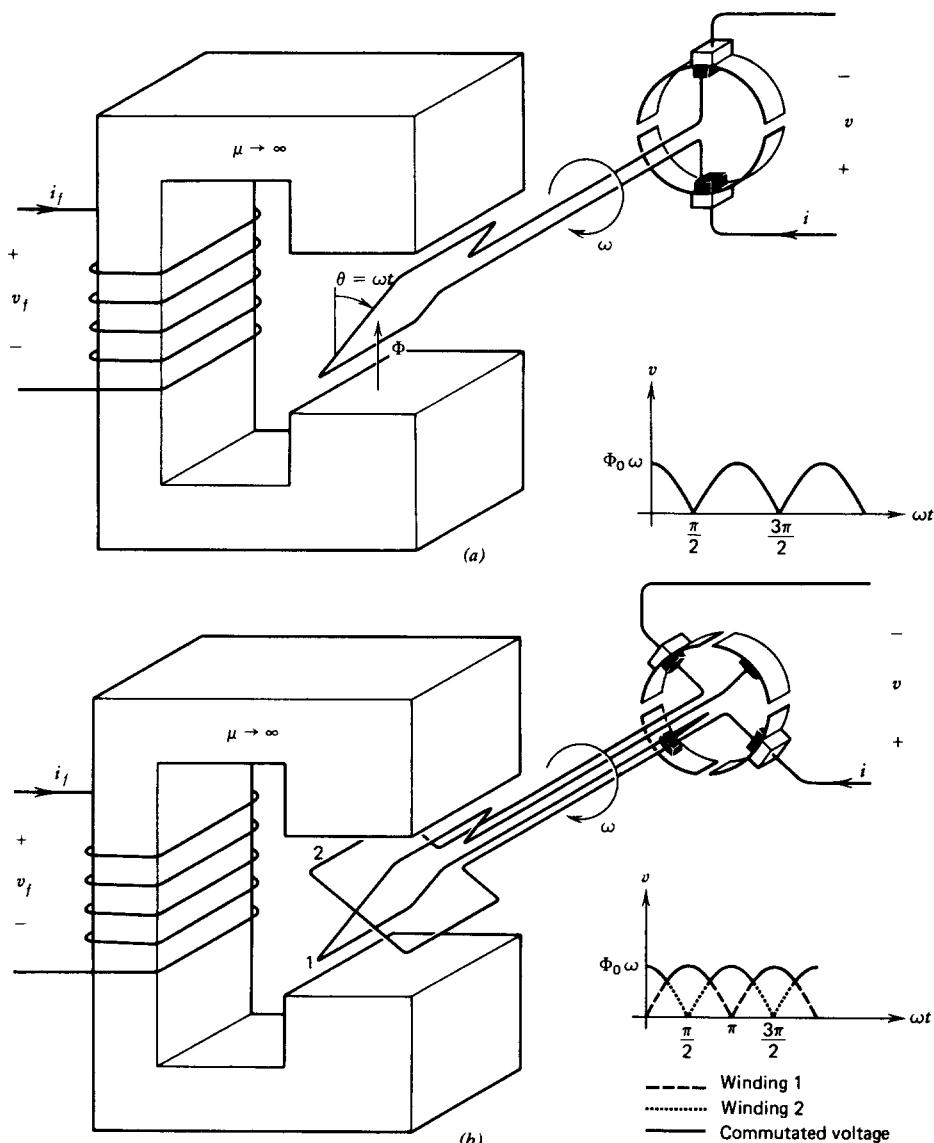


Figure 6-20 (a) If the slip rings are split so that when the voltage tends to change sign the terminals are also reversed, the resulting voltage is of one polarity. (b) The voltage waveform can be smoothed out by placing a second coil at right angles to the first and using a four-section commutator.

The voltage waveform can be smoothed out by using a four-section commutator and placing a second coil perpendicular to the first, as in Figure 6-20b. This second coil now generates its peak voltage when the first coil generates zero voltage. With more commutator sections and more coils, the dc voltage can be made as smooth as desired.

### 6-3-5 MHD Machines

Magnetohydrodynamic machines are based on the same principles as rotating machines, replacing the rigid rotor by a conducting fluid. For the linear machine in Figure 6-21, a fluid with Ohmic conductivity  $\sigma$  flowing with velocity  $v_y$ , moves perpendicularly to an applied magnetic field  $B_0\mathbf{i}_z$ . The terminal voltage  $V$  is related to the electric field and current as

$$\mathbf{E} = \mathbf{i}_x \frac{V}{s}, \quad \mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = \sigma \left( \frac{V}{s} + v_y B_0 \right) \mathbf{i}_x = \frac{i}{Dd} \mathbf{i}_x \quad (41)$$

which can be rewritten as

$$V = iR - v_y B_0 s \quad (42)$$

which has a similar equivalent circuit as for the homopolar generator.

The force on the channel is then

$$\begin{aligned} \mathbf{f} &= \int_V \mathbf{J} \times \mathbf{B} dV \\ &= -iB_0 s \mathbf{i}_y, \end{aligned} \quad (43)$$

again opposite to the fluid motion.

### 6-3-6 Paradoxes

Faraday's law is prone to misuse, which has led to numerous paradoxes. The confusion arises because the same

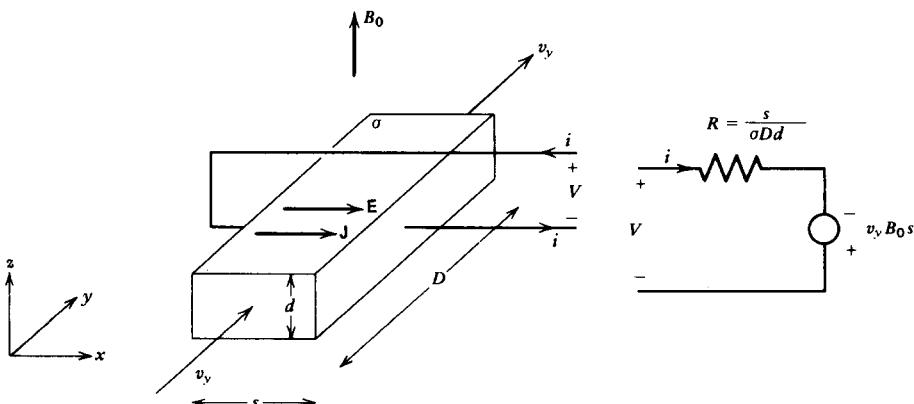


Figure 6-21 An MHD (magnetohydrodynamic) machine replaces a rotating conductor by a moving fluid.

contribution can arise from either the electromotive force side of the law, as a speed voltage when a conductor moves orthogonal to a magnetic field, or as a time rate of change of flux through the contour. This flux term itself has two contributions due to a time varying magnetic field or due to a contour that changes its shape, size, or orientation. With all these potential contributions it is often easy to miss a term or to double count.

### (a) A Commutatorless dc Machine\*

Many persons have tried to make a commutatorless dc machine but to no avail. One novel unsuccessful attempt is illustrated in Figure 6-22, where a highly conducting wire is vibrated within the gap of a magnetic circuit with sinusoidal velocity:

$$v_x = v_0 \sin \omega t \quad (44)$$

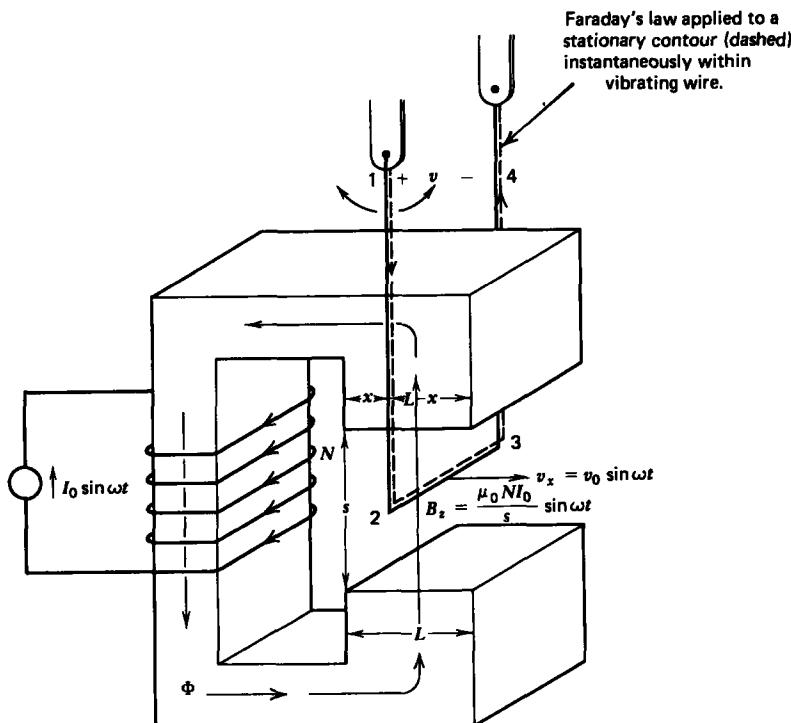


Fig 6-22 It is impossible to design a commutatorless dc machine. Although the speed voltage alone can have a dc average, it will be canceled by the transformer electromotive force due to the time rate of change of magnetic flux through the loop. The total terminal voltage will always have a zero time average.

\* H. Sohon, *Electrical Essays for Recreation*. Electrical Engineering, May (1946), p. 294.

The sinusoidal current imposes the air gap flux density at the same frequency  $\omega$ :

$$B_z = B_0 \sin \omega t, \quad B_0 = \mu_0 N I_0 / s \quad (45)$$

Applying Faraday's law to a stationary contour instantaneously within the open circuited wire yields

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \int_1^2 \mathbf{E}'^0 \cdot d\mathbf{l} + \int_2^3 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{\mathbf{E} = -v \times \mathbf{B}} + \int_3^4 \mathbf{E}'^0 \cdot d\mathbf{l} + \int_4^1 \underbrace{\mathbf{E} \cdot d\mathbf{l}}_{-v}$$

$$= -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (46)$$

where the electric field within the highly conducting wire as measured by an observer moving with the wire is zero. The electric field on the 2-3 leg within the air gap is given by (11), where  $\mathbf{E}' = 0$ , while the 4-1 leg defines the terminal voltage. If we erroneously argue that the flux term on the right-hand side is zero because the magnetic field  $\mathbf{B}$  is perpendicular to  $d\mathbf{S}$ , the terminal voltage is

$$v = v_x B_z l = v_0 B_0 l \sin^2 \omega t \quad (47)$$

which has a dc time-average value. Unfortunately, this result is not complete because we forgot to include the flux that turns the corner in the magnetic core and passes perpendicularly through our contour. Only the flux to the right of the wire passes through our contour, which is the fraction  $(L-x)/L$  of the total flux. Then the correct evaluation of (46) is

$$-v + v_x B_z l = +\frac{d}{dt} [(L-x) B_z l] \quad (48)$$

where  $x$  is treated as a constant because the contour is stationary. The change in sign on the right-hand side arises because the flux passes through the contour in the direction opposite to its normal defined by the right-hand rule. The voltage is then

$$v = v_x B_z l - (L-x) l \frac{dB_z}{dt} \quad (49)$$

where the wire position is obtained by integrating (44),

$$x = \int v_x dt = -\frac{v_0}{\omega} (\cos \omega t - 1) + x_0 \quad (50)$$

and  $x_0$  is the wire's position at  $t = 0$ . Then (49) becomes

$$\begin{aligned} v &= l \frac{d}{dt}(xB_z) - Ll \frac{dB_z}{dt} \\ &= B_0 lv_0 \left[ \left( \frac{x_0 \omega}{v_0} + 1 \right) \cos \omega t - \cos 2\omega t \right] - Ll B_0 \omega \cos \omega t \quad (51) \end{aligned}$$

which has a zero time average.

### (b) Changes in Magnetic Flux Due to Switching

Changing the configuration of a circuit using a switch does not result in an electromotive force unless the magnetic flux itself changes.

In Figure 6-23a, the magnetic field through the loop is externally imposed and is independent of the switch position. Moving the switch does not induce an EMF because the magnetic flux through any surface remains unchanged.

In Figure 6-23b, a dc current source is connected to a circuit through a switch  $S$ . If the switch is instantaneously moved from contact 1 to contact 2, the magnetic field due to the source current  $I$  changes. The flux through any fixed area has thus changed resulting in an EMF.

### (c) Time Varying Number of Turns on a Coil\*

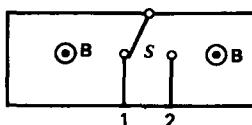
If the number of turns on a coil is changing with time, as in Figure 6-24, the voltage is equal to the time rate of change of flux through the coil. Is the voltage then

$$v \stackrel{?}{=} N \frac{d\Phi}{dt} \quad (52)$$

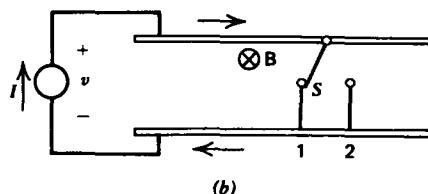
or

$$v \stackrel{?}{=} \frac{d}{dt}(N\Phi) = N \frac{d\Phi}{dt} + \Phi \frac{dN}{dt} \quad (53)$$

No current is induced by switching.



(a)



(b)

Figure 6-23 (a) Changes in a circuit through the use of a switch does not by itself generate an EMF. (b) However, an EMF can be generated if the switch changes the magnetic field.

\* L. V. Bewley, Flux Linkages and Electromagnetic Induction. Macmillan, New York, 1952.

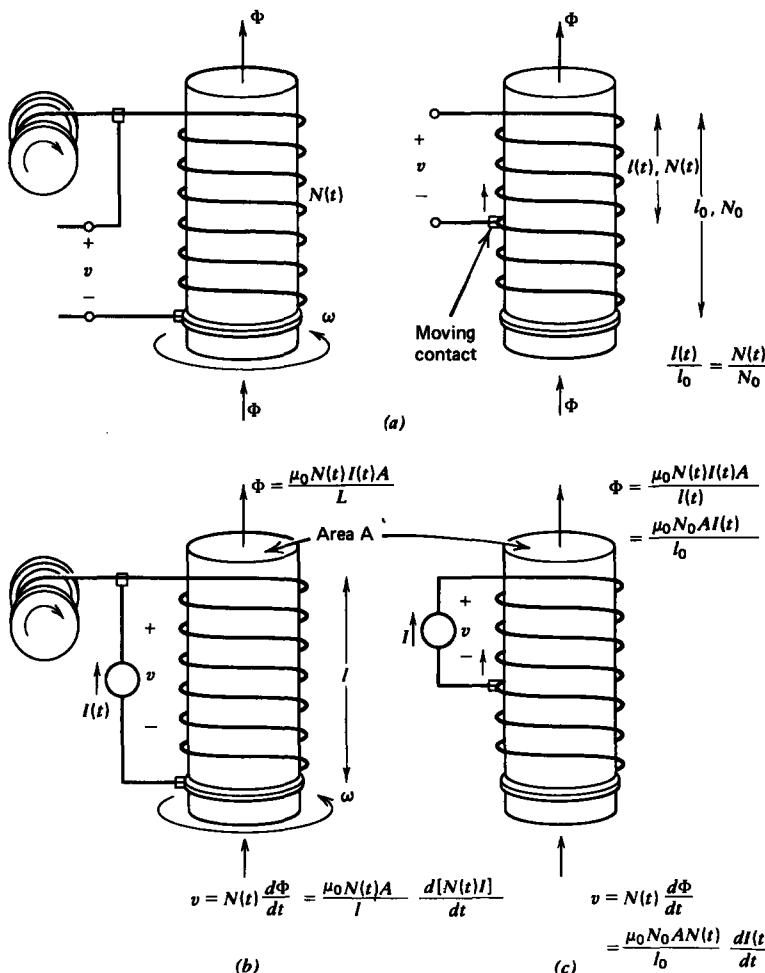


Figure 6-24 (a) If the number of turns on a coil is changing with time, the induced voltage is  $v = N(t) d\Phi/dt$ . A constant flux does not generate any voltage. (b) If the flux itself is proportional to the number of turns, a dc current can generate a voltage. (c) With the tap changing coil, the number of turns per unit length remains constant so that a dc current generates no voltage because the flux does not change with time.

For the first case a dc flux generates no voltage while the second does.

We use Faraday's law with a stationary contour instantaneously within the wire. Because the contour is stationary, its area of  $NA$  is not changing with time and so can be taken outside the time derivative in the flux term of Faraday's law so that the voltage is given by (52) and (53) is wrong. Note that there is no speed voltage contribution in the electromotive force because the velocity of the wire is in the same direction as the contour ( $\mathbf{v} \times \mathbf{B} \cdot d\mathbf{l} = 0$ ).

If the flux  $\Phi$  itself depends on the number of turns, as in Figure 6-24b, there may be a contribution to the voltage even if the exciting current is dc. This is true for the turns being wound onto the cylinder in Figure 6-24b. For the tap changing configuration in Figure 6-24c, with uniformly wound turns, the ratio of turns to effective length is constant so that a dc current will still not generate a voltage.

## 6-4 MAGNETIC DIFFUSION INTO AN OHMIC CONDUCTOR\*

If the current distribution is known, the magnetic field can be directly found from the Biot-Savart or Ampere's laws. However, when the magnetic field varies with time, the generated electric field within an Ohmic conductor induces further currents that also contribute to the magnetic field.

### 6-4-1 Resistor-Inductor Model

A thin conducting shell of radius  $R_i$ , thickness  $\Delta$ , and depth  $l$  is placed within a larger conducting cylinder, as shown in Figure 6-25. A step current  $I_0$  is applied at  $t = 0$  to the larger cylinder, generating a surface current  $\mathbf{K} = (I_0/l)\mathbf{i}_\phi$ . If the length  $l$  is much greater than the outer radius  $R_o$ , the magnetic field is zero outside the cylinder and uniform inside for  $R_i < r < R_o$ . Then from the boundary condition on the discontinuity of tangential  $\mathbf{H}$  given in Section 5-6-1, we have

$$\mathbf{H}_0 = \frac{I_0}{l} \mathbf{i}_z, \quad R_i < r < R_o \quad (1)$$

The magnetic field is different inside the conducting shell because of the induced current, which from Lenz's law, flows in the opposite direction to the applied current. Because the shell is assumed to be very thin ( $\Delta \ll R_i$ ), this induced current can be considered a surface current related to the volume current and electric field in the conductor as

$$\mathbf{K}_\phi = J_\phi \Delta = (\sigma \Delta) \mathbf{E}_\phi \quad (2)$$

The product  $(\sigma \Delta)$  is called the surface conductivity. Then the magnetic fields on either side of the thin shell are also related by the boundary condition of Section 5-6-1:

$$\mathbf{H}_i - \mathbf{H}_0 = \mathbf{K}_\phi = (\sigma \Delta) \mathbf{E}_\phi \quad (3)$$

\* Much of the treatment of this section is similar to that of H. H. Woodson and J. R. Melcher, Electromechanical Dynamics, Part II, Wiley, N.Y., 1968, Ch. 7.

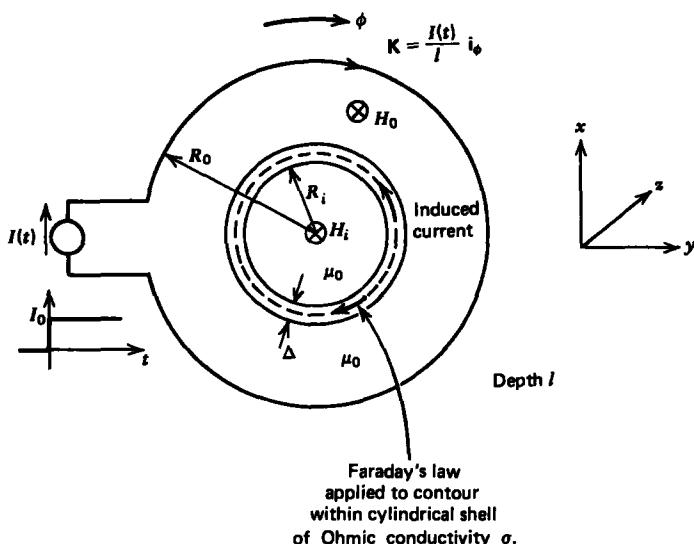


Figure 6-25 A step change in magnetic field causes the induced current within an Ohmic conductor to flow in the direction where its self-flux opposes the externally imposed flux. Ohmic dissipation causes the induced current to exponentially decay with time with a  $L/R$  time constant.

Applying Faraday's law to a contour within the conducting shell yields

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \Rightarrow E_\phi 2\pi R_i = -\mu_0 \pi R_i^2 \frac{dH_i}{dt} \quad (4)$$

where only the magnetic flux due to  $H_i$  passes through the contour. Then using (1)–(3) in (4) yields a single equation in  $H_i$ :

$$\frac{dH_i}{dt} + \frac{H_i}{\tau} = \frac{I(t)}{l\tau}, \quad \tau = \frac{\mu_0 R_i \sigma \Delta}{2} \quad (5)$$

where we recognize the time constant  $\tau$  as just being the ratio of the shell's self-inductance to resistance:

$$L = \frac{\Phi}{K_\phi l} = \frac{\mu_0 \pi R_i^2}{l}, \quad R = \frac{2\pi R_i}{\sigma l \Delta}, \quad \tau = \frac{L}{R} = \frac{\mu_0 R_i \sigma \Delta}{2} \quad (6)$$

The solution to (5) for a step current with zero initial magnetic field is

$$H_i = \frac{I_0}{l} (1 - e^{-t/\tau}) \quad (7)$$

Initially, the magnetic field is excluded from inside the conducting shell by the induced current. However, Ohmic

dissipation causes the induced current to decay with time so that the magnetic field may penetrate through the shell with characteristic time constant  $\tau$ .

#### 6-4-2 The Magnetic Diffusion Equation

The transient solution for a thin conducting shell could be solved using the integral laws because the geometry constrained the induced current to flow azimuthally with no radial variations. If the current density is not directly known, it becomes necessary to self-consistently solve for the current density with the electric and magnetic fields:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (\text{Faraday's law}) \quad (8)$$

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad (\text{Ampere's law}) \quad (9)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Gauss's law}) \quad (10)$$

For linear magnetic materials with constant permeability  $\mu$  and constant Ohmic conductivity  $\sigma$  moving with velocity  $\mathbf{U}$ , the constitutive laws are

$$\mathbf{B} = \mu \mathbf{H}, \quad \mathbf{J}_f = \sigma(\mathbf{E} + \mathbf{U} \times \mu \mathbf{H}) \quad (11)$$

We can reduce (8)–(11) to a single equation in the magnetic field by taking the curl of (9), using (8) and (11) as

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{H}) &= \nabla \times \mathbf{J}_f \\ &= \sigma[\nabla \times \mathbf{E} + \mu \nabla \times (\mathbf{U} \times \mathbf{H})] \\ &= \mu \sigma \left( -\frac{\partial \mathbf{H}}{\partial t} + \nabla \times (\mathbf{U} \times \mathbf{H}) \right) \end{aligned} \quad (12)$$

The double cross product of  $\mathbf{H}$  can be simplified using the vector identity

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{H}) &= \nabla(\nabla \cdot \overset{\circ}{\mathbf{H}}) - \nabla^2 \mathbf{H} \\ \Rightarrow \frac{1}{\mu \sigma} \nabla^2 \mathbf{H} &= \frac{\partial \mathbf{H}}{\partial t} - \nabla \times (\mathbf{U} \times \mathbf{H}) \end{aligned} \quad (13)$$

where  $\mathbf{H}$  has no divergence from (10). Remember that the Laplacian operator on the left-hand side of (13) also differentiates the directionally dependent unit vectors in cylindrical ( $\mathbf{i}_r$  and  $\mathbf{i}_\phi$ ) and spherical ( $\mathbf{i}_r$ ,  $\mathbf{i}_\theta$ , and  $\mathbf{i}_\phi$ ) coordinates.

### 6-4-3 Transient Solution with $\mathbf{J}_0$ Motion ( $\mathbf{U} = 0$ )

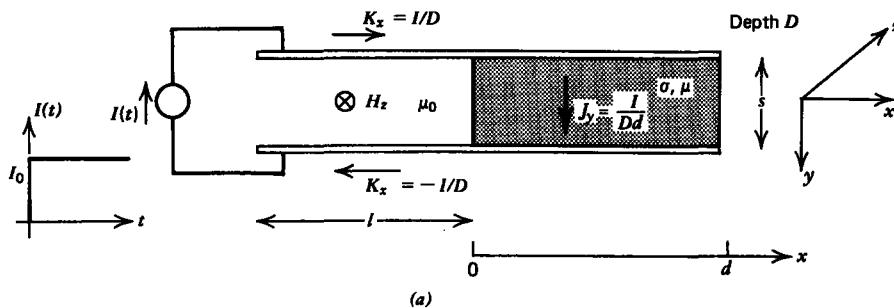
A step current is turned on at  $t = 0$ , in the parallel plate geometry shown in Figure 6-26. By the right-hand rule and with the neglect of fringing, the magnetic field is in the  $z$  direction and only depends on the  $x$  coordinate,  $B_z(x, t)$ , so that (13) reduces to

$$\frac{\partial^2 B_z}{\partial x^2} - \sigma\mu \frac{\partial B_z}{\partial t} = 0 \quad (14)$$

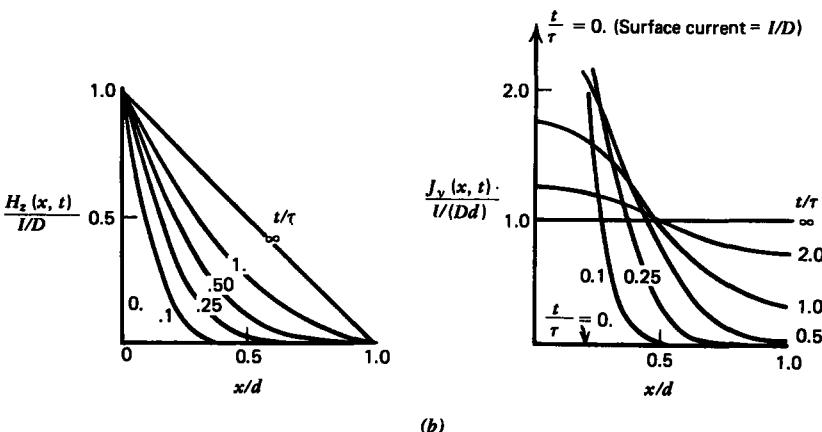
which is similar in form to the diffusion equation of a distributed resistive-capacitive cable developed in Section 3-6-4.

In the dc steady state, the second term is zero so that the solution in each region is of the form

$$\frac{\partial^2 H_z}{\partial x^2} = 0 \Rightarrow H_z = ax + b \quad (15)$$



(a)



(b)

Figure 6-26 (a) A current source is instantaneously turned on at  $t = 0$ . The resulting magnetic field within the Ohmic conductor remains continuous and is thus zero at  $t = 0$  requiring a surface current at  $x = 0$ . (b) For later times the magnetic field and current diffuse into the conductor with longest time constant  $\tau = \sigma\mu d^2/\pi^2$  towards a steady state of uniform current with a linear magnetic field.

where  $a$  and  $b$  are found from the boundary conditions. The current on the electrodes immediately spreads out to a uniform surface distribution  $\pm(I/D)\mathbf{i}_x$  traveling from the upper to lower electrode uniformly through the Ohmic conductor. Then, the magnetic field is uniform in the free space region, decreasing linearly to zero within the Ohmic conductor being continuous across the interface at  $x = 0$ :

$$\lim_{t \rightarrow \infty} H_z(x) = \begin{cases} \frac{I}{D}, & -l \leq x \leq 0 \\ \frac{I}{Dd}(d-x), & 0 \leq x \leq d \end{cases} \quad (16)$$

In the free space region where  $\sigma = 0$ , the magnetic field remains constant for all time. Within the conducting slab, there is an initial charging transient as the magnetic field builds up to the linear steady-state distribution in (16). Because (14) is a linear equation, for the total solution of the magnetic field as a function of time and space, we use superposition and guess a solution that is the sum of the steady-state solution in (16) and a transient solution which dies off with time:

$$H_z(x, t) = \frac{I}{Dd}(d-x) + \hat{H}(x) e^{-\alpha t} \quad (17)$$

We follow the same procedures as for the lossy cable in Section 3-6-4. At this point we do not know the function  $\hat{H}(x)$  or the parameter  $\alpha$ . Substituting the assumed solution of (17) back into (14) yields the ordinary differential equation

$$\frac{d^2 \hat{H}(x)}{dx^2} + \sigma \mu \alpha \hat{H}(x) = 0 \quad (18)$$

which has the trigonometric solutions

$$\hat{H}(x) = A_1 \sin \sqrt{\sigma \mu \alpha} x + A_2 \cos \sqrt{\sigma \mu \alpha} x \quad (19)$$

Since the time-independent part in (17) already meets the boundary conditions of

$$\begin{aligned} H_z(x=0) &= I/D \\ H_z(x=d) &= 0 \end{aligned} \quad (20)$$

the transient part of the solution must be zero at the ends

$$\begin{aligned} \hat{H}(x=0) = 0 &\Rightarrow A_2 = 0 \\ \hat{H}(x=d) = 0 &\Rightarrow A_1 \sin \sqrt{\sigma \mu \alpha} d = 0 \end{aligned} \quad (21)$$

which yields the allowed values of  $\alpha$  as

$$\sqrt{\sigma \mu \alpha} d = n\pi \Rightarrow \alpha_n = \frac{1}{\mu \sigma} \left( \frac{n\pi}{d} \right)^2, \quad n = 1, 2, 3, \dots \quad (22)$$

Since there are an infinite number of allowed values of  $\alpha$ , the most general solution is the superposition of all allowed solutions:

$$H_z(x, t) = \frac{I}{Dd} (d - x) + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} e^{-\alpha_n t} \quad (23)$$

This relation satisfies the boundary conditions but not the initial conditions at  $t = 0$  when the current is first turned on. Before the current takes its step at  $t = 0$ , the magnetic field is zero in the slab. Right after the current is turned on, the magnetic field must remain zero. Faraday's law would otherwise make the electric field and thus the current density infinite within the slab, which is nonphysical. Thus we impose the initial condition

$$H_z(x, t = 0) = 0 = \frac{I}{Dd} (d - x) + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{d} \quad (24)$$

which will allow us to solve for the amplitudes  $A_n$  by multiplying (24) through by  $\sin(m\pi x/d)$  and then integrating over  $x$  from 0 to  $d$ :

$$0 = \frac{I}{Dd} \int_0^d (d - x) \sin \frac{m\pi x}{d} dx + \sum_{n=1}^{\infty} A_n \int_0^d \sin \frac{n\pi x}{d} \sin \frac{m\pi x}{d} dx \quad (25)$$

The first term on the right-hand side is easily integrable\* while the product of sine terms integrates to zero unless  $m = n$ , yielding

$$A_m = -\frac{2I}{m\pi D} \quad (26)$$

The total solution is thus

$$H_z(x, t) = \frac{I}{D} \left( 1 - \frac{x}{d} - 2 \sum_{n=1}^{\infty} \frac{\sin(n\pi x/d)}{n\pi} e^{-n^2 t/\tau} \right) \quad (27)$$

where we define the fundamental continuum magnetic diffusion time constant  $\tau$  as

$$\tau = \frac{1}{\alpha_1} = \frac{\mu\sigma d^2}{\pi^2} \quad (28)$$

analogous to the lumped parameter time constant of (5) and (6).

\*  $\int_0^d (d - x) \sin \frac{m\pi x}{d} dx = \frac{d^2}{m\pi}$

The magnetic field approaches the steady state in times long compared to  $\tau$ . For a perfect conductor ( $\sigma \rightarrow \infty$ ), this time is infinite and the magnetic field is forever excluded from the slab. The current then flows only along the  $x = 0$  surface. However, even for copper ( $\sigma \approx 6 \times 10^7$  siemens/m) 10-cm thick, the time constant is  $\tau \approx 80$  msec, which is fast for many applications. The current then diffuses into the conductor where the current density is easily obtained from Ampere's law as

$$\begin{aligned}\mathbf{J}_f &= \nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x} \mathbf{i}_y \\ &= \frac{I}{Dd} \left( 1 + 2 \sum_{n=1}^{\infty} \cos \frac{n\pi x}{d} e^{-n^2 t/\tau} \right) \mathbf{i}_y\end{aligned}\quad (29)$$

The diffusion of the magnetic field and current density are plotted in Figure 6-26b for various times

The force on the conducting slab is due to the Lorentz force tending to expand the loop and a magnetization force due to the difference of permeability of the slab and the surrounding free space as derived in Section 5-8-1:

$$\begin{aligned}\mathbf{F} &= \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} + \mu_0 \mathbf{J}_f \times \mathbf{H} \\ &= (\mu - \mu_0)(\mathbf{H} \cdot \nabla)\mathbf{H} + \mu_0 \mathbf{J}_f \times \mathbf{H}\end{aligned}\quad (30)$$

For our case with  $\mathbf{H} = H_z(x)\mathbf{i}_z$ , the magnetization force density has no contribution so that (30) reduces to

$$\begin{aligned}\mathbf{F} &= \mu_0 \mathbf{J}_f \times \mathbf{H} \\ &= \mu_0(\nabla \times \mathbf{H}) \times \mathbf{H} \\ &= \mu_0(\mathbf{H} \cdot \nabla)\mathbf{H} - \nabla(\frac{1}{2}\mu_0 \mathbf{H} \cdot \mathbf{H}) \\ &= -\frac{d}{dx}(\frac{1}{2}\mu_0 H_z^2) \mathbf{i}_x\end{aligned}\quad (31)$$

Integrating (31) over the slab volume with the magnetic field independent of  $y$  and  $z$ ,

$$\begin{aligned}f_x &= - \int_0^d s D \frac{d}{dx}(\frac{1}{2}\mu_0 H_z^2) dx \\ &= -\frac{1}{2}\mu_0 H_z^2 s D \Big|_0^d \\ &= \frac{1}{2} \frac{\mu_0 I^2 s}{D}\end{aligned}\quad (32)$$

gives us a constant force with time that is independent of the permeability. Note that our approach of expressing the current density in terms of the magnetic field in (31) was easier than multiplying the infinite series of (27) and (29), as the

result then only depended on the magnetic field at the boundaries that are known from the boundary conditions of (20). The resulting integration in (32) was easy because the force density in (31) was expressed as a pure derivative of  $x$ .

#### 6-4-4 The Sinusoidal Steady State (Skin Depth)

We now place an infinitely thick conducting slab a distance  $d$  above a sinusoidally varying current sheet  $K_0 \cos \omega t i_y$ , which lies on top of a perfect conductor, as in Figure 6-27a. The

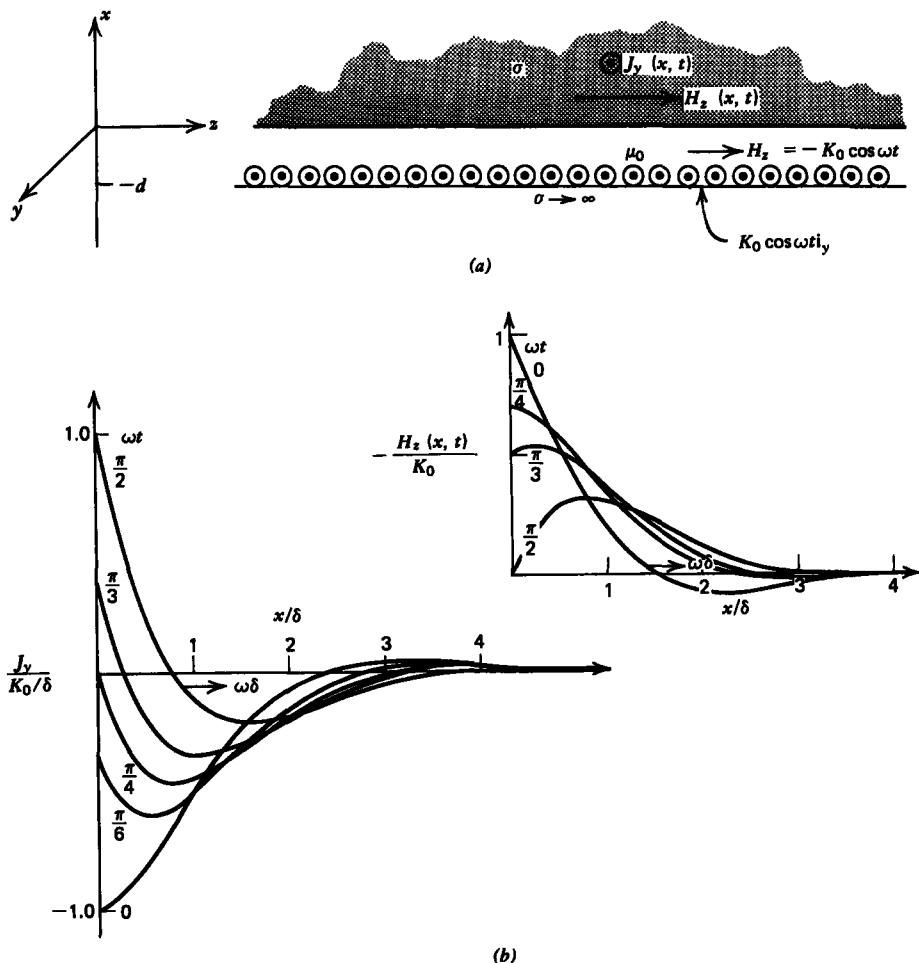


Figure 6.27 (a) A stationary conductor lies above a sinusoidal surface current placed upon a perfect conductor so that  $\mathbf{H} = 0$  for  $x < -d$ . (b) The magnetic field and current density propagates and decays into the conductor with the same characteristic length given by the skin depth  $\delta = \sqrt{2/(\omega\mu\sigma)}$ . The phase speed of the wave is  $\omega\delta$ .

magnetic field within the conductor is then also sinusoidally varying with time:

$$H_z(x, t) = \operatorname{Re} [\hat{H}_z(x) e^{j\omega t}] \quad (33)$$

Substituting (33) into (14) yields

$$\frac{d^2 \hat{H}_z}{dx^2} - j\omega\mu\sigma\hat{H}_z = 0 \quad (34)$$

with solution

$$\hat{H}_z(x) = A_1 e^{(1+j)x/\delta} + A_2 e^{-(1+j)x/\delta} \quad (35)$$

where the skin depth  $\delta$  is defined as

$$\delta = \sqrt{2/(\omega\mu\sigma)} \quad (36)$$

Since the magnetic field must remain finite far from the current sheet,  $A_1$  must be zero. The magnetic field is also continuous across the  $x=0$  boundary because there is no surface current, so that the solution is

$$\begin{aligned} H_z(x, t) &= \operatorname{Re} [-K_0 e^{-(1+j)x/\delta} e^{j\omega t}] \\ &= -K_0 \cos(\omega t - x/\delta) e^{-x/\delta}, \quad x \geq 0 \end{aligned} \quad (37)$$

where the magnetic field in the gap is uniform, determined by the discontinuity in tangential  $\mathbf{H}$  at  $x = -d$  to be  $H_z = -K$ , for  $-d < x \leq 0$  since within the perfect conductor ( $x < -d$ )  $\mathbf{H} = 0$ . The magnetic field diffuses into the conductor as a strongly damped propagating wave with characteristic penetration depth  $\delta$ . The skin depth  $\delta$  is also equal to the propagating wavelength, as drawn in Figure 6-27b. The current density within the conductor

$$\begin{aligned} \mathbf{J}_f &= \nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x} \mathbf{i}, \\ &= +\frac{K_0 e^{-x/\delta}}{\delta} \left[ \sin\left(\omega t - \frac{x}{\delta}\right) - \cos\left(\omega t - \frac{x}{\delta}\right) \right] \mathbf{i}, \end{aligned} \quad (38)$$

is also drawn in Figure 6-27b at various times in the cycle, being confined near the interface to a depth on the order of  $\delta$ . For a perfect conductor,  $\delta \rightarrow 0$ , and the volume current becomes a surface current.

Seawater has a conductivity of  $\approx 4$  siemens/m so that at a frequency of  $f = 1$  MHz ( $\omega = 2\pi f$ ) the skin depth is  $\delta \approx 0.25$  m. This is why radio communications to submarines are difficult. The conductivity of copper is  $\sigma \approx 6 \times 10^7$  siemens/m so that at 60 Hz the skin depth is  $\delta \approx 8$  mm. Power cables with larger radii have most of the current confined near the surface so that the center core carries very little current. This

reduces the cross-sectional area through which the current flows, raising the cable resistance leading to larger power dissipation.

Again, the magnetization force density has no contribution to the force density since  $H_z$  only depends on  $x$ :

$$\begin{aligned}\mathbf{F} &= \mu_0(\mathbf{M} \cdot \nabla)\mathbf{H} + \mu_0\mathbf{J}_f \times \mathbf{H} \\ &= \mu_0(\nabla \times \mathbf{H}) \times \mathbf{H} \\ &= -\nabla\left(\frac{1}{2}\mu_0\mathbf{H} \cdot \mathbf{H}\right)\end{aligned}\quad (39)$$

The total force per unit area on the slab obtained by integrating (39) over  $x$  depends only on the magnetic field at  $x = 0$ :

$$\begin{aligned}f_x &= -\int_0^\infty \frac{d}{dx}\left(\frac{\mu_0}{2}H_z^2\right) dx \\ &= -\frac{1}{2}\mu_0 H_z|_0^\infty \\ &= \frac{1}{2}\mu_0 K_0^2 \cos^2 \omega t\end{aligned}\quad (40)$$

because again  $\mathbf{H}$  is independent of  $y$  and  $z$  and the  $x$  component of the force density of (39) was written as a pure derivative with respect to  $x$ . Note that this approach was easier than integrating the cross product of (38) with (37).

This force can be used to levitate the conductor. Note that the region for  $x > \delta$  is dead weight, as it contributes very little to the magnetic force.

#### 6-4-5 Effects of Convection

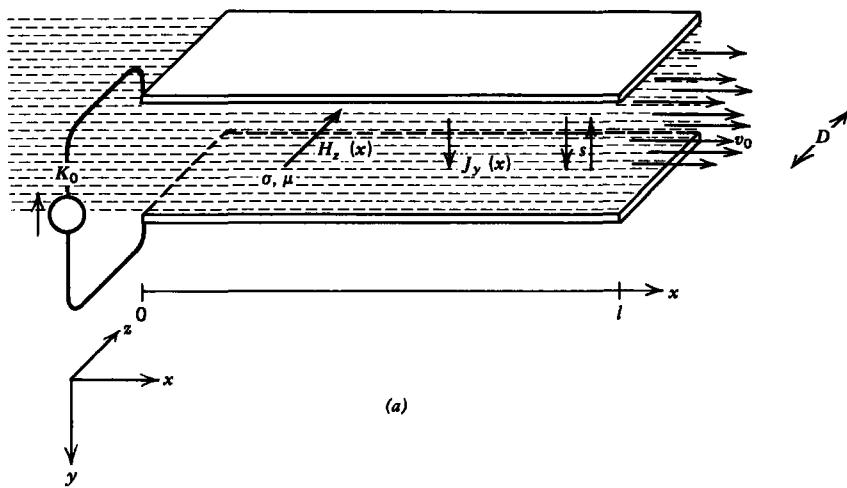
A distributed dc surface current  $-K_0 i$ , at  $x = 0$  flows along parallel electrodes and returns via a conducting fluid moving to the right with constant velocity  $v_0 i_x$ , as shown in Figure 6-28a. The flow is not impeded by the current source at  $x = 0$ . With the neglect of fringing, the magnetic field is purely  $z$  directed and only depends on the  $x$  coordinate, so that (13) in the dc steady state, with  $\mathbf{U} = v_0 i_x$  being a constant, becomes\*

$$\frac{d^2 H_z}{dx^2} - \mu\sigma v_0 \frac{dH_z}{dx} = 0 \quad (41)$$

Solutions of the form

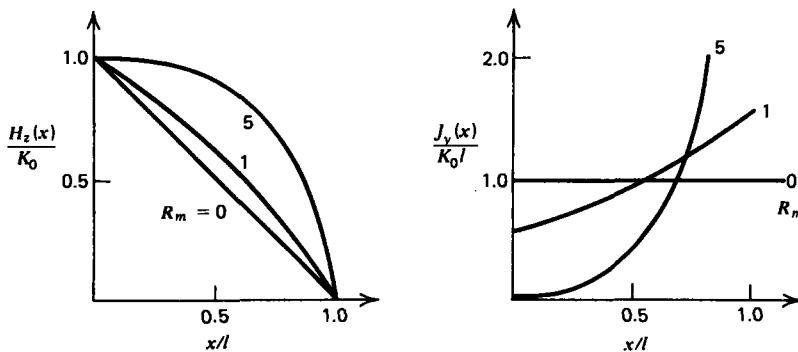
$$H_z(x) = A e^{kx} \quad (42)$$

\*  $\nabla \times (\mathbf{U} \times \mathbf{H}) = \mathbf{U} (\nabla \cdot \mathbf{H}) - \mathbf{H} (\nabla \cdot \mathbf{U}) + (\mathbf{H} \cdot \nabla) \mathbf{U} - (\mathbf{U} \cdot \nabla) \mathbf{H} = -v_0 \frac{d\mathbf{H}}{dx}$



$$H_z(x) = \frac{K_0}{1 - e^{-R_m}} (e^{R_m x/l} - e^{-R_m})$$

$$J_y(x) = -\frac{K_0}{1 - e^{-R_m}} \frac{R_m}{l} e^{R_m x/l}$$



(b)

Figure 6-28 (a) A conducting material moving through a magnetic field tends to pull the magnetic field and current density with it. (b) The magnetic field and current density are greatly disturbed by the flow when the magnetic Reynolds number is large,  $R_m = \sigma \mu U l \gg 1$ .

when substituted back into (41) yield two allowed values of  $p$ ,

$$p^2 - \mu \sigma v_0 p = 0 \Rightarrow p = 0, \quad p = \mu \sigma v_0 \quad (43)$$

Since (41) is linear, the most general solution is just the sum of the two allowed solutions,

$$H_z(x) = A_1 e^{R_m x/l} + A_2 \quad (44)$$

where the magnetic Reynold's number is defined as

$$R_m = \sigma \mu v_0 l = \frac{\sigma \mu l^2}{l/v_0} \quad (45)$$

and represents the ratio of a representative magnetic diffusion time given by (28) to a fluid transport time ( $l/v_0$ ). The boundary conditions are

$$H_z(x=0) = K_0, \quad H_z(x=l) = 0 \quad (46)$$

so that the solution is

$$H_z(x) = \frac{K_0}{1 - e^{R_m}} (e^{R_m x/l} - e^{R_m}) \quad (47)$$

The associated current distribution is then

$$\begin{aligned} \mathbf{J}_f &= \nabla \times \mathbf{H} = -\frac{\partial H_z}{\partial x} \mathbf{i}_y \\ &= -\frac{K_0}{1 - e^{R_m}} \frac{R_m}{l} e^{R_m x/l} \mathbf{i}_y, \end{aligned} \quad (48)$$

The field and current distributions plotted in Figure 6-28b for various  $R_m$  show that the magnetic field and current are pulled along in the direction of flow. For small  $R_m$  the magnetic field is hardly disturbed from the zero flow solution of a linear field and constant current distribution. For very large  $R_m \gg 1$ , the magnetic field approaches a uniform distribution while the current density approaches a surface current at  $x = l$ .

The force on the moving fluid is independent of the flow velocity:

$$\begin{aligned} \mathbf{f} &= \int_0^l \mathbf{J} \times \mu_0 \mathbf{H} s D dx \\ &= -\frac{K_0^2}{(1 - e^{R_m})^2} \mu_0 \frac{R_m}{l} s D \int_0^l e^{R_m x/l} (e^{R_m x/l} - e^{R_m}) dx \mathbf{i}_x \\ &= -\frac{K_0^2 \mu_0 s D}{(1 - e^{R_m})^2} e^{R_m x/l} \left( \frac{e^{R_m x/l}}{2} - e^{R_m} \right) \Big|_0^l \mathbf{i}_x \\ &= \frac{1}{2} \mu_0 K_0^2 s D \mathbf{i}_x \end{aligned} \quad (49)$$

#### 6-4-6 A Linear Induction Machine

The induced currents in a conductor due to a time varying magnetic field give rise to a force that can cause the conductor to move. This describes a motor. The inverse effect is when we cause a conductor to move through a time varying

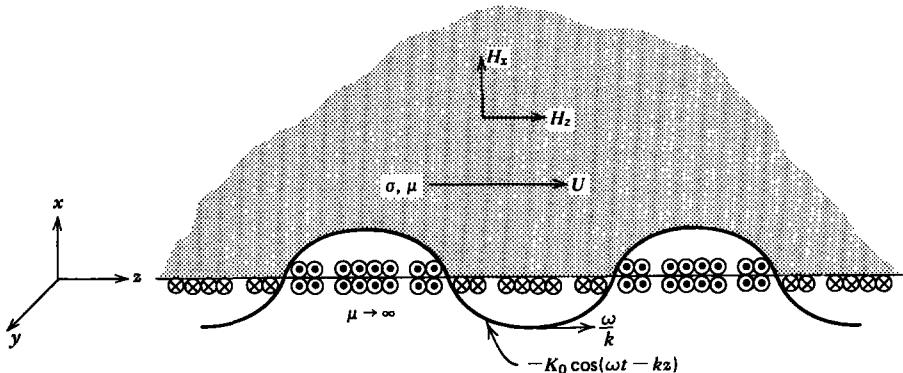
magnetic field generating a current, which describes a generator.

The linear induction machine shown in Figure 6-29a assumes a conductor moves to the right at constant velocity  $Ui_z$ . Directly below the conductor with no gap is a surface current placed on top of an infinitely permeable medium

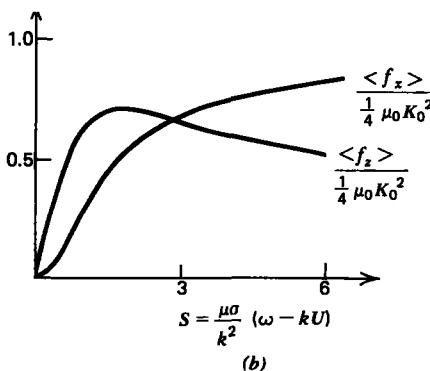
$$\mathbf{K}(t) = -K_0 \cos(\omega t - kz) \mathbf{i}_y = \operatorname{Re} [-K_0 e^{j(\omega t - kz)} \mathbf{i}_y] \quad (50)$$

which is a traveling wave moving to the right at speed  $\omega/k$ . For  $x > 0$ , the magnetic field will then have  $x$  and  $z$  components of the form

$$\begin{aligned} H_z(x, z, t) &= \operatorname{Re} [\hat{H}_z(x) e^{j(\omega t - kz)}] \\ H_x(x, z, t) &= \operatorname{Re} [\hat{H}_x(x) e^{j(\omega t - kz)}] \end{aligned} \quad (51)$$



(a)



(b)

Figure 6-29 (a) A traveling wave of surface current induces currents in a conductor that is moving at a velocity  $U$  different from the wave speed  $\omega/k$ . (b) The resulting forces can levitate and propel the conductor as a function of the slip  $S$ , which measures the difference in speeds of the conductor and traveling wave.

where (10) ( $\nabla \cdot \mathbf{B} = 0$ ) requires these components to be related as

$$\frac{d\hat{H}_x}{dx} - jk\hat{H}_z = 0 \quad (52)$$

The  $z$  component of the magnetic diffusion equation of (13) is

$$\frac{d^2\hat{H}_z}{dx^2} - k^2\hat{H}_z = j\mu\sigma(\omega - kU)\hat{H}_z \quad (53)$$

which can also be written as

$$\frac{d^2\hat{H}_z}{dx^2} - \gamma^2\hat{H}_z = 0 \quad (54)$$

where

$$\gamma^2 = k^2(1 + jS), \quad S = \frac{\mu\sigma}{k^2}(\omega - kU) \quad (55)$$

and  $S$  is known as the slip. Solutions of (54) are again exponential but complex because  $\gamma$  is complex:

$$\hat{H}_z = A_1 e^{-\gamma x} + A_2 e^{-\gamma x} \quad (56)$$

Because  $\hat{H}_z$  must remain finite far from the current sheet,  $A_1 = 0$ , so that using (52) the magnetic field is of the form

$$\hat{\mathbf{H}} = K_0 e^{-\gamma x} \left( \mathbf{i}_z - \frac{jk}{\gamma} \mathbf{i}_x \right) \quad (57)$$

where we use the fact that the tangential component of  $\mathbf{H}$  is discontinuous in the surface current, with  $\mathbf{H} = 0$  for  $x < 0$ .

The current density in the conductor is

$$\begin{aligned} \mathbf{J}_f &= \nabla \times \mathbf{H} = \mathbf{i}_y \left( \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \Rightarrow \hat{J}_y = -jk\hat{H}_x - \frac{d\hat{H}_z}{dx} \\ &= K_0 e^{-\gamma x} \frac{(\gamma^2 - k^2)}{\gamma} \\ &= \frac{K_0 k^2 j S e^{-\gamma x}}{\gamma} \end{aligned} \quad (58)$$

If the conductor and current wave travel at the same speed ( $\omega/k = U$ ), no current is induced as the slip is zero. Currents are only induced if the conductor and wave travel at different velocities. This is the principle of all induction machines.

The force per unit area on the conductor then has  $x$  and  $z$  components:

$$\begin{aligned} \mathbf{f} &= \int_0^\infty \mathbf{J} \times \mu_0 \mathbf{H} dx \\ &= \int_0^\infty \mu_0 J_y (H_z \mathbf{i}_x - H_x \mathbf{i}_z) dx \end{aligned} \quad (59)$$

These integrations are straightforward but lengthy because first the instantaneous field and current density must be found from (51) by taking the real parts. More important is the time-average force per unit area over a period of excitation:

$$\langle \mathbf{f} \rangle = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathbf{f} dt \quad (60)$$

Since the real part of a complex quantity is equal to half the sum of the quantity and its complex conjugate,

$$\begin{aligned} A &= \operatorname{Re} [\hat{A} e^{j\omega t}] = \frac{1}{2} (\hat{A} e^{j\omega t} + \hat{A}^* e^{-j\omega t}) \\ B &= \operatorname{Re} [\hat{B} e^{j\omega t}] = \frac{1}{2} (\hat{B} e^{j\omega t} + \hat{B}^* e^{-j\omega t}) \end{aligned} \quad (61)$$

the time-average product of two quantities is

$$\begin{aligned} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} AB dt &= \frac{1}{4} \frac{\omega}{2\pi} \int_0^{2\pi/\omega} (\hat{A} \hat{B} e^{2j\omega t} + \hat{A}^* \hat{B} + \hat{A} \hat{B}^* \\ &\quad + \hat{A}^* \hat{B}^* e^{-2j\omega t}) dt \\ &= \frac{1}{4} (\hat{A}^* \hat{B} + \hat{A} \hat{B}^*) \\ &= \frac{1}{2} \operatorname{Re} (\hat{A} \hat{B}^*) \end{aligned} \quad (62)$$

which is a formula often used for the time-average power in circuits where  $A$  and  $B$  are the voltage and current.

Then using (62) in (59), the  $x$  component of the time-average force per unit area is

$$\begin{aligned} \langle f_x \rangle &= \frac{1}{2} \operatorname{Re} \left( \int_0^\infty \mu_0 \hat{J}_y \hat{H}_z^* dx \right) \\ &= \frac{\mu_0}{2} K_0^2 k^2 S \operatorname{Re} \left( \frac{j}{\gamma} \int_0^\infty e^{-(\gamma + \gamma^*)x} dx \right) \\ &= \frac{\mu_0}{2} K_0^2 k^2 S \operatorname{Re} \left( \frac{j}{\gamma(\gamma + \gamma^*)} \right) \\ &= \frac{1}{4} \frac{\mu_0 K_0^2 S^2}{[1 + S^2 + (1 + S^2)^{1/2}]} = \frac{1}{4} \mu_0 K_0^2 \left( \frac{\sqrt{1 + S^2} - 1}{\sqrt{1 + S^2}} \right) \end{aligned} \quad (63)$$

where the last equalities were evaluated in terms of the slip  $S$  from (55).

We similarly compute the time-average shear force per unit area as

$$\begin{aligned}
 \langle f_z \rangle &= -\frac{1}{2} \operatorname{Re} \left( \int_0^\infty \mu_0 J_y H_x^* dx \right) \\
 &= \frac{\mu_0}{2} \frac{K_0^2 k^3 S}{\gamma \gamma^*} \operatorname{Re} \left( \int_0^\infty e^{-(\gamma + \gamma^*)x} dx \right) \\
 &= \frac{\mu_0}{2} \frac{k^3 K_0^2 S}{\gamma \gamma^*} \operatorname{Re} \left( \frac{1}{(\gamma + \gamma^*)} \right) \\
 &= \frac{\mu_0 K_0^2 S}{4\sqrt{1+S^2} \operatorname{Re}(\sqrt{1+jS})} \tag{64}
 \end{aligned}$$

When the wave speed exceeds the conductor's speed ( $\omega/k > U$ ), the force is positive as  $S > 0$  so that the wave pulls the conductor along. When  $S < 0$ , the slow wave tends to pull the conductor back as  $\langle f_z \rangle < 0$ . The forces of (63) and (64), plotted in Figure 6-29b, can be used to simultaneously lift and propel a conducting material. There is no force when the wave and conductor travel at the same speed ( $\omega/k = U$ ) as the slip is zero ( $S = 0$ ). For large  $S$ , the levitating force  $\langle f_z \rangle$  approaches the constant value  $\frac{1}{4}\mu_0 K_0^2$  while the shear force approaches zero. There is an optimum value of  $S$  that maximizes  $\langle f_z \rangle$ . For smaller  $S$ , less current is induced while for larger  $S$  the phase difference between the imposed and induced currents tend to decrease the time-average force.

#### 6-4-7 Superconductors

In the limit of infinite Ohmic conductivity ( $\sigma \rightarrow \infty$ ), the diffusion time constant of (28) becomes infinite while the skin depth of (36) becomes zero. The magnetic field cannot penetrate a perfect conductor and currents are completely confined to the surface.

However, in this limit the Ohmic conduction law is no longer valid and we should use the superconducting constitutive law developed in Section 3-2-2d for a single charge carrier:

$$\frac{\partial \mathbf{J}}{\partial t} = \omega_p^2 \epsilon \mathbf{E} \tag{65}$$

Then for a stationary medium, following the same procedure as in (12) and (13) with the constitutive law of (65), (8)–(11) reduce to

$$\nabla^2 \frac{\partial \mathbf{H}}{\partial t} - \omega_p^2 \epsilon \mu \frac{\partial \mathbf{H}}{\partial t} = 0 \Rightarrow \nabla^2 (\mathbf{H} - \mathbf{H}_0) - \omega_p^2 \epsilon \mu (\mathbf{H} - \mathbf{H}_0) = 0 \tag{66}$$

where  $\mathbf{H}_0$  is the instantaneous magnetic field at  $t = 0$ . If the superconducting material has no initial magnetic field when an excitation is first turned on, then  $\mathbf{H}_0 = 0$ .

If the conducting slab in Figure 6-27a becomes superconducting, (66) becomes

$$\frac{d^2 H_z}{dx^2} - \frac{\omega_p^2}{c^2} H_z = 0, \quad c = \frac{1}{\sqrt{\epsilon \mu}} \quad (67)$$

where  $c$  is the speed of light in the medium.

The solution to (67) is

$$\begin{aligned} H_z &= A_1 e^{\omega_p x/c} + A_2 e^{-\omega_p x/c} \\ &= -K_0 \cos \omega t e^{-\omega_p x/c} \end{aligned} \quad (68)$$

where we use the boundary condition of continuity of tangential  $\mathbf{H}$  at  $x = 0$ .

The current density is then

$$\begin{aligned} J_y &= -\frac{\partial H_z}{\partial x} \\ &= -\frac{K_0 \omega_p}{c} \cos \omega t e^{-\omega_p x/c} \end{aligned} \quad (69)$$

For any frequency  $\omega$ , including dc ( $\omega = 0$ ), the field and current decay with characteristic length:

$$l_c = c/\omega_p \quad (70)$$

Since the plasma frequency  $\omega_p$  is typically on the order of  $10^{15}$  radian/sec, this characteristic length is very small,  $l_c \approx 3 \times 10^8 / 10^{15} \approx 3 \times 10^{-7}$  m. Except for this thin sheath, the magnetic field is excluded from the superconductor while the volume current is confined to this region near the interface.

There is one experimental exception to the governing equation in (66), known as the Meissner effect. If an ordinary conductor is placed within a dc magnetic field  $\mathbf{H}_0$  and then cooled through the transition temperature for superconductivity, the magnetic flux is pushed out except for a thin sheath of width given by (70). This is contrary to (66), which allows the time-independent solution  $\mathbf{H} = \mathbf{H}_0$ , where the magnetic field remains trapped within the superconductor. Although the reason is not well understood, superconductors behave as if  $\mathbf{H}_0 = 0$  no matter what the initial value of magnetic field.

## 6-5 ENERGY STORED IN THE MAGNETIC FIELD

### 6-5-1 A Single Current Loop

The differential amount of work necessary to overcome the electric and magnetic forces on a charge  $q$  moving an

incremental distance  $\mathbf{ds}$  at velocity  $\mathbf{v}$  is

$$dW_q = -q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{ds} \quad (1)$$

### (a) Electrical Work

If the charge moves solely under the action of the electrical and magnetic forces with no other forces of mechanical origin, the incremental displacement in a small time  $dt$  is related to its velocity as

$$\mathbf{ds} = \mathbf{v} dt \quad (2)$$

Then the magnetic field cannot contribute to any work on the charge because the magnetic force is perpendicular to the charge's displacement:

$$dW_q = -q\mathbf{v} \cdot \mathbf{E} dt \quad (3)$$

and the work required is entirely due to the electric field. Within a charge neutral wire, the electric field is not due to Coulombic forces but rather arises from Faraday's law. The moving charge constitutes an incremental current element,

$$q\mathbf{v} = i\mathbf{dl} \Rightarrow dW_q = -i\mathbf{E} \cdot \mathbf{dl} dt \quad (4)$$

so that the total work necessary to move all the charges in the closed wire is just the sum of the work done on each current element,

$$\begin{aligned} dW &= \oint_L dW_q = -i dt \oint_L \mathbf{E} \cdot \mathbf{dl} \\ &= i dt \frac{d}{dt} \int_S \mathbf{B} \cdot \mathbf{dS} \\ &= i dt \frac{d\Phi}{dt} \\ &= i d\Phi \end{aligned} \quad (5)$$

which through Faraday's law is proportional to the change of flux through the current loop. This flux may be due to other currents and magnets (mutual flux) as well as the self-flux due to the current  $i$ . Note that the third relation in (5) is just equivalent to the circuit definition of electrical power delivered to the loop:

$$P = \frac{dW}{dt} = i \frac{d\Phi}{dt} = vi \quad (6)$$

All of this energy supplied to accelerate the charges in the wire is stored as no energy is dissipated in the lossless loop and no mechanical work is performed if the loop is held stationary.

### (b) Mechanical Work

The magnetic field contributed no work in accelerating the charges. This is not true when the current-carrying wire is itself moved a small vector displacement  $\mathbf{ds}$  requiring us to perform mechanical work,

$$\begin{aligned} dW &= -(\mathbf{i} d\mathbf{l} \times \mathbf{B}) \cdot \mathbf{ds} = \mathbf{i} (\mathbf{B} \times d\mathbf{l}) \cdot \mathbf{ds} \\ &= \mathbf{i} \mathbf{B} \cdot (d\mathbf{l} \times \mathbf{ds}) \end{aligned} \quad (7)$$

where we were able to interchange the dot and the cross using the scalar triple product identity proved in Problem 1-10a. We define  $S_1$  as the area originally bounding the loop and  $S_2$  as the bounding area after the loop has moved the distance  $\mathbf{ds}$ , as shown in Figure 6-30. The incremental area  $dS_3$  is then the strip joining the two positions of the loop defined by the bracketed quantity in (7):

$$dS_3 = d\mathbf{l} \times \mathbf{ds} \quad (8)$$

The flux through each of the contours is

$$\Phi_1 = \int_{S_1} \mathbf{B} \cdot d\mathbf{S}, \quad \Phi_2 = \int_{S_2} \mathbf{B} \cdot d\mathbf{S} \quad (9)$$

where their difference is just the flux that passes outward through  $dS_3$ :

$$d\Phi = \Phi_2 - \Phi_1 = \mathbf{B} \cdot dS_3 \quad (10)$$

The incremental mechanical work of (7) necessary to move the loop is then identical to (5):

$$dW = i \mathbf{B} \cdot dS_3 = i d\Phi \quad (11)$$

Here there was no change of electrical energy input, with the increase of stored energy due entirely to mechanical work in moving the current loop.

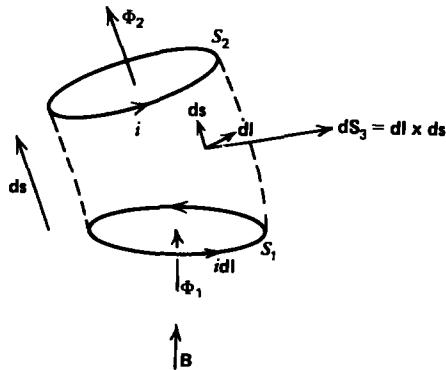


Figure 6-30 The mechanical work necessary to move a current-carrying loop is stored as potential energy in the magnetic field.

### 6-5-2 Energy and Inductance

If the loop is isolated and is within a linear permeable material, the flux is due entirely to the current, related through the self-inductance of the loop as

$$\Phi = Li \quad (12)$$

so that (5) or (11) can be integrated to find the total energy in a loop with final values of current  $I$  and flux  $\Phi$ :

$$\begin{aligned} W &= \int_0^\Phi i d\Phi \\ &= \int_0^\Phi \frac{\Phi}{L} d\Phi \\ &= \frac{1}{2} \frac{\Phi^2}{L} = \frac{1}{2} LI^2 = \frac{1}{2} I\Phi \end{aligned} \quad (13)$$

### 6-5-3 Current Distributions

The results of (13) are only true for a single current loop. For many interacting current loops or for current distributions, it is convenient to write the flux in terms of the vector potential using Stokes' theorem:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l} \quad (14)$$

Then each incremental-sized current element carrying a current  $I$  with flux  $d\Phi$  has stored energy given by (13):

$$dW = \frac{1}{2} I d\Phi = \frac{1}{2} I \cdot \mathbf{A} dl \quad (15)$$

For  $N$  current elements, (15) generalizes to

$$\begin{aligned} W &= \frac{1}{2} (\mathbf{I}_1 \cdot \mathbf{A}_1 dl_1 + \mathbf{I}_2 \cdot \mathbf{A}_2 dl_2 + \dots + \mathbf{I}_N \cdot \mathbf{A}_N dl_N) \\ &= \frac{1}{2} \sum_{n=1}^N \mathbf{I}_n \cdot \mathbf{A}_n dl_n \end{aligned} \quad (16)$$

If the current is distributed over a line, surface, or volume, the summation is replaced by integration:

$$W = \begin{cases} \frac{1}{2} \int_L \mathbf{I}_f \cdot \mathbf{A} dl & \text{(line current)} \\ \frac{1}{2} \int_S \mathbf{K}_f \cdot \mathbf{A} dS & \text{(surface current)} \\ \frac{1}{2} \int_V \mathbf{J}_f \cdot \mathbf{A} dV & \text{(volume current)} \end{cases} \quad (17)$$

Remember that in (16) and (17) the currents and vector potentials are all evaluated at their final values as opposed to (11), where the current must be expressed as a function of flux.

#### 6-5-4 Magnetic Energy Density

This stored energy can be thought of as being stored in the magnetic field. Assuming that we have a free volume distribution of current  $\mathbf{J}_f$ , we use (17) with Ampere's law to express  $\mathbf{J}_f$  in terms of  $\mathbf{H}$ ,

$$W = \frac{1}{2} \int_V \mathbf{J}_f \cdot \mathbf{A} dV = \frac{1}{2} \int_V (\nabla \times \mathbf{H}) \cdot \mathbf{A} dV \quad (18)$$

where the volume  $V$  is just the volume occupied by the current. Larger volumes (including all space) can be used in (18), for the region outside the current has  $\mathbf{J}_f = 0$  so that no additional contributions arise.

Using the vector identity

$$\begin{aligned} \nabla \cdot (\mathbf{A} \times \mathbf{H}) &= \mathbf{H} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{H}) \\ &= \mathbf{H} \cdot \mathbf{B} - \mathbf{A} \cdot (\nabla \times \mathbf{H}) \end{aligned} \quad (19)$$

we rewrite (18) as

$$W = \frac{1}{2} \int_V [\mathbf{H} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{H})] dV \quad (20)$$

The second term on the right-hand side can be converted to a surface integral using the divergence theorem:

$$\int_V \nabla \cdot (\mathbf{A} \times \mathbf{H}) dV = \oint_S (\mathbf{A} \times \mathbf{H}) \cdot d\mathbf{S} \quad (21)$$

It now becomes convenient to let the volume extend over all space so that the surface is at infinity. If the current distribution does not extend to infinity the vector potential dies off at least as  $1/r$  and the magnetic field as  $1/r^2$ . Then, even though the area increases as  $r^2$ , the surface integral in (21) decreases at least as  $1/r$  and thus is zero when  $S$  is at infinity. Then (20) becomes simply

$$W = \frac{1}{2} \int_V \mathbf{H} \cdot \mathbf{B} dV = \frac{1}{2} \int_V \mu H^2 dV = \frac{1}{2} \int_V \frac{B^2}{\mu} dV \quad (22)$$

where the volume  $V$  now extends over all space. The magnetic energy density is thus

$$w = \frac{1}{2} \mathbf{H} \cdot \mathbf{B} = \frac{1}{2} \mu H^2 = \frac{1}{2} \frac{B^2}{\mu} \quad (23)$$

These results are only true for linear materials where  $\mu$  does not depend on the magnetic field, although it can depend on position.

For a single coil, the total energy in (22) must be identical to (13), which gives us an alternate method to calculating the self-inductance from the magnetic field.

### 6-5-5 The Coaxial Cable

#### (a) External Inductance

A typical cable geometry consists of two perfectly conducting cylindrical shells of radii  $a$  and  $b$  and length  $l$ , as shown in Figure 6-31. An imposed current  $I$  flows axially as a surface current in opposite directions on each cylinder. We neglect fringing field effects near the ends so that the magnetic field is the same as if the cylinder were infinitely long. Using Ampere's law we find that

$$H_\phi = \frac{I}{2\pi r}, \quad a < r < b \quad (24)$$

The total magnetic flux between the two conductors is

$$\begin{aligned} \Phi &= \int_a^b \mu_0 H_\phi l \, dr \\ &= \frac{\mu_0 I l}{2\pi} \ln \frac{b}{a} \end{aligned} \quad (25)$$

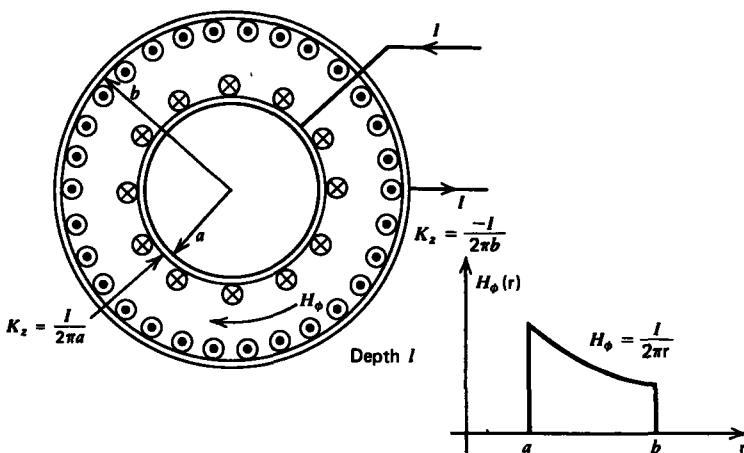


Figure 6-31 The magnetic field between two current-carrying cylindrical shells forming a coaxial cable is confined to the region between cylinders.

giving the self-inductance as

$$L = \frac{\Phi}{I} = \frac{\mu_0 l}{2\pi} \ln \frac{b}{a} \quad (26)$$

The same result can just as easily be found by computing the energy stored in the magnetic field

$$\begin{aligned} W &= \frac{1}{2} L I^2 = \frac{1}{2} \mu_0 \int_a^b H_\phi^2 2\pi r l dr \\ &= \frac{\mu_0 l I^2}{4\pi} \ln \frac{b}{a} \Rightarrow L = \frac{2W}{I^2} = \frac{\mu_0 l \ln(b/a)}{2\pi} \end{aligned} \quad (27)$$

### (b) Internal Inductance

If the inner cylinder is now solid, as in Figure 6-32, the current at low enough frequencies where the skin depth is much larger than the radius, is uniformly distributed with density

$$J_z = \frac{I}{\pi a^2} \quad (28)$$

so that a linearly increasing magnetic field is present within the inner cylinder while the outside magnetic field is

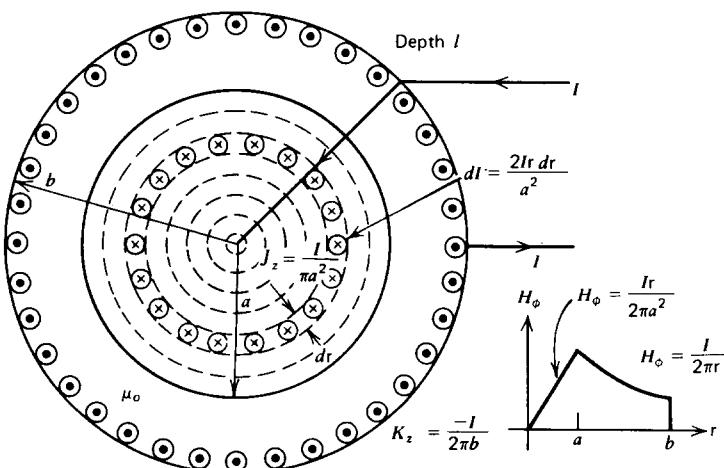


Figure 6-32 At low frequencies the current in a coaxial cable is uniformly distributed over the solid center conductor so that the internal magnetic field increases linearly with radius. The external magnetic field remains unchanged. The inner cylinder can be thought of as many incremental cylindrical shells of thickness  $dr$  carrying a fraction of the total current. Each shell links its own self-flux as well as the mutual flux of the other shells of smaller radius. The additional flux within the current-carrying conductor results in the internal inductance of the cable.

unchanged from (24):

$$H_\phi = \begin{cases} \frac{Ir}{2\pi a^2}, & 0 < r < a \\ \frac{I}{2\pi r}, & a < r < b \end{cases} \quad (29)$$

The self-inductance cannot be found using the flux per unit current definition for a current loop since the current is not restricted to a thin filament. The inner cylinder can be thought of as many incremental cylindrical shells, as in Figure 6-32, each linking its own self-flux as well as the mutual flux of the other shells of smaller radius. Note that each shell is at a different voltage due to the differences in enclosed flux, although the terminal wires that are in a region where the magnetic field is negligible have a well-defined unique voltage difference.

The easiest way to compute the self-inductance as seen by the terminal wires is to use the energy definition of (22):

$$\begin{aligned} W &= \frac{1}{2}\mu_0 \int_0^b H_\phi^2 2\pi lr dr \\ &= \pi l \mu_0 \left[ \int_0^a \left( \frac{Ir}{2\pi a^2} \right)^2 r dr + \int_a^b \left( \frac{I}{2\pi r} \right)^2 r dr \right] \\ &= \frac{\mu_0 l I^2}{4\pi} \left( \frac{1}{4} + \ln \frac{b}{a} \right) \end{aligned} \quad (30)$$

which gives the self-inductance as

$$L = \frac{2W}{I^2} = \frac{\mu_0 l}{2\pi} \left( \frac{1}{4} + \ln \frac{b}{a} \right) \quad (31)$$

The additional contribution of  $\mu_0 l / 8\pi$  is called the internal inductance and is due to the flux within the current-carrying conductor.

### 6-5-6 Self-Inductance, Capacitance, and Resistance

We can often save ourselves further calculations for the external self-inductance if we already know the capacitance or resistance for the same two-dimensional geometry composed of highly conducting electrodes with no internal inductance contribution. For the arbitrary geometry shown in Figure 6-33 of depth  $d$ , the capacitance, resistance, and inductance

are defined as the ratios of line and surface integrals:

$$\begin{aligned} C &= \frac{\epsilon d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds}{\int_L \mathbf{E} \cdot d\mathbf{l}} \\ R &= \frac{\int_L \mathbf{E} \cdot d\mathbf{l}}{\sigma d \oint_S \mathbf{E} \cdot \mathbf{n}_s \, ds} \\ L &= \frac{\mu d \int_L \mathbf{H} \cdot \mathbf{n}_l \, dl}{\oint_S \mathbf{H} \cdot d\mathbf{s}} \end{aligned} \quad (32)$$

Because the homogeneous region between electrodes is charge and current free, both the electric and magnetic fields can be derived from a scalar potential that satisfies Laplace's equation. However, the electric field must be incident normally onto the electrodes while the magnetic field is incident tangentially so that  $\mathbf{E}$  and  $\mathbf{H}$  are perpendicular everywhere, each being along the potential lines of the other. This is accounted for in (32) and Figure 6-33 by having  $\mathbf{n}_s \, ds$  perpendicular to  $d\mathbf{s}$  and  $\mathbf{n}_l \, dl$  perpendicular to  $d\mathbf{l}$ . Then since  $C$ ,  $R$ , and  $L$  are independent of the field strengths, we can take  $\mathbf{E}$  and  $\mathbf{H}$  to both have unit magnitude so that in the products of  $LC$  and  $L/R$  the line and surface integrals cancel:

$$\begin{aligned} LC &= \epsilon \mu d^2 = d^2/c^2, \quad c = 1/\sqrt{\epsilon \mu} \\ L/R &= \mu \sigma d^2, \quad RC = \epsilon / \sigma \end{aligned} \quad (33)$$

These products are then independent of the electrode geometry and depend only on the material parameters and the depth of the electrodes.

We recognize the  $L/R$  ratio to be proportional to the magnetic diffusion time of Section 6-4-3 while  $RC$  is just the charge relaxation time of Section 3-6-1. In Chapter 8 we see that the  $\sqrt{LC}$  product is just equal to the time it takes an electromagnetic wave to propagate a distance  $d$  at the speed of light  $c$  in the medium.

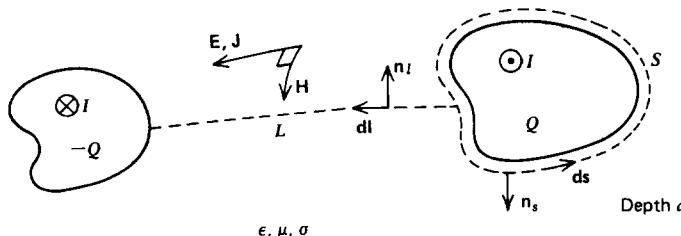


Figure 6-33 The electric and magnetic fields in the two-dimensional homogeneous charge and current-free region between hollow electrodes can be derived from a scalar potential that obeys Laplace's equation. The electric field lines are along the magnetic potential lines and vice versa so  $\mathbf{E}$  and  $\mathbf{H}$  are perpendicular. The inductance-capacitance product is then a constant.

## 6-6 THE ENERGY METHOD FOR FORCES

### 6-6-1 The Principle of Virtual Work

In Section 6-5-1 we calculated the energy stored in a current-carrying loop by two methods. First we calculated the electric energy input to a loop with no mechanical work done. We then obtained the same answer by computing the mechanical work necessary to move a current-carrying loop in an external field with no further electrical inputs. In the most general case, an input of electrical energy can result in stored energy  $dW$  and mechanical work by the action of a force  $f_x$  causing a small displacement  $dx$ :

$$i d\Phi = dW + f_x dx \quad (1)$$

If we knew the total energy stored in the magnetic field as a function of flux and position, the force is simply found as

$$f_x = -\frac{\partial W}{\partial x} \Big|_{\Phi} \quad (2)$$

We can easily compute the stored energy by realizing that no matter by what process or order the system is assembled, if the final position  $x$  and flux  $\Phi$  are the same, the energy is the same. Since the energy stored is independent of the order that we apply mechanical and electrical inputs, we choose to mechanically assemble a system first to its final position  $x$  with no electrical excitations so that  $\Phi = 0$ . This takes no work as with zero flux there is no force of electrical origin. Once the system is mechanically assembled so that its position remains constant, we apply the electrical excitation to bring the system to its final flux value. The electrical energy required is

$$W = \int_{x \text{ const}} i d\Phi \quad (3)$$

For linear materials, the flux and current are linearly related through the inductance that can now be a function of  $x$  because the inductance depends on the geometry:

$$i = \Phi/L(x) \quad (4)$$

Using (4) in (3) allows us to take the inductance outside the integral because  $x$  is held constant so that the inductance is also constant:

$$\begin{aligned} W &= \frac{1}{L(x)} \int_0^\Phi \Phi d\Phi \\ &= \frac{\Phi^2}{2L(x)} = \frac{1}{2} L(x) i^2 \end{aligned} \quad (5)$$

The stored energy is the same as found in Section 6-5-2 even when mechanical work is included and the inductance varies with position:

To find the force on the moveable member, we use (2) with the energy expression in (5), which depends only on flux and position:

$$\begin{aligned}
 f_x &= -\frac{\partial W}{\partial x} \Big|_{\Phi} \\
 &= -\frac{\Phi^2}{2} \frac{d[1/L(x)]}{dx} \\
 &= \frac{1}{2} \frac{\Phi^2}{L^2(x)} \frac{dL(x)}{dx} \\
 &= \frac{1}{2} i^2 \frac{dL(x)}{dx}
 \end{aligned} \tag{6}$$

### 6-6-2 Circuit Viewpoint

This result can also be obtained using a circuit description with the linear flux-current relation of (4):

$$\begin{aligned}
 v &= \frac{d\Phi}{dt} \\
 &= L(x) \frac{di}{dt} + i \frac{dL(x)}{dt} \\
 &= L(x) \frac{di}{dt} + i \frac{dL(x)}{dx} \frac{dx}{dt}
 \end{aligned} \tag{7}$$

The last term, proportional to the speed of the moveable member, just adds to the usual inductive voltage term. If the geometry is fixed and does not change with time, there is no electromechanical coupling term.

The power delivered to the system is

$$p = vi = i \frac{d}{dt} [L(x)i] \tag{8}$$

which can be expanded as

$$p = \frac{d}{dt} \left( \frac{1}{2} L(x) i^2 \right) + \frac{1}{2} i^2 \frac{dL(x)}{dx} \frac{dx}{dt} \tag{9}$$

This is in the form

$$p = \frac{dW}{dt} + f_x \frac{dx}{dt}, \quad \begin{cases} W = \frac{1}{2} L(x) i^2 \\ f_x = \frac{1}{2} i^2 \frac{dL(x)}{dx} \end{cases} \tag{10}$$

which states that the power delivered to the inductor is equal to the sum of the time rate of energy stored and mechanical power performed on the inductor. This agrees with the energy method approach. If the inductance does not change with time because the geometry is fixed, all the input power is stored as potential energy  $W$ .

### Example 6-2 MAGNETIC FIELDS AND FORCES

#### (a) Relay

Find the force on the moveable slug in the magnetic circuit shown in Figure 6-34.

#### SOLUTION

---

It is necessary to find the inductance of the system as a function of the slug's position so that we can use (6). Because of the infinitely permeable core and slug, the  $H$  field is non-zero only in the air gap of length  $x$ . We use Ampere's law to obtain

$$H = NI/x$$

The flux through the gap

$$\Phi = \mu_0 N I A / x$$

is equal to the flux through each turn of the coil yielding the inductance as

$$L(x) = \frac{N\Phi}{I} = \frac{\mu_0 N^2 A}{x}$$

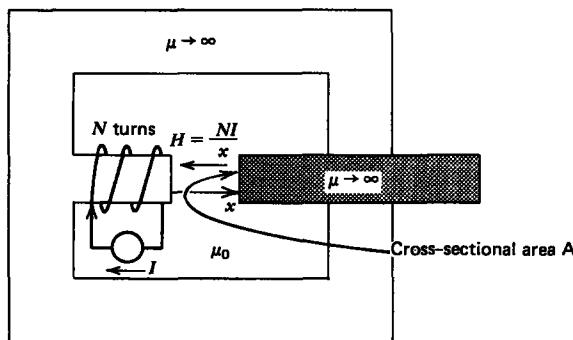


Figure 6-34 The magnetic field exerts a force on the moveable member in the relay pulling it into the magnetic circuit.

The force is then

$$\begin{aligned} f_x &= \frac{1}{2} I^2 \frac{dL(x)}{dx} \\ &= -\frac{\mu_0 N^2 A I^2}{2x^2} \end{aligned}$$

The minus sign means that the force is opposite to the direction of increasing  $x$ , so that the moveable piece is attracted to the coil.

### (b) One Turn Loop

Find the force on the moveable upper plate in the one turn loop shown in Figure 6-35.

### SOLUTION

---

The current distributes itself uniformly as a surface current  $K = I/D$  on the moveable plate. If we neglect nonuniform field effects near the corners, the  $\mathbf{H}$  field being tangent to the conductors just equals  $K$ :

$$H_z = I/D$$

The total flux linked by the current source is then

$$\begin{aligned} \Phi &= \mu_0 H_z xl \\ &= \frac{\mu_0 xl}{D} I \end{aligned}$$

which gives the inductance as

$$L(x) = \frac{\Phi}{I} = \frac{\mu_0 xl}{D}$$

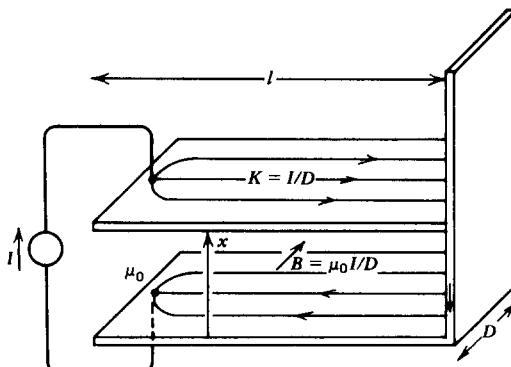


Figure 6-35 The magnetic force on a current-carrying loop tends to expand the loop.

The force is then constant

$$f_x = \frac{1}{2} I^2 \frac{dL(x)}{dx}$$

$$= \frac{1}{2} \frac{\mu_0 l I^2}{D}$$

### 6-6-3 Magnetization Force

A material with permeability  $\mu$  is partially inserted into the magnetic circuit shown in Figure 6-36. With no free current in the moveable material, the  $x$ -directed force density from Section 5-8-1 is

$$F_x = \mu_0 (\mathbf{M} \cdot \nabla) H_x$$

$$= (\mu - \mu_0) (\mathbf{H} \cdot \nabla) H_x$$

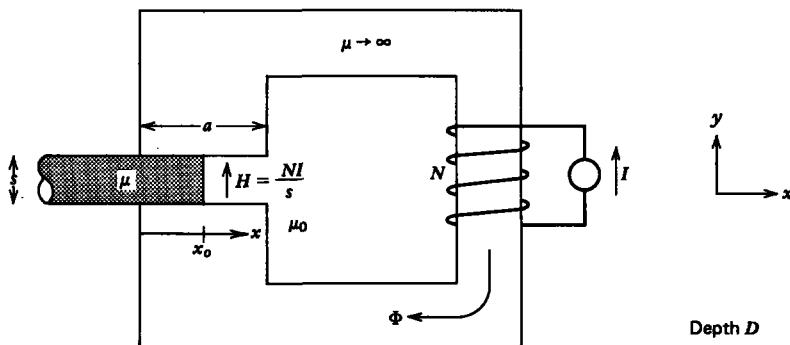
$$= (\mu - \mu_0) \left( H_x \frac{\partial H_x}{\partial x} + H_y \frac{\partial H_x}{\partial y} \right) \quad (11)$$

where we neglect variations with  $z$ . This force arises in the fringing field because within the gap the magnetic field is essentially uniform:

$$H_y = NI/s \quad (12)$$

Because the magnetic field in the permeable block is curl free,

$$\nabla \times \mathbf{H} = 0 \Rightarrow \frac{\partial H_x}{\partial y} = \frac{\partial H_y}{\partial x} \quad (13)$$



(a)

Figure 6-36 · A permeable material tends to be pulled into regions of higher magnetic field.

(11) can be rewritten as

$$F_x = \frac{(\mu - \mu_0)}{2} \frac{\partial}{\partial x} (H_x^2 + H_y^2) \quad (14)$$

The total force is then

$$\begin{aligned} f_x &= sD \int_{-\infty}^{x_0} F_x dx \\ &= \frac{(\mu - \mu_0)}{2} sD (H_x^2 + H_y^2) \Big|_{-\infty}^{x_0} \\ &= \frac{(\mu - \mu_0)}{2} \frac{N^2 I^2 D}{s} \end{aligned} \quad (15)$$

where the fields at  $x = -\infty$  are zero and the field at  $x = x_0$  is given by (12). High permeability material is attracted to regions of stronger magnetic field. It is this force that causes iron materials to be attracted towards a magnet. Diamagnetic materials ( $\mu < \mu_0$ ) will be repelled.

This same result can more easily be obtained using (6) where the flux through the gap is

$$\Phi = HD[\mu x + \mu_0(a - x)] = \frac{NID}{s} [(\mu - \mu_0)x + a\mu_0] \quad (16)$$

so that the inductance is

$$L = \frac{N\Phi}{I} = \frac{N^2 D}{s} [(\mu - \mu_0)x + a\mu_0] \quad (17)$$

Then the force obtained using (6) agrees with (15)

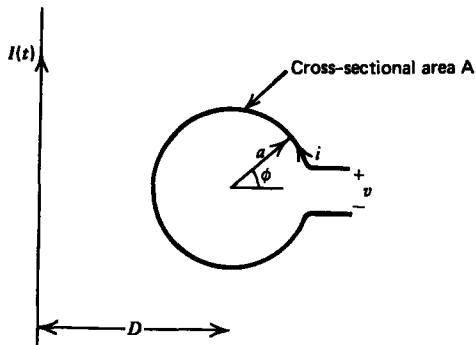
$$\begin{aligned} f_x &= \frac{1}{2} I^2 \frac{dL(x)}{dx} \\ &= \frac{(\mu - \mu_0)}{2s} N^2 I^2 D \end{aligned} \quad (18)$$

## PROBLEMS

---

### Section 6-1

1. A circular loop of radius  $a$  with Ohmic conductivity  $\sigma$  and cross-sectional area  $A$  has its center a small distance  $D$  away from an infinitely long time varying current.



- (a) Find the mutual inductance  $M$  and resistance  $R$  of the loop. Hint:

$$\int \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \left[ \frac{\sqrt{a^2-b^2} \tan(x/2)}{a+b} \right],$$

$$\int \frac{r dr}{\sqrt{D^2-r^2}} = -\sqrt{D^2-r^2}$$

- (b) This loop is stationary and has a self-inductance  $L$ . What is the time dependence of the induced short circuit current when the line current is instantaneously stepped on to a dc level  $I$  at  $t=0$ ?

- (c) Repeat (b) when the line current has been on a long time and is suddenly turned off at  $t=T$ .

- (d) If the loop has no resistance and is moving with radial velocity  $v_r = dr/dt$ , what is the short circuit current and open circuit voltage for a dc line current?

- (e) What is the force on the loop when it carries a current  $i$ ? Hint:

$$\int \frac{\cos \phi d\phi}{D+a \cos \phi} = -\frac{1}{a} \sin^{-1} [\cos \phi]$$

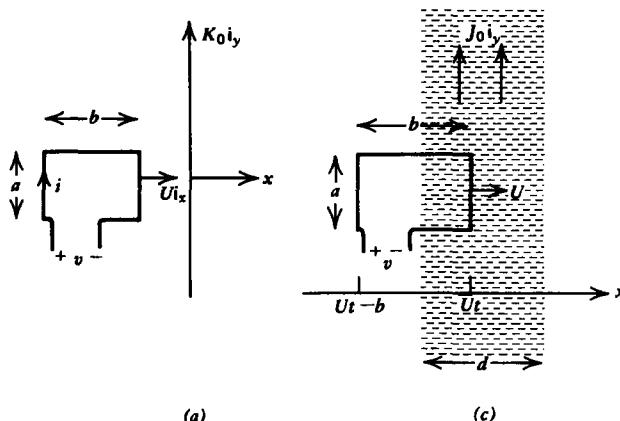
$$+ \frac{D}{a\sqrt{D^2-a^2}} \sin^{-1} \left( \frac{a+D \cos \phi}{D+a \cos \phi} \right)$$

2. A rectangular loop at the far left travels with constant velocity  $Ui_x$  towards and through a dc surface current sheet  $K_0 i_y$  at  $x=0$ . The right-hand edge of the loop first reaches the current sheet at  $t=0$ .

- (a) What is the loop's open circuit voltage as a function of time?

- (b) What is the short circuit current if the loop has self-inductance  $L$  and resistance  $R$ ?

- (c) Find the open circuit voltage if the surface current is replaced by a fluid with uniformly distributed volume current. The current is undisturbed as the loop passes through.

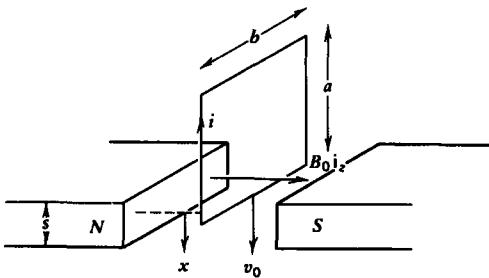


(a)

(c)

Specifically consider the case when  $d > b$  and then sketch the results when  $d = b$  and  $d < b$ . The right edge of the current loop reaches the volume current at  $t = 0$ .

3. A short circuited rectangular loop of mass  $m$  and self-inductance  $L$  is dropped with initial velocity  $v_{0i}$  between the pole faces of a magnet that has a concentrated uniform magnetic field  $B_{0i}$ . Neglect gravity.



- (a) What is the imposed flux through the loop as a function of the loop's position  $x$  ( $0 < x < s$ ) within the magnet?

(b) If the wire has conductivity  $\sigma$  and cross-sectional area  $A$ , what equation relates the induced current  $i$  in the loop and the loop's velocity?

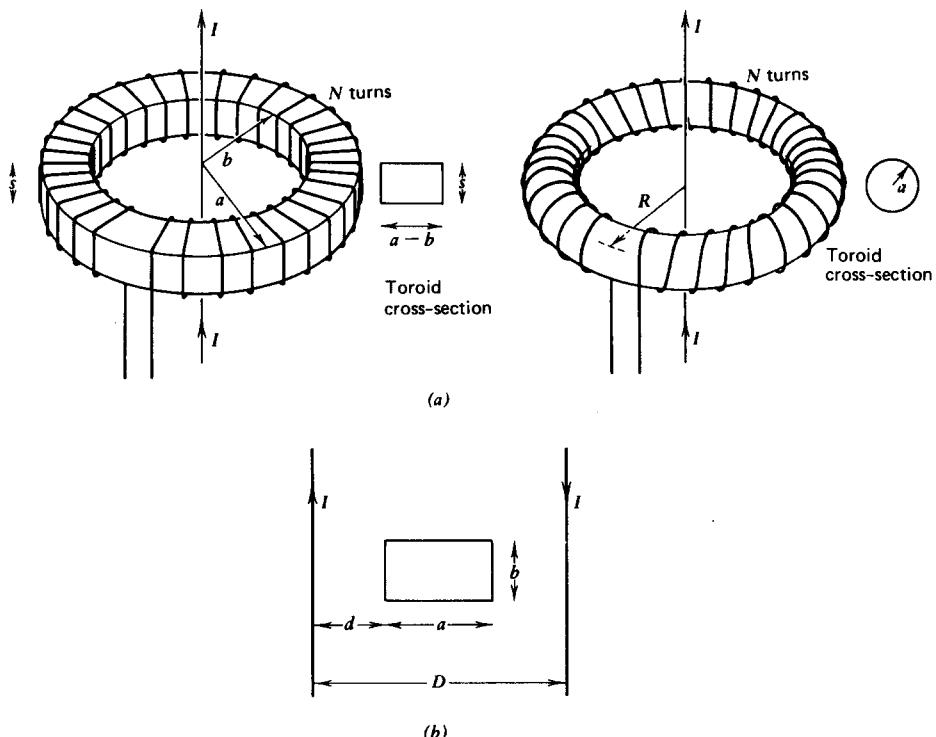
(c) What is the force on the loop in terms of  $i$ ? Obtain a single equation for the loop's velocity. (Hint: Let  $\omega_0^2 = B_0^2 b^2 / m L$ ,  $\alpha = R/L$ .)

(d) How does the loop's velocity and induced current vary with time?

(e) If  $\sigma \rightarrow \infty$ , what minimum initial velocity is necessary for the loop to pass through the magnetic field?

4. Find the mutual inductance between the following currents:

(a) Toroidal coil of rectangular or circular cross section



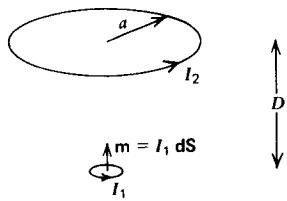
coaxially centered about an infinitely long line current. Hint:

$$\int \frac{dx}{a+b \cos x} = \frac{2}{\sqrt{a^2-b^2}} \tan^{-1} \left\{ \frac{\sqrt{a^2-b^2} \tan(x/2)}{a+b} \right\},$$

$$\int \frac{r dr}{\sqrt{R^2-r^2}} = -\sqrt{R^2-r^2}$$

- (b) A very long rectangular current loop, considered as two infinitely long parallel line currents, a distance  $D$  apart, carrying the same current  $I$  in opposite directions near a small rectangular loop of width  $a$ , which is a distance  $d$  away from the left line current. Consider the cases  $d+a < D$ ,  $d < D < d+a$ , and  $d > D$ .

5. A circular loop of radius  $a$  is a distance  $D$  above a point magnetic dipole of area  $dS$  carrying a current  $I_1$ .



(a) What is the vector potential due to the dipole at all points on the circular loop? (Hint: See Section 5-5-1.)

(b) How much flux of the dipole passes through the circular loop?

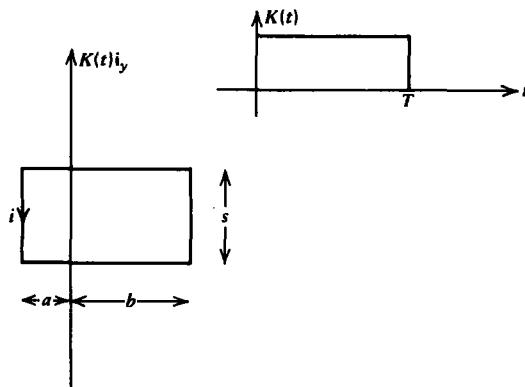
(c) What is the mutual inductance between the dipole and the loop?

(d) If the loop carries a current  $I_2$ , what is the magnetic field due to  $I_2$  at the position of the point dipole? (Hint: See Section 5-2-4a.)

(e) How much flux due to  $I_2$  passes through the magnetic dipole?

(f) What is the mutual inductance? Does your result agree with (c)?

6. A small rectangular loop with self-inductance  $L$ , Ohmic conductivity  $\sigma$ , and cross-sectional area  $A$  straddles a current sheet.



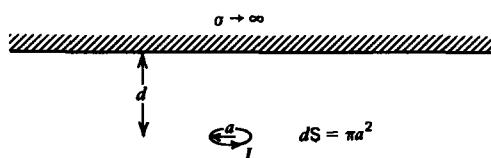
(a) The current sheet is instantaneously turned on to a dc level  $K_0$ , at  $t = 0$ . What is the induced loop current?

(b) After a long time  $T$  the sheet current is instantaneously set to zero. What is the induced loop current?

(c) What is the induced loop current if the current sheet varies sinusoidally with time as  $K_0 \cos \omega t$ .

7. A point magnetic dipole with area  $dS$  lies a distance  $d$  below a perfectly conducting plane of infinite extent. The dipole current  $I$  is instantaneously turned on at  $t = 0$ .

(a) Using the method of images, find the magnetic field everywhere along the conducting plane. (Hint:  $\mathbf{i}_r \cdot \mathbf{i}_r = \sin \theta$ ,



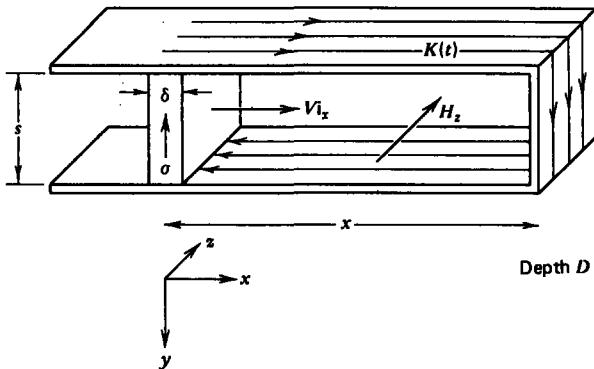
$$\mathbf{i}_\theta \cdot \mathbf{i}_r = \cos \theta.)$$

- (b) What is the surface current distribution?  
(c) What is the force on the plane? Hint:

$$\int \frac{r^3 dr}{(r^2 + d^2)^5} = -\frac{(r^2 + d^2/4)}{6(r^2 + d^2)^4}$$

(d) If the plane has a mass  $M$  in the gravity field  $g$ , what current  $I$  is necessary to just lift the conductor? Evaluate for  $M = 10^{-3}$  kg,  $d = 10^{-2}$  m, and  $a = 10^{-3}$  m.

8. A thin block with Ohmic conductivity  $\sigma$  and thickness  $\delta$  moves with constant velocity  $V_i x$  between short circuited perfectly conducting parallel plates. An initial surface current  $K_0$  is imposed at  $t = 0$  when  $x = x_0$ , but the source is then removed.



(a) The surface current on the plates  $K(t)$  will vary with time. What is the magnetic field in terms of  $K(t)$ ? Neglect fringing effects.

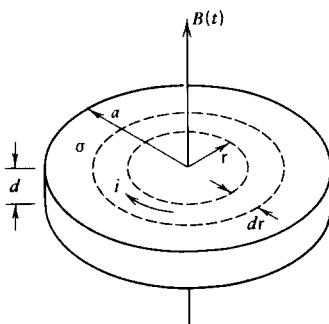
(b) Because the moving block is so thin, the current is uniformly distributed over the thickness  $\delta$ . Using Faraday's law, find  $K(t)$  as a function of time.

(c) What value of velocity will just keep the magnetic field constant with time until the moving block reaches the end?

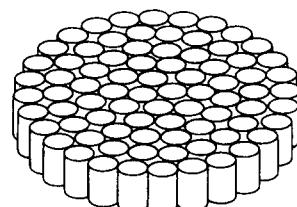
(d) What happens to the magnetic field for larger and smaller velocities?

9. A thin circular disk of radius  $a$ , thickness  $d$ , and conductivity  $\sigma$  is placed in a uniform time varying magnetic field  $B(t)$ .

(a) Neglecting the magnetic field of the eddy currents, what is the current induced in a thin circular filament at radius  $r$  of thickness  $dr$ .



(a)



(d)

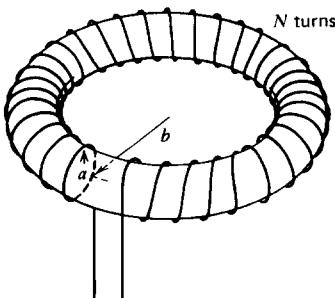
- (b) What power is dissipated in this incremental current loop?
- (c) How much power is dissipated in the whole disk?
- (d) If the disk is instead cut up into  $N$  smaller circular disks with negligible wastage, what is the approximate radius of each smaller disk?
- (e) If these  $N$  smaller disks are laminated together to form a thin disk of closely packed cylindrical wires, what is the power dissipated?

### Section 6-2

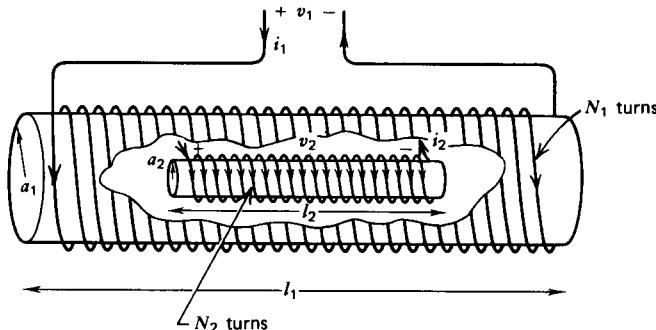
10. Find the self-inductance of an  $N$  turn toroidal coil of circular cross-sectional radius  $a$  and mean radius  $b$ . **Hint:**

$$\int \frac{d\theta}{b+r \cos \theta} = \frac{2}{\sqrt{b^2-r^2}} \tan^{-1} \frac{\sqrt{b^2-r^2} \tan(\theta/2)}{b+r}$$

$$\int \frac{r dr}{\sqrt{b^2-r^2}} = -\sqrt{b^2-r^2}$$



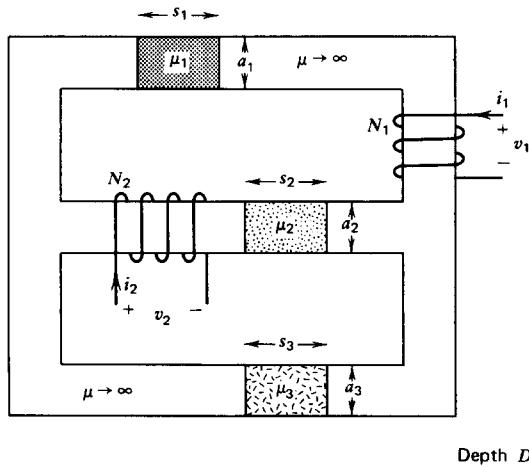
11. A large solenoidal coil of long length  $l_1$ , radius  $a_1$ , and number of turns  $N_1$  coaxially surrounds a smaller coil of long length  $l_2$ , radius  $a_2$ , and turns  $N_2$ .



- (a) Neglecting fringing field effects find the self-inductances and mutual inductances of each coil.  
**(Hint:** Assume the magnetic field is essentially uniform within the cylinders.)
- (b) What is the voltage across each coil in terms of  $i_1$  and  $i_2$ ?
- (c) If the coils are connected in series so that  $i_1 = i_2$  with the fluxes of each coil in the same direction, what is the total self-inductance?
- (d) Repeat (c) if the series connection is reversed so that  $i_1 = -i_2$  and the fluxes due to each coil are in opposite directions.
- (e) What is the total self-inductance if the coils are connected in parallel so that  $v_1 = v_2$  or  $v_1 = -v_2$ ?

12. The iron core shown with infinite permeability has three gaps filled with different permeable materials.

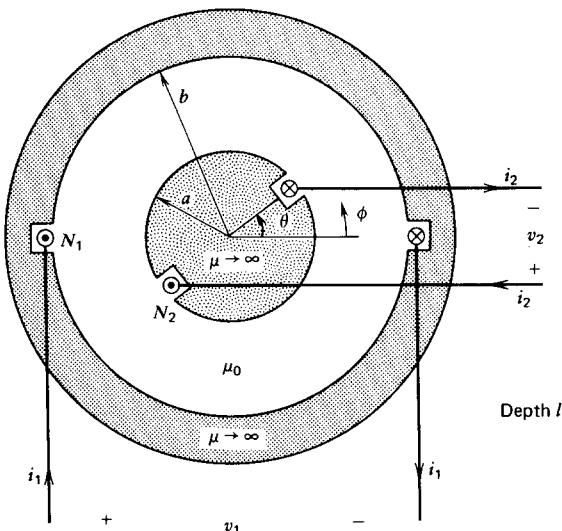
- (a) What is the equivalent magnetic circuit?  
(b) Find the magnetic flux everywhere in terms of the gap reluctances.



Depth  $D$

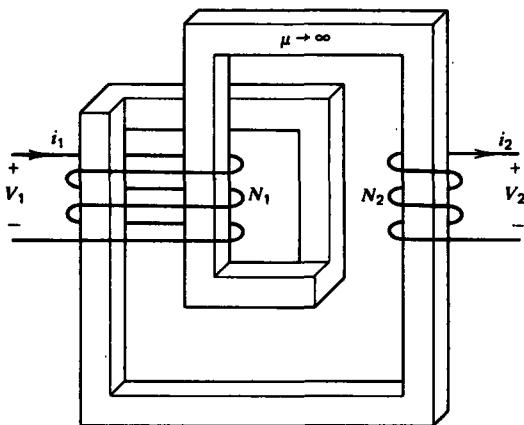
- (c) What is the total magnetic flux through each winding?  
 (d) What is the self-inductance and mutual inductance of each winding?

13. A cylindrical shell of infinite permeability, length  $l$  and inner radius  $b$  coaxially surrounds a solid cylinder also with infinite permeability and length  $l$  but with smaller radius  $a$  so that there is a small gap  $g = b - a$ . An  $N_1$  turn coil carrying a current  $I_1$  is placed within two slots on the inner surface of the outer cylinder.

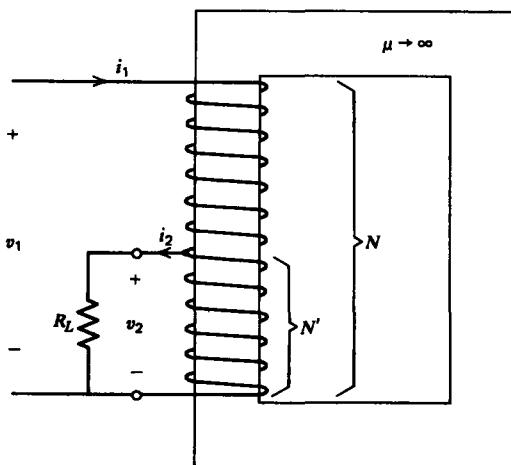


- (a) What is the magnetic field everywhere? Neglect all radial variations in the narrow air gap. (**Hint:** Separately consider  $0 < \phi < \pi$  and  $\pi < \phi < 2\pi$ .)  
 (b) What is the self-inductance of the coil?  
 (c) A second coil with  $N_2$  turns carrying a current  $I_2$  is placed in slots on the inner cylinder that is free to rotate. When the rotor is at angle  $\theta$ , what is the total magnetic field due to currents  $I_1$  and  $I_2$ ? (**Hint:** Separately consider  $0 < \phi < \theta$ ,  $\theta < \phi < \pi$ ,  $\pi < \phi < \pi + \theta$ , and  $\pi + \theta < \phi < 2\pi$ .)  
 (d) What is the self-inductance and mutual inductance of coil 2 as a function of  $\theta$ ?  
 (e) What is the torque on the rotor coil?

14. (a) What is the ratio of terminal voltages and currents for the odd twisted ideal transformer shown?  
 (b) A resistor  $R_L$  is placed across the secondary winding ( $v_2, i_2$ ). What is the impedance as seen by the primary winding?



15. An  $N$  turn coil is wound onto an infinitely permeable magnetic core. An autotransformer is formed by connecting a tap of  $N'$  turns.

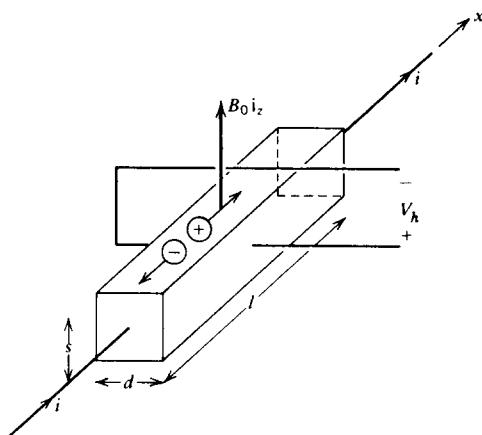


(a) What are the terminal voltage ( $v_2/v_1$ ) and current ( $i_2/i_1$ ) ratios?

(b) A load resistor  $R_L$  is connected across the terminals of the tap. What is the impedance as seen by the input terminals?

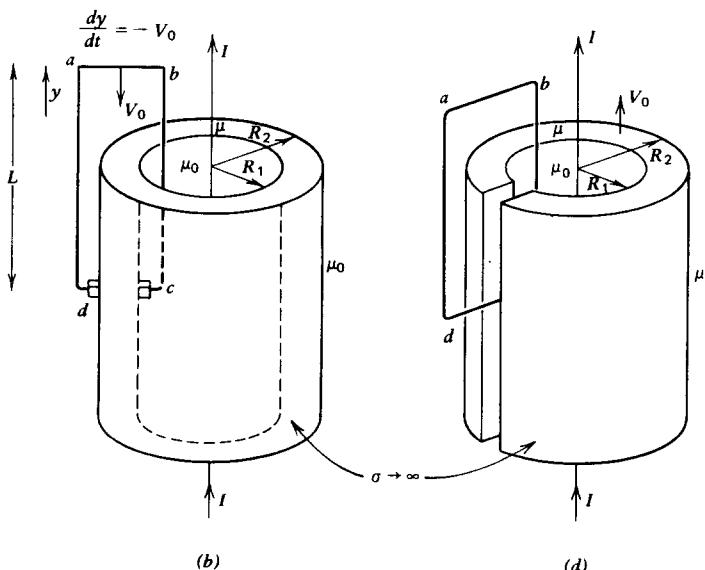
### Section 6-3

16. A conducting material with current density  $J_z \mathbf{i}_z$  has two species of charge carriers with respective mobilities  $\mu_+$  and  $\mu_-$  and number densities  $n_+$  and  $n_-$ . A magnetic field  $B_0 \mathbf{i}_z$  is imposed perpendicular to the current flow.



- (a) What is the open circuit Hall voltage? (**Hint:** The transverse current of each carrier must be zero.)  
 (b) What is the short circuit Hall current?

17. A highly conducting hollow iron cylinder with permeability  $\mu$  and inner and outer radii  $R_1$  and  $R_2$  is concentric to an infinitely long dc line current (adapted from L. V. Bewley, *Flux Linkages and Electromagnetic Induction*. Macmillan, New York, 1952, pp. 71-77).



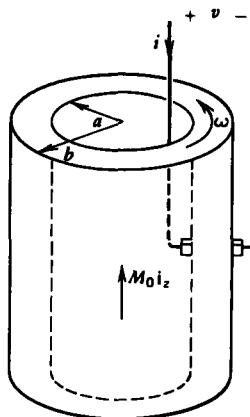
- (a) What is the magnetic flux density everywhere? Find the electromotive force (EMF) of the loop for each of the following cases.

(b) A highly conducting circuit  $abcd$  is moving downward with constant velocity  $V_0$  making contact with the surfaces of the cylinders via sliding brushes. The circuit is completed from  $c$  to  $d$  via the iron cylinder.

(c) Now the circuit remains stationary and the iron cylinder moves upwards at velocity  $V_0$ .

(d) Now a thin axial slot is cut in the cylinder so that it can slip by the complete circuit  $abcd$ , which remains stationary as the cylinder moves upwards at speed  $V_0$ . The brushes are removed and a highly conducting wire completes the  $c-d$  path.

18. A very long permanently magnetized cylinder  $M_0 i_z$  rotates on a shaft at constant angular speed  $\omega$ . The inner and outer surfaces at  $r = a$  and  $r = b$  are perfectly conducting, so that brushes can make electrical contact.



(a) If the cylinder is assumed very long compared to its radius, what are the approximate values of  $\mathbf{B}$  and  $\mathbf{H}$  in the magnet?

(b) What is the open circuit voltage?

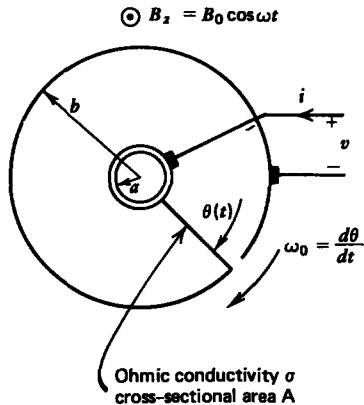
(c) If the magnet has an Ohmic conductivity  $\sigma$ , what is the equivalent circuit of this generator?

(d) What torque is required to turn the magnet if the terminals are short circuited?

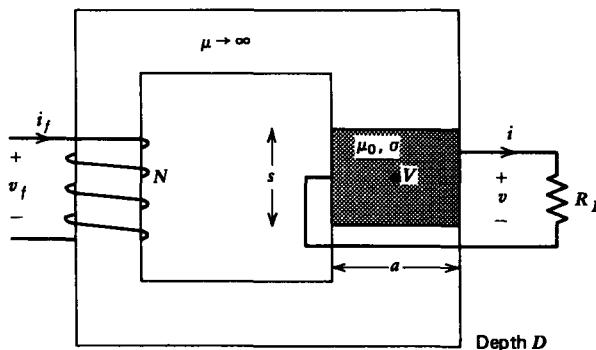
19. A single spoke wheel has a perfectly conducting cut rim. The spoke has Ohmic conductivity  $\sigma$  and cross-sectional area  $A$ . The wheel rotates at constant angular speed  $\omega_0$  in a sinusoidally varying magnetic field  $B_z = B_0 \cos \omega t$ .

(a) What is the open circuit voltage and short circuit current?

(b) What is the equivalent circuit?



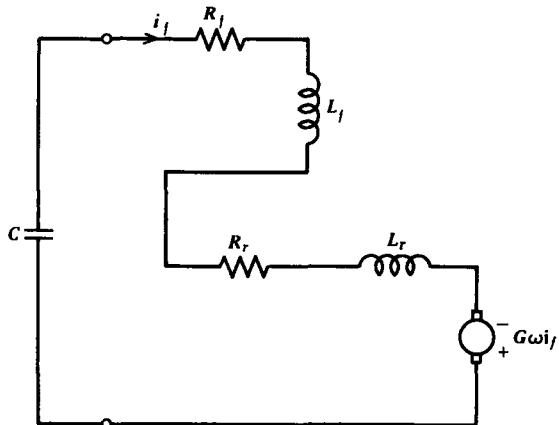
20. An MHD machine is placed within a magnetic circuit.



- (a) A constant dc current  $i_f = I_0$  is applied to the  $N$  turn coil. How much power is delivered to the load resistor  $R_L$ ?
- (b) The MHD machine and load resistor  $R_L$  are now connected in series with the  $N$  turn coil that has a resistance  $R_f$ . No current is applied. For what minimum flow speed can the MHD machine be self-excited?

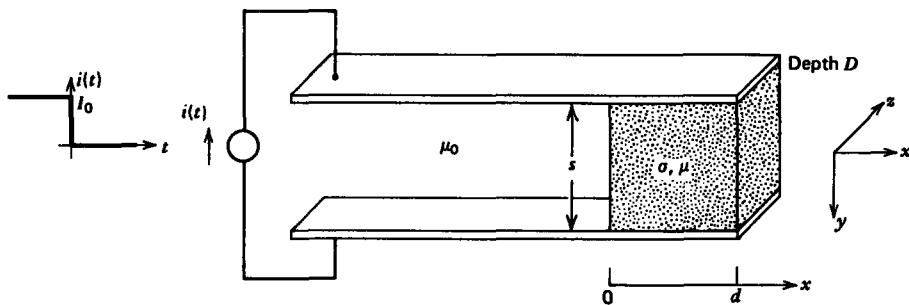
21. The field winding of a homopolar generator is connected in series with the rotor terminals through a capacitor  $C$ . The rotor is turned at constant speed  $\omega$ .

- (a) For what minimum value of rotor speed is the system self-excited?
- (b) For the self-excited condition of (a) what range of values of  $C$  will result in dc self-excitation or in ac self-excitation?
- (c) What is the frequency for ac self-excitation?



## Section 6-4

22. An Ohmic block separates two perfectly conducting parallel plates. A dc current that has been applied for a long time is instantaneously turned off at  $t = 0$ .



(a) What are the initial and final magnetic field distributions? What are the boundary conditions?

(b) What are the transient magnetic field and current distributions?

(c) What is the force on the block as a function of time?

23. A block of Ohmic material is placed within a magnetic circuit. A step current  $I_0$  is applied at  $t = 0$ .

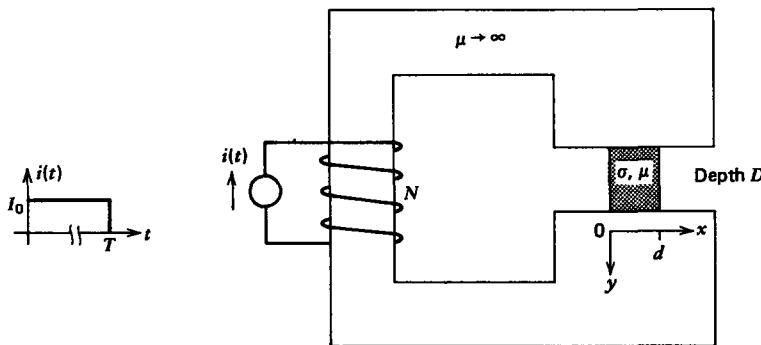
(a) What is the dc steady-state solution for the magnetic field distribution?

(b) What are the boundary and initial conditions for the magnetic field in the conducting block?

(c) What are the transient field and current distributions?

(d) What is the time dependence of the force on the conductor?

(e) The current has been on a long time so that the system is in the dc steady state found in (a) when at  $t = T$  the current

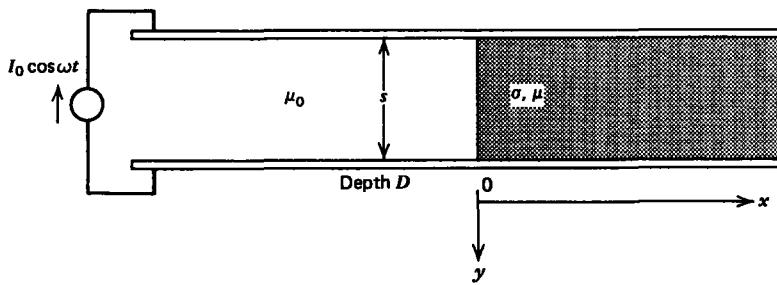


is instantaneously turned off. What are the transient field and current distributions in the conductor?

(f) If the applied coil current varies sinusoidally with time as  $i(t) = I_0 \cos \omega t$ , what are the sinusoidal steady-state field and current distributions? (Hint: Leave your answer in terms of complex amplitudes.)

(g) What is the force on the block?

24. A semi-infinite conducting block is placed between parallel perfect conductors. A sinusoidal current source is applied.



(a) What are the magnetic field and current distributions within the conducting block?

(b) What is the total force on the block?

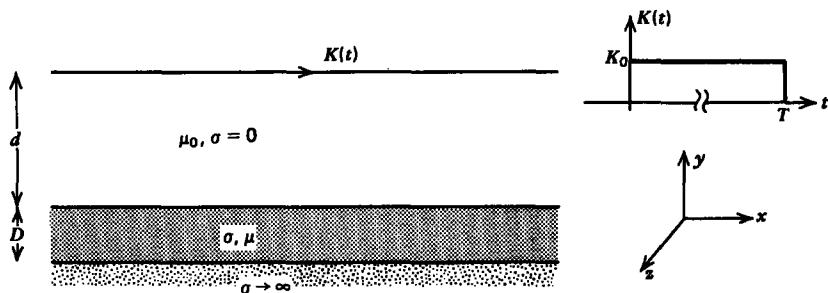
(c) Repeat (a) and (b) if the block has length  $d$ .

25. A current sheet that is turned on at  $t = 0$  lies a distance  $d$  above a conductor of thickness  $D$  and conductivity  $\sigma$ . The conductor lies on top of a perfectly conducting plane.

(a) What are the initial and steady-state solutions? What are the boundary conditions?

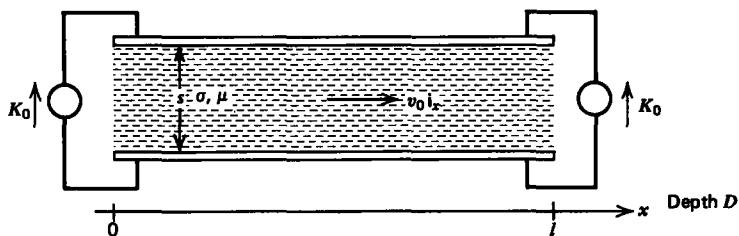
(b) What are the transient magnetic field and current distributions?

(c) After a long time  $T$ , so that the system has reached the dc steady state, the surface current is set to zero. What are the subsequent field and current distributions?



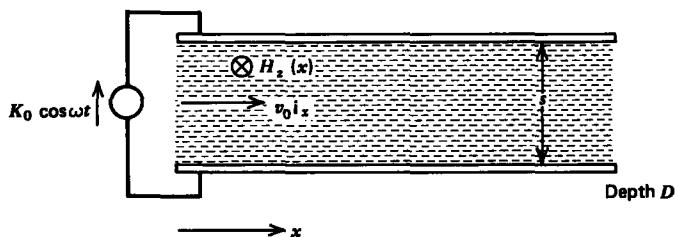
(d) What are the field and current distributions if the current sheet varies as  $K_0 \cos \omega t$ ?

26. Distributed dc currents at  $x = 0$  and  $x = l$  flow through a conducting fluid moving with constant velocity  $v_0 i_x$ .



- (a) What are the magnetic field and current distributions?  
 (b) What is the force on the fluid?

27. A sinusoidal surface current at  $x = 0$  flows along parallel electrodes and returns through a conducting fluid moving to the right with constant velocity  $v_0 i_x$ . The flow is not impeded by the current source. The system extends to  $x = \infty$ .



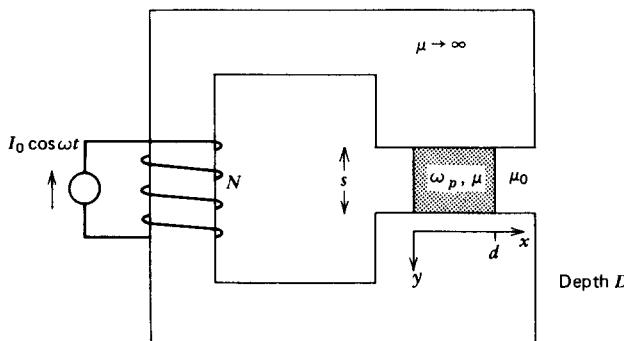
- (a) What are the magnetic field and current density distributions?  
 (b) What is the time-average force on the fluid?

28. The surface current for the linear induction machine treated in Section 6-4-6 is now put a distance  $s$  below the conductor.

(a) What are the magnetic field and current distributions in each region of space? (**Hint:** Check your answer with Section 6-4-6 when  $s = 0$ .)

(b) Repeat (a) if  $s$  is set to zero but the conductor has a finite thickness  $d$ .

29. A superconducting block with plasma frequency  $\omega_p$  is placed within a magnetic circuit with exciting current  $I_0 \cos \omega t$ .

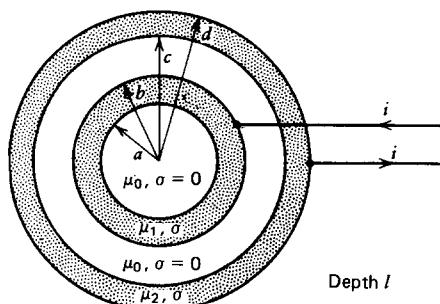


(a) What are the magnetic field and current distributions in the superconductor?

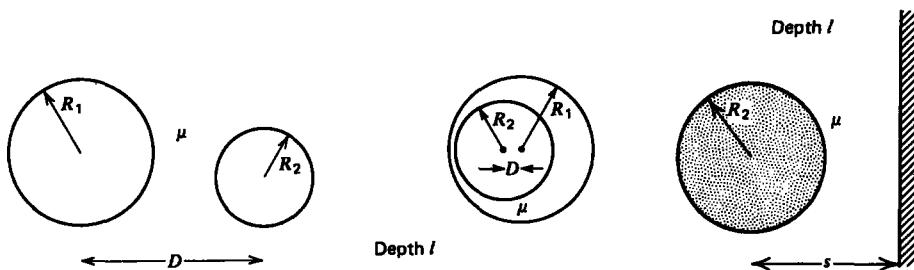
(b) What is the force on the block?

### Section 6.5

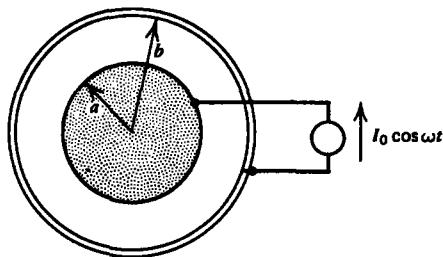
30. Find the magnetic energy stored and the self-inductance for the geometry below where the current in each shell is uniformly distributed.



31. Find the external self-inductance of the two wire lines shown. (**Hint:** See Section 2-6-4c.)



32. A coaxial cable with solid inner conductor is excited by a sinusoidally varying current  $I_0 \cos \omega t$  at high enough frequency so that the skin depth is small compared to the radius  $a$ . The current is now nonuniformly distributed over the inner conductor.



(a) Assuming that  $\mathbf{H} = H_\phi(r)\mathbf{i}_\phi$ , what is the governing equation for  $H_\phi(r)$  within the inner cylinder. (Hint:  $\nabla^2 \mathbf{H} = \nabla(\nabla \cdot \mathbf{H}) - \nabla \times (\nabla \times \mathbf{H})$ .)

(b) Solve (a) for solutions of the form

$$H_\phi(r) = \operatorname{Re} [\hat{H}_\phi(r) e^{j\omega t}]$$

**Hint:** Bessel's equation is

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - p^2)y = 0$$

with solutions

$$y = A_1 J_p(x) + A_2 Y_p(x)$$

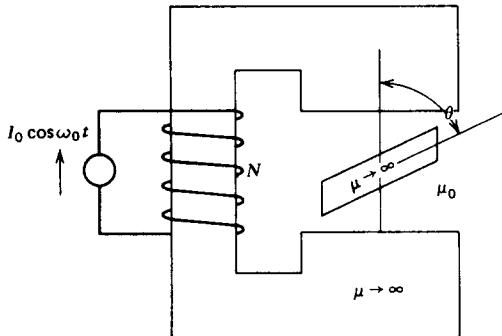
where  $Y_p$  is singular at  $x = 0$ .

(c) What is the current distribution? Hint:

$$\frac{d}{dx} [J_1(x)] + \frac{1}{x} J_1(x) = J_0(x)$$

### Section 6-6

33. A reluctance motor is made by placing a high permeability material, which is free to rotate, in the air gap of a magnetic circuit excited by a sinusoidal current  $I_0 \cos \omega_0 t$ .



The inductance of the circuit varies as

$$L(\theta) = L_0 + L_1 \cos 2\theta$$

where the maximum inductance  $L_0 + L_1$  occurs when  $\theta = 0$  or  $\theta = \pi$  and the minimum inductance  $L_0 - L_1$  occurs when  $\theta = \pm\pi/2$ .

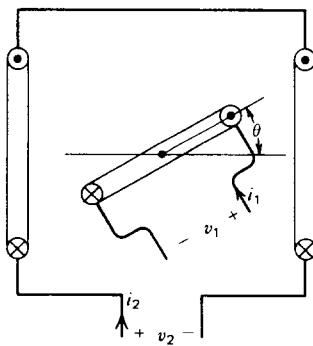
(a) What is the torque on the slab as a function of the angle  $\theta$ ?

(b) The rotor is rotating at constant speed  $\omega$ , where  $\theta = \omega t + \delta$  and  $\delta$  is the angle of the rotor at  $t = 0$ . At what value of  $\omega$  does the torque have a nonzero time average. The reluctance motor is then a synchronous machine. **Hint:**

$$\begin{aligned} \cos^2 \omega_0 t \sin 2\theta &= \frac{1}{2}[\sin 2\theta + \cos 2\omega_0 t \sin 2\theta] \\ &= \frac{1}{2}\{\sin 2\theta + \frac{1}{2}[\sin 2(\omega_0 t + \theta) + \sin 2(\theta - \omega_0 t)]\} \end{aligned}$$

(c) What is the maximum torque that can be delivered by the shaft and at what angle  $\delta$  does it occur?

34. A system of two coupled coils have the following flux-current relations:



(c)

$$\Phi_1 = L_1(\theta)i_1 + M(\theta)i_2$$

$$\Phi_2 = M(\theta)i_1 + L_2(\theta)i_2$$

- (a) What is the power  $p$  delivered to the coils?  
 (b) Write this power in the form

$$p = \frac{dW}{dt} + T \frac{d\theta}{dt}$$

What are  $W$  and  $T$ ?

- (c) A small coil is free to rotate in the uniform magnetic field produced by another coil. The flux-current relation is

$$\Phi_1 = L_1 i_1 + M_0 i_2 \sin \theta$$

$$\Phi_2 = M_0 i_1 \sin \theta + L_2 i_2$$

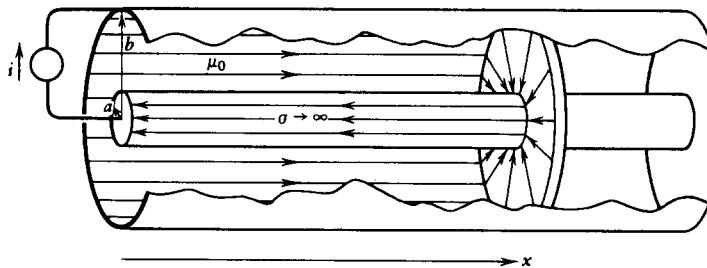
The coils are excited by dc currents  $I_1$  and  $I_2$ . What is the torque on the small coil?

- (d) If the small coil has conductivity  $\sigma$ , cross-sectional area  $A$ , total length  $l$ , and is short circuited, what differential equation must the current  $i_1$  obey if  $\theta$  is a function of time? A dc current  $I_2$  is imposed in coil 2.

- (e) The small coil has moment of inertia  $J$ . Consider only small motions around  $\theta = 0$  so that  $\cos \theta \approx 1$ . With the torque and current equations linearized, try exponential solutions of the form  $e^{st}$  and solve for the natural frequencies.

- (f) The coil is released from rest at  $\theta = \theta_0$ . What is  $\theta(t)$  and  $i_1(t)$ ? Under what conditions are the solutions oscillatory? Damped?

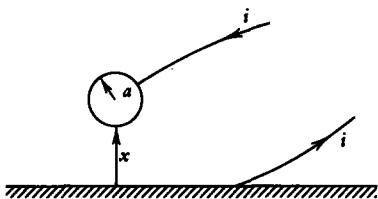
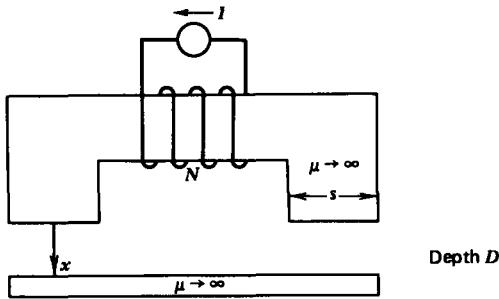
35. A coaxial cable has its short circuited end free to move.



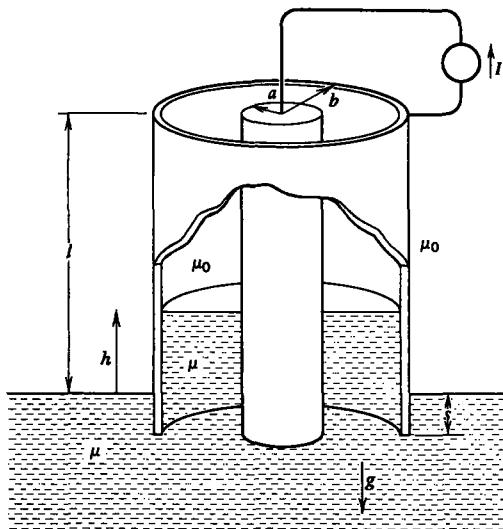
- (a) What is the inductance of the cable as a function of  $x$ ?  
 (b) What is the force on the end?

36. For the following geometries, find:

- (a) The inductance;  
 (b) The force on the moveable member.



37. A coaxial cylinder is dipped into a magnetizable fluid with permeability  $\mu$  and mass density  $\rho_m$ . How high  $h$  does the fluid rise within the cylinder?



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