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SOLUTIONS TO CHAPTER 6

6.1 POLARIZATION DENSITY

6.1.1 (a) From (6.1.6), the polarization charge density is

$$\rho_p = -\nabla \cdot \mathbf{P} = P_o \beta \sin \beta x \quad (1)$$

(b) The polarization surface charge density at the respective surfaces follows from (6.1.7) evaluated at the respective interfaces.

$$\begin{aligned} \sigma_{sp} &= -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) \\ &= \begin{cases} -(0 - P_o \cos \beta x) = P_o \cos \beta x; & y = d \\ -(P_o \cos \beta x - 0) = -P_o \cos \beta x; & y = 0 \end{cases} \end{aligned} \quad (2)$$

6.2 LAWS AND CONTINUITY CONDITIONS WITH POLARIZATION

6.2.1 (a) Given the polarization density, the polarization current density follows from (6.2.9).

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t} = \frac{dP_o}{dt} \cos \beta x (\mathbf{i}_x + \mathbf{i}_y) \quad (1)$$

The polarization charge density is as found in Prob. 6.1.1.

(b) Substitution of these quantities into (6.2.10) gives

$$\frac{\partial \rho_p}{\partial t} + \nabla \cdot \mathbf{J}_p = \frac{\partial P_o}{\partial t} \beta \sin \beta x - \frac{dP_o}{dt} \beta \sin \beta x = 0 \quad (2)$$

6.3 PERMANENT POLARIZATION

6.3.1 (a) The polarization charge density between the electrodes is

$$\rho_p = -\nabla \cdot \mathbf{P} = P_o \beta \sin \beta x \quad (1)$$

Thus, at each point between the electrodes,

$$\nabla^2 \Phi = -\frac{\rho_p}{\epsilon_o} = -\frac{P_o \beta}{\epsilon_o} \sin \beta x \quad (2)$$

and a particular solution is gotten from

$$\frac{\partial^2 \Phi}{\partial x^2} = -\frac{P_o \beta}{\epsilon_o} \sin \beta x \Rightarrow \Phi_p = \frac{P_o}{\beta \epsilon_o} \sin \beta x \quad (3)$$

To satisfy the boundary conditions, use the homogeneous solutions Ax

$$\Phi = Ax + \frac{P_o}{\beta \epsilon_o} \sin \beta x \quad (4)$$

which satisfies $\Phi(x=0) = 0$. To make $\Phi(x=a) = -V$,

$$A = -\frac{V}{a} - \frac{P_o}{\beta \epsilon_o a} \sin \beta a \quad (5)$$

so that (6.3.4) becomes

$$\Phi = \frac{P_o}{\beta \epsilon_o} (\sin \beta x - \frac{x}{a} \sin \beta a) - V \frac{x}{a} \quad (6)$$

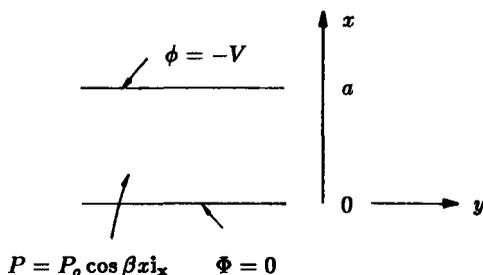


Figure S6.3.1

(b) In this case,

$$\rho_p = -\nabla \cdot \mathbf{P} = \beta P_o \sin \beta y \quad (7)$$

and

$$\frac{\partial^2 \Phi_p}{\partial y^2} = -\frac{\beta P_o}{\epsilon_o} \sin \beta y \Rightarrow \Phi_p = \frac{P_o}{\epsilon_o \beta} \sin \beta y \quad (8)$$

Boundary conditions at $x=0$ and $x=a$, are satisfied by $\Phi = -Vx/a$. Thus, we let

$$\Phi = \frac{P_o}{\epsilon_o \beta} \sin \beta y - V \frac{x}{a} + \Phi_1 \quad (9)$$

Then Φ_1 must be $-(P_o/\epsilon_o \beta) \sin \beta y$ at $x=0$ and $x=a$. Such a solution to Laplace's equation is symmetric about $x=a/2$;

$$\Phi = \frac{P_o}{\epsilon_o \beta} \sin \beta y - \frac{Vx}{a} + A \cosh \beta \left(x - \frac{a}{2} \right) \sin \beta y \quad (10)$$

To satisfy boundary conditions

$$A = -\frac{P_o}{\epsilon_o \beta} \frac{1}{\cosh \left(\frac{\beta a}{2} \right)} \quad (11)$$

Thus, from (6.3.4) and (6.3.5),

$$\Phi = \frac{P_o}{\epsilon_o \beta} \sin \beta y \left[1 - \frac{\cosh \beta \left(x - \frac{a}{2} \right)}{\cosh \left(\frac{\beta a}{2} \right)} \right] - \frac{Vx}{a}$$

6.3.2 The polarization charge density inside the rectangular region is

$$\rho_p = -\nabla \cdot \mathbf{P} = P_o \frac{\pi}{a} \sin \frac{\pi}{a} x \quad (1)$$

The potential is therefore a solution to

$$\nabla^2 \Phi = -P_o \frac{\pi}{a \epsilon_o} \sin \frac{\pi}{a} x \quad (2)$$

that satisfies the zero potential boundary conditions. Two of these conditions are satisfied by the particular solution to (6.3.2) that follows from assuming that it, like the charge distribution, only depends on x .

$$\frac{d^2 \Phi_p}{dx^2} = -P_o \frac{\pi}{a \epsilon_o} \sin \frac{\pi}{a} x \quad (3)$$

Two integrations, with the integration constants adjusted to make the potential at $x = a$ and $x = 0$ zero then give the particular solution

$$\Phi_p = P_o \frac{a}{\pi \epsilon_o} \sin \frac{\pi}{a} x \quad (4)$$

The homogeneous solution must also be zero on these boundaries and cancel this particular solution when evaluated at $y = \pm b$.

$$\Phi_h(x, \pm b) = -P_o \frac{a}{\pi \epsilon_o} \sin \frac{\pi}{a} x \quad (5)$$

Because these conditions are even in y , and because of the former boundary conditions, the potential is therefore taken as having a cosh dependence on x and the potential distribution suggested by the conditions of (6.3.5) at $y = \pm b$.

$$\Phi_h = -P_o \frac{a}{\pi \epsilon_o} \sin \frac{\pi}{a} x \frac{\cosh \frac{\pi}{a} y}{\cosh \frac{\pi}{a} b} \quad (6)$$

The required potential is then the sum of the particular and homogeneous solutions, (6.3.4) and (6.3.6).

$$\Phi = P_o \frac{a}{\pi \epsilon_o} \sin \frac{\pi}{a} x \left[1 - \frac{\cosh \frac{\pi}{a} y}{\cosh \frac{\pi}{a} b} \right] \quad (7)$$

6.3.3 (a) First,

$$\rho_p = -\nabla \cdot \mathbf{P} = -\frac{\partial P_y}{\partial y} = 0 \quad (1)$$

and

$$\sigma_{sp} = -\mathbf{n} \cdot [\mathbf{P}^a - \mathbf{P}^b] = P_o \cos \left[\left(\frac{2\pi}{\Lambda} \right) x \right] \quad (2)$$

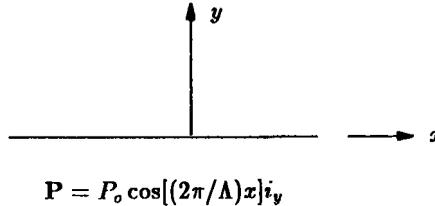


Figure S6.3.3

(b) The Φ above and below must satisfy Laplace's equation and the boundary conditions that at $y = 0$

$$\Phi^a = \Phi^b \quad (3)$$

$$\epsilon_o E_x^a - \epsilon_o E_x^b = -\epsilon_o \left. \frac{\partial \Phi^a}{\partial y} \right|_{x=0} + \epsilon_o \left. \frac{\partial \Phi^b}{\partial y} \right|_{x=0} = P_o \cos \left[\left(\frac{2\pi}{\Lambda} \right) x \right] \quad (4)$$

To these ends, and to make $\Phi \rightarrow 0$ at $y \rightarrow \pm\infty$, make

$$\Phi = \frac{P_o \Lambda}{4\epsilon_o \pi} \cos \left(\frac{2\pi}{\Lambda} x \right) e^{\mp(2\pi y/\Lambda)} \quad (5)$$

6.3.4 In the region $-a < y < 0$, the divergence of the polarization density is zero and so the polarization charge density is zero as well. Thus, in both regions (a) and (b), the potential must satisfy Laplace's equation. Boundary conditions on the potential are that it be the given values at $y = \pm a$, that it be continuous at $y = 0$ and that it satisfy Gauss' continuity condition at $y = 0$. This condition requires that

$$-\mathbf{n} \cdot \left[\frac{\partial \Phi^a}{\partial y} - \frac{\partial \Phi^b}{\partial y} \right]_{y=0} = \sigma_{sp} \quad (1)$$

where the polarization surface charge density follows from

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) = -[0 - P_o \sin \beta(x - x_o)] = P_o \sin \beta(x - x_o) \quad (2)$$

Thus, the fields are the same as if there were an unpaired surface charge density $\sigma_{su} = P_o \sin \beta(x - x_o)$ in the plane $y = 0$. With the identification of $\sigma_o \rightarrow P_o$, the physical situation is the same as considered in Prob. 5.6.12 and the solution as outlined there.

6.3.5 The given polarization density is uniform, so there is no volume polarization charge density. The polarization surface charge density at the cavity interface is

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) = -\mathbf{i}_r \cdot \mathbf{i}_z P_o = -P_o \cos \theta \quad (1)$$

Thus, the boundary conditions at $r = R$ are

$$\Phi^a = \Phi^b \quad (2)$$

$$\mathbf{n} \cdot (\epsilon_o \mathbf{E}^a - \epsilon_o \mathbf{E}^b) = \epsilon_o E_r^a - \epsilon_o E_r^b = \sigma_o - P_o \cos \theta \quad (3)$$

On the right in this last expression is the sum of the polarization and unpaired surface charge densities. Superposition can be used to find the potentials due to the respective terms on the right and then their sum can be taken. Symmetry and Gauss' integral law give the electric field due to the uniform unpaired surface charge density.

$$4\pi\epsilon_o r^2 E_r^a = 4\pi R^2 \sigma_o \Rightarrow E_r^a = \frac{1}{\epsilon_o} (R/r)^2 \sigma_o \quad (4)$$

There is no electric field intensity inside, so the potential there is what it is on the surface. Thus,

$$\Phi = \begin{cases} \frac{R\sigma_o}{\epsilon_o}; & r \leq R \\ \frac{R^2\sigma_o}{r}; & r \geq R \end{cases} \quad (5)$$

To find the potential from the second term in (6.3.3), assume that

$$\Phi = \begin{cases} A \frac{r}{R} \cos \theta \\ A \frac{R^2}{r^2} \cos \theta \end{cases} \quad (6)$$

where the coefficient has been adjusted to satisfy (6.3.2). Substitution of these expressions into (6.3.3) then gives

$$A = -\frac{P_o R}{3\epsilon_o} \quad (7)$$

The sum of (6.3.5) and (6.3.6) with the latter evaluated using (6.3.7) is the given potential.

6.3.6 In polar coordinates, the uniform y directed polarization density is

$$\mathbf{P} = P_o \mathbf{i}_y = P_o (\cos \phi \mathbf{i}_r - \sin \phi \mathbf{i}_\phi) \quad (1)$$

Because the divergence of \mathbf{P} is zero in the volume, the only polarization charge is a surface charge density at $r = R$. This is

$$\sigma_{sp} = -\mathbf{n} \cdot \cos(\mathbf{P}^a - \mathbf{P}^b) = -P_o \cos \phi \quad (2)$$

The equipotential boundary condition in the plane $y = 0$ is met by assuming solutions

$$\Phi^a = \frac{A}{r} \cos \phi; \quad \Phi^b = Br \cos \phi \quad (3)$$

Boundary conditions at $r = R$ are

$$-\left[\frac{\partial \Phi^a}{\partial r} - \frac{\partial \Phi^b}{\partial r} \right]_{r=R} = \frac{\sigma_{sp}}{\epsilon_o} \Rightarrow \left(\frac{A}{R^2} + B \right) \cos \phi = -P_o \cos \phi \quad (4)$$

$$\frac{A}{R} = BR \quad (5)$$

Simultaneous solution for the coefficients gives

$$A = -\frac{P_o R^2}{2}; \quad B = -\frac{P_o}{2} \quad (6)$$

and hence,

$$\Phi^a = -\frac{P_o R}{2} \frac{R}{r} \cos \phi; \quad \Phi^b = -\frac{P_o R}{2} \frac{r}{R} \cos \phi \quad (7)$$

6.3.7 The fields in regions (a) and (b), respectively above and below the interface, are taken as uniform. Because the line integral of \mathbf{E} between the electrodes is zero,

$$aE_x^a + bE_x^b = 0 \quad (1)$$

At the interface, there is a polarization surface charge density

$$\sigma_{sp} = -(P_x^a - P_x^b) = P_o \quad (2)$$

Thus, Gauss' continuity condition requires that

$$\epsilon_o(E_x^a - E_x^b) = P_o \quad (3)$$

Solution of (6.3.1) and (6.3.3) then gives

$$E_x^a = \frac{P_o}{\epsilon_o} \frac{1}{\left(1 + \frac{a}{b}\right)}; \quad E_x^b = -\frac{P_o}{\epsilon_o} \frac{1}{\left(\frac{b}{a} + 1\right)} \quad (4)$$

6.3.8 (a) The polarization charge density is

$$\rho_p = -\nabla \cdot \mathbf{P} = -\nabla \cdot \nabla \psi = -\nabla^2 \psi \quad (1)$$

where $\psi = P_o r \cos(\phi - \alpha)$ is a solution to Laplace's equation. Thus, $\rho_p = 0$.

(b) The surface polarization charge density at $r = b$ is

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^I - \mathbf{P}^{II}) = \mathbf{i}_r \cdot \nabla P_o r \cos(\phi - \alpha) = P_o \cos(\phi - \alpha) \quad (2)$$

It is assumed that there is no unpaired surface charge density on this interface, so the boundary conditions are

$$\Phi^I(r = a) = 0 \quad (3)$$

$$\Phi^I(r = b) = \Phi^{II}(r = b) \quad (4)$$

$$\epsilon_o E_r^I - \epsilon_o E_r^{II} = P_o \cos(\phi - \alpha) \quad (5)$$

Solutions to Laplace's equation that have the same dependence as the right hand side of (6.3.5) take the form

$$\Phi = \begin{cases} A[(r/a) - (a/r)] \cos(\phi - \alpha) \\ B(r/b) \cos(\phi - \alpha) \end{cases} \quad (6)$$

Here, the solution that is infinite at the origin has been omitted and the two contributions to the outer potential adjusted to satisfy (6.3.3). Substitution of (6.3.6) into (6.3.4) and (6.3.5) then gives

$$B = A \left(\frac{b}{a} - \frac{a}{b} \right) = -\frac{P_o b^2}{2\epsilon_o a} \left(\frac{b}{a} - \frac{a}{b} \right) \quad (7)$$

Thus, (6.3.6) and (6.3.7) are the given potentials.

6.3.9 (a) Note that the scalar function inside the gradient operator is a solution to Laplace's equation. Thus,

$$\rho_p = -\nabla \cdot \mathbf{P} = -\nabla^2 [P_o (r^m / r^{m-1}) \cos m\phi] = 0 \quad (1)$$

and there is no polarization charge density in the volume of the rotor. However, at the interface there is a surface polarization charge density given by

$$\sigma_{sp} = -\mathbf{n} \cdot (\mathbf{P}^a - \mathbf{P}^b) = P_r^b(r = b) = P_o m \cos m\phi \quad (2)$$

(b) Boundary and continuity conditions are

$$\Phi^a(r = a) = 0; \quad \Phi^a(r = b) = \Phi^b(r = b) \quad (2)$$

$$-\epsilon_o \left[\frac{\partial \Phi^a}{\partial r}(r=b) - \frac{\partial \Phi^b}{\partial r}(r=b) \right] = P_o m \cos m\phi \quad (3)$$

Given the ϕ dependence of σ_{sp} , solutions to Laplace's equation are assumed to take the form

$$\Phi^a = A[(r/a)^m - (a/r)^m] \cos m\phi \quad (4a)$$

$$\Phi^b = A(r/b)^m [(b/a)^m - (a/b)^m] \cos m\phi \quad (4b)$$

where a linear combination of solutions has been selected in the annular region, (a), that satisfies the zero potential condition at $r = a$ and the coefficients have been arranged so that the potential is continuous at $r = b$. The last of the boundary conditions then determines A .

$$P_o m \cos m\phi = -\epsilon_o A \frac{2m}{b} (a/b)^m \cos m\phi \Rightarrow A = -\frac{bP_o}{2\epsilon_o} (b/a)^m \quad (5)$$

Thus, the potential is

$$\Phi = -\frac{bP_o}{2\epsilon_o} (b/a)^m \cos m\phi \begin{cases} [(r/a)^m - (a/r)^m]; & b < r < a \\ (r/b)^m [(b/a)^m - (a/b)^m]; & r < b \end{cases} \quad (6)$$

- (c) With the substitution $\phi \rightarrow \phi - \Omega t$, at a given instant in time there is only a shift in the origin of ϕ . Because the field laws do not involve a time rate of change, they are satisfied by the new solution. To stay at a point of constant $\phi - \Omega t$ and hence constant \mathbf{P} requires being at the angular position $\phi = \Omega t + \text{constant}$. Thus, the new solution is one that represents the fields associated with a rotor having the angular velocity Ω .

- (d) From (6.3.6), the surface charge density on the wall at $r = a$ is

$$\sigma_{su} = \epsilon_o \frac{\partial \Phi^a}{\partial r}(r=a) = -\frac{bP_o}{2} (b/a)^m \frac{2m}{a} \cos m(\phi - \Omega t) \quad (7)$$

The net charge on the segment is then

$$\begin{aligned} q &= l \int_{-\pi/m}^0 \sigma_{su} a d\phi = -lbP_o (b/a)^m [\sin(-m\Omega t) - \sin m(-\frac{\pi}{m} - \Omega t)] \\ &= lbP_o (b/a)^m [\sin(m\Omega t) - \sin(\pi + m\Omega t)] \\ &= 2lbP_o (b/a)^m \sin(m\Omega t) \end{aligned} \quad (8)$$

and hence the output voltage is

$$v_o = -R \frac{dq}{dt} = -2lbP_o R (b/a)^m m\Omega \cos(m\Omega t) \quad (9)$$

- 6.3.10 (a) The potential in regions (a) and (b), respectively $0 < x$ and $x < 0$, take the form of (6.6.26) and (6.6.27) with $V = 0$ in the latter because both of the electrodes are grounded over their full length in the x direction and $a \rightarrow d$.

$$\Phi^a = \sum_{n=1}^{\infty} V_n e^{-\frac{n\pi}{d}x} \sin \frac{n\pi}{d}y \quad (1)$$

$$\Phi^b = \sum_{n=1}^{\infty} V_n e^{\frac{n\pi}{d}x} \sin \frac{n\pi}{d}y \quad (2)$$

The coefficient in these expressions, V_n , has been adjusted so that the potential is continuous at the interface. Given $V(y)$, these coefficients follow by evaluating either of these expressions at $y = 0$

$$\sum_{n=1}^{\infty} V_n \sin \frac{n\pi}{d}y = V(y) \quad (3)$$

multiplying by $\sin(m\pi y/d)$ and integrating from $y = 0$ to $y = d$.

$$V_m = \frac{2}{d} \int_0^d V(y) \sin \frac{m\pi}{d}y dy \quad (4)$$

Given $V(y)$, this integral can be evaluated and the coefficients needed to complete (1) and (2) determined.

- (b) In addition to the continuity of potential which is already satisfied by (1) and (2), the continuity condition at $x = 0$ is

$$-\epsilon_o \frac{\partial \Phi^a}{\partial x} + \epsilon_o \frac{\partial \Phi^b}{\partial x} = \sigma_{sp} = P_o \quad (5)$$

- (c) With P_o now the given quantity, substitution of (1) and (2) into (5) gives

$$\sum_{n=1}^{\infty} V_n (2\epsilon_o \frac{n\pi}{d}) \sin \frac{n\pi}{d}y = P_o \quad (6)$$

The coefficients are evaluated in this case by the same procedure as leading to (4).

$$V_m = \frac{2}{d[2\epsilon_o(n\pi/d)]} \int_0^d P_o \sin \frac{n\pi}{d}y dy = \frac{dP_o}{\epsilon_o} \frac{[1 - (-1)^m]}{(m\pi)^2} \quad (7)$$

Evaluated using this coefficient, (1) and (2) become the given potential.

6.3.11 In region (b), the potential must satisfy Poisson's equation with the charge density found in Prob. 6.1.1.

$$\nabla^2 \Phi^b = -\frac{P_o \beta}{\epsilon_o} \sin \beta x \quad (1)$$

while in region (a) it satisfies Laplace's equation. At the interface, the potential must be continuous and satisfy Poisson's continuity condition for the polarization surface charge density found in Prob. 6.1.1.

$$-\epsilon_o \left[\frac{\partial \Phi^a}{\partial y}(y=d) - \frac{\partial \Phi^b}{\partial y}(y=d) \right] = P_o \cos \beta x \quad (2)$$

Finally, the potential must go to zero as $y \rightarrow \infty$ and be zero in the plane $y = 0$. The particular solution to (1) is taken as depending only on x . Thus, two integrations give

$$\Phi_p^b = \frac{P_o}{\epsilon_o \beta} \sin \beta x \quad (3)$$

The x dependence of the potential due to the surface charge density is $\cos(\beta x)$ while that due to the volume charge density is $\sin(\beta x)$. The potential is taken as the sum of potentials due to these two sources, $\Phi = \Phi_s + \Phi_v$. The potential due to the surface charge satisfies Laplace's equation in each region and takes the form

$$\Phi_s = \begin{cases} A_s e^{-\beta(y-d)} \cos \beta x \\ A_s \frac{\sinh(\beta y)}{\sinh(\beta d)} \cos \beta x \end{cases} \quad (4)$$

Here, the coefficients have been adjusted to make the potential continuous at $y = d$, while the sinh function satisfies the zero potential boundary condition at $y = 0$. The coefficient is determined by requiring that Gauss' continuity condition be satisfied with the surface charge density given by (2).

$$\begin{aligned} -\epsilon_o A_s [-\beta - \beta \coth(\beta d)] \cos(\beta x) &= P_o \cos(\beta x) \Rightarrow A_s \\ &= \frac{P_o}{\epsilon_o \beta [1 + \coth(\beta d)]} \end{aligned} \quad (5)$$

The part of the potential due to the bulk charge takes the form

$$\Phi_v = \begin{cases} A_v e^{-\beta(y-d)} \sin(\beta x) \\ \frac{P_o}{\epsilon_o \beta} [1 + B_v \sinh(\beta y) + C_v \cosh(\beta y)] \sin(\beta x) \end{cases} \quad (6)$$

where the solution in the lower region has been taken as the sum of the particular solution, (3), and two solutions to Laplace's equation. This part of the potential must also be zero at $y = 0$, so $C = -1$. In addition both the potential and its normal derivative must be continuous at $y = d$.

$$A_v = \frac{P_o}{\epsilon_o \beta} [1 + B_v \sinh(\beta d) - \cosh(\beta d)] \quad (7)$$

$$-\beta A_v = \frac{P_o}{\epsilon_o} [B_v \cosh(\beta d) - \sinh(\beta d)] \quad (8)$$

Simultaneous solution of these expressions gives ($\sinh^2 x - \cosh^2 x = -1$).

$$A_v = \frac{P_o}{\epsilon_o \beta} \frac{[\cosh(\beta d) - 1]}{[\cosh(\beta d) + \sinh(\beta d)]} \quad (9)$$

$$B_v = \frac{\cosh(\beta d) + \sinh(\beta d) - 1}{\cosh(\beta d) + \sinh(\beta d)} \quad (10)$$

Finally, the total solution is the sum of (4) and (6) with the coefficients given by (5), (9), and (10).

6.4 POLARIZATION CONSTITUTIVE LAWS

6.4.1 In terms of the number density N , the polarization density is given by

$$\mathbf{P} = Nq\mathbf{d} = (\epsilon - \epsilon_o)\mathbf{E} \quad (1)$$

It follows that the separation d of single electronic charges needed to account for the given polarization is

$$d = \frac{(\epsilon - \epsilon_o)\mathbf{E}}{Ng} = \frac{(1.5)(8.85 \times 10^{-12})(10^7)}{(6 \times 10^{26}/8)10^3(1.6 \times 10^{-19})} = 1.1 \times 10^{-14} \quad (2)$$

This is less than 1/1000 of a dimension typical of an atom.

6.5 FIELDS IN THE PRESENCE OF ELECTRICALLY LINEAR DIELECTRICS

6.5.1 (a) The divergence of $\epsilon\mathbf{E}$ is zero

$$\nabla \cdot \epsilon\mathbf{E} = \frac{\partial}{\partial y} \left[\epsilon(x) \frac{v}{d} \right] = 0 \quad (1)$$

and the curl of \mathbf{E} , a uniform field, is as well. Given that the field is normal to the perfectly conducting boundaries, which extend to infinity, it follows that the solution, which does indeed satisfy the relevant field laws, is uniquely specified.

(b) On the upper surface of the lower electrode in the regions to right and left,

$$\sigma_s = \epsilon_a \frac{v}{d}; \quad \epsilon_b \frac{v}{d} \quad (2)$$

It follows that the net charge on the lower electrode is

$$q = ac\epsilon_a \frac{v}{d} + bc\epsilon_b \frac{v}{d} = \frac{c}{d}(a\epsilon_a + b\epsilon_b)v \quad (3)$$

and hence the capacitance is as given.

(c) In this case, the surface charge density on the lower electrode is

$$\sigma_s = \epsilon(x) \frac{v}{d} \quad (4)$$

and so the net charge on the lower electrode is

$$q = c \int_0^l \sigma_s dx = \frac{cv}{d} \int_0^l \epsilon_0 \left(1 + \frac{x}{l}\right) dx = \left(\frac{\epsilon_0 cl}{d} \frac{3}{2}\right)v \quad (5)$$

so that C is as given.

6.5.2 A uniform electric field, $\mathbf{E} = (v/d)\mathbf{i}_y$ is irrotational and satisfies Gauss' law with the permittivity varying with x , a direction perpendicular to the proposed electric field.

$$\nabla \cdot \epsilon \mathbf{E} = \frac{\partial}{\partial y} [\epsilon_a (1 + \alpha \cos \beta x) \frac{v}{d}] = 0 \quad (1)$$

Thus, \mathbf{E} is indeed uniform and

$$D_y = \epsilon_a (1 + \alpha \cos \beta x) \frac{v}{d} \quad (2)$$

This is also the density of unpaired surface charge on the lower electrode, so the total charge on that electrode is

$$\begin{aligned} q &= c \int_0^l \epsilon_a (1 + \alpha \cos \beta x) \frac{v}{d} dx \\ &= \frac{c\epsilon_a}{d} \left[x + \frac{\alpha}{\beta} \sin \beta x \right]_0^l v = Cv \end{aligned} \quad (3a)$$

$$C \equiv \frac{c\epsilon_a}{d} \left[l + \frac{\alpha}{\beta} \sin \beta l \right] \quad (3b)$$

6.5.3 (a) Because the field is independent of x and z ,

$$\nabla \cdot \mathbf{D} = \frac{\partial D_y}{\partial y} = 0 \quad (1)$$

and from this it follows that $D_y = D_y(t)$.

(b) In terms of the given distribution of permittivity,

$$D_y = \epsilon_o \left[1 + \chi_a \left(1 + \frac{y}{l} \right) \right] E_y \quad (2)$$

This expression can be solved for E_y and hence for the y dependence of E_y . To determine the unknown D_y , that expression is integrated from the lower to the upper electrode and the result equated to the voltage.

$$\int_0^l E_y dy = v = \int_0^l \frac{D_y dy}{\epsilon_o \left[1 + \chi_a \left(1 + \frac{y}{l} \right) \right]} = \frac{D_y l}{\epsilon_o \chi_a} \ln \frac{(1 + 2\chi_a)}{(1 + \chi_a)} \quad (3)$$

The total charge on the lower electrode, and hence C , follows from this result

$$q = AD_y = \left[\frac{\epsilon_o \chi_a A}{l} / \ln \frac{(1 + 2\chi_a)}{(1 + \chi_a)} \right] v \quad (4)$$

6.5.4 Because the field is independent of x and z ,

$$\nabla \cdot \mathbf{D} = \frac{\partial D_y}{\partial y} = 0 \quad (1)$$

and from this it follows that $D_y = D_y(t)$. This means that

$$D_y = \epsilon_p e^{-y/d} E_y(y, t) = D_y(t) \quad (2)$$

is independent of y and can be solved for E_y . The voltage is then

$$v = \int_0^l E_y dy = \frac{D_y}{\epsilon_p} \int_0^l e^{-y/d} dy = \frac{D_y d}{\epsilon_p} (1 - e^{-l/d}) \quad (3)$$

and this expression can be solved for D_y , which is the surface charge density on the lower electrode.

$$q = AD_y = Cv; \quad C \equiv \frac{A\epsilon_p}{d(1 - e^{-l/d})} \quad (4)$$

6.5.5 (a) For each, the electric field intensity in each region takes the form $\mathbf{E} = \mathbf{i}_r A/r$ [the potential takes the form $\Phi = A \ln(r)$]. In the first case, the integral of this field between the electrodes must be the same whether it is taken in the dielectric or in the free space region. Thus, in the first case,

$$v = \int_b^a \frac{A}{r} dr = A \ln(a/b) \Rightarrow \mathbf{E} = \mathbf{i}_r v / r \ln(a/b) \quad (1)$$

Note that this solution satisfies the conditions that the tangential field be continuous at the dielectric-free space interfaces and that the normal \mathbf{D} be

continuous (there is no normal \mathbf{D}). The field is normal to the circular cylinder electrodes and so these are equipotentials, as required. In the second case, the coefficient A has a different value in each of the regions. The two coefficients are found by requiring that

$$v = \int_b^R \frac{A_1}{r} dr + \int_R^a \frac{A_2}{r} dr = A_1 \ln(R/b) + A_2 \ln(a/r) \quad (2)$$

and that at the interface

$$\epsilon \frac{A_1}{R} = \epsilon_o \frac{A_2}{R} \quad (3)$$

Thus, in the second case,

$$\mathbf{E} = \frac{\mathbf{i}_r v}{[\ln(R/b) + \frac{\epsilon}{\epsilon_o} \ln(a/R)]r} \begin{cases} 1; & b < r < R \\ \epsilon/\epsilon_o; & R < r < a \end{cases} \quad (4)$$

(b) The capacitance follows from integrating the surface charge density over the inner electrode. In the first case,

$$q = l[\alpha b \epsilon E_r(r=b) + (2\pi - \alpha)b \epsilon_o E_r(r=b)] = Cv; \quad (5a)$$

$$C \equiv l[\alpha \epsilon + (2\pi - \alpha)\epsilon_o]/\ln(a/b) \quad (5b)$$

while in the second case

$$q = l2\pi b D_r(r=b) \equiv Cv; \quad C \equiv 2\pi l \epsilon / [\ln(R/b) + \frac{\epsilon}{\epsilon_o} \ln(a/R)] \quad (6)$$

6.5.6

Based on experience in the special case where the wedge is of uniform permittivity (so that the spatial variations in permittivity are bumps at the interfaces) postulate that the electric field is no different than if the dielectric wedge were not present.

$$\mathbf{E} = [v/r \ln(a/b)] \mathbf{i}_r \quad (1)$$

Because the electric field is perpendicular to the gradient in permittivity, there is no induced polarization charge, (6.5.9), and hence no distortion of this field by the dielectric. The field of (1) has no divergence (and of course no curl) and hence does satisfy the bulk conditions throughout the volume. It also has no tangential value on the boundaries, as required. The given capacitance follows from integrating the unpaired surface charge over the surface of the inner electrode.

$$\begin{aligned} q &= \int_0^l \int_0^{2\pi} (\epsilon_a + \epsilon_b \cos^2 \phi) \frac{v}{b \ln(a/b)} b d\phi dz \\ &= v \left[\frac{2\pi \epsilon_a + \epsilon_b \pi}{\ln(a/b)} \right] l \end{aligned}$$

6.6 PIECE-WISE UNIFORM ELECTRICALLY LINEAR DIELECTRICS

6.6.1 Given that the imposed potential takes the form

$$\Phi(r \rightarrow \infty) = -E_0 r \cos \theta \quad (1)$$

assume potentials of the form

$$\Phi = -E_0 r \cos \theta + \frac{A}{r^2} \cos \theta; \quad r > R \quad (2)$$

$$\Phi = B \frac{r}{R} \cos \theta = \left(E_0 R + \frac{A}{R^2} \right) \frac{r}{R} \cos \theta; \quad r < R \quad (3)$$

Here, the coefficients have already been adjusted to make the potential continuous at $r = R$. The remaining condition is that

$$-\epsilon_0 \frac{\partial \Phi}{\partial r} + \epsilon_s \frac{\partial \Phi}{\partial r} = 0 \quad (4)$$

from which it follows that

$$A = \frac{E_0 R^3 (\epsilon_s - \epsilon_0)}{(2\epsilon_0 + \epsilon_s)} \quad (5)$$

The given potentials follow from substitution of (5) into (2) and (3).

6.6.2 (a) Assume a potential within the cavity that is consistent with the dipole being at the origin with the addition term satisfying Laplace's equation while having the same θ dependence as the dipole and being finite at the origin. Outside the cavity, the potential again has the θ dependence of the dipole and goes to zero at infinity.

$$\Phi = \begin{cases} \frac{p \cos \theta}{4\pi\epsilon_0 r^2} + Br \cos \theta; & r < a \\ A \frac{\cos \theta}{r^2}; & a < r \end{cases} \quad (1)$$

Potential continuity and continuity of normal \mathbf{D} at $r = a$ requires that the coefficients A and B satisfy

$$\begin{bmatrix} \frac{1}{a^2} & -a \\ \frac{2\epsilon}{a^3} & \epsilon_0 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} \frac{p}{4\pi\epsilon_0 a^2} \\ \frac{2p}{4\pi a^3} \end{bmatrix} \quad (2)$$

Thus,

$$A = \frac{3p}{4\pi(\epsilon_0 + 2\epsilon)}; \quad B = \frac{-2p}{4\pi a^3} \left(\frac{\epsilon}{\epsilon_0} - 1 \right) \quad (3)$$

so that the required potential is

$$\Phi = \frac{p \cos \theta}{4\pi\epsilon_0} \begin{cases} \frac{1}{r^2} - \frac{2}{a^3} \left(\frac{\epsilon}{\epsilon_0} - 1 \right) r; & r < a \\ \frac{3}{1+2\frac{\epsilon}{\epsilon_0}} \frac{1}{r^2}; & a < r \end{cases} \quad (4)$$

The electric field follows as

$$\mathbf{E} = \frac{p}{4\pi\epsilon_o} \begin{cases} \left[\frac{2}{r^3} + \frac{2}{a^3} \frac{(\epsilon_o - 1)}{(1 + 2\epsilon)} \right] \cos\theta \mathbf{i}_r + \left[\frac{1}{r^3} - \frac{2(\epsilon_o - 1)}{a^3(1 + 2\epsilon)} \right] \sin\theta \mathbf{i}_\theta & r < a \\ \frac{6}{1 + 2\epsilon} \frac{\cos\theta}{r^3} \mathbf{i}_r + \frac{3}{1 + 2\epsilon} \frac{\sin\theta}{r^3} \mathbf{i}_\theta; & a < r \end{cases} \quad (5)$$

(b) In the limit $\epsilon \rightarrow \infty$, the tangential electric field at $r = a$ becomes

$$\lim_{\epsilon \rightarrow \infty} \frac{1}{a^3} - \frac{2(\frac{\epsilon}{\epsilon_o} - 1)}{a^3(1 + \frac{2\epsilon}{\epsilon_o})} = \frac{1}{a^3} - \frac{1}{a^3} = 0 \quad (6)$$

and the potential inside the cavity, (1a), becomes

$$\lim_{\epsilon \rightarrow \infty} \Phi^b = \frac{p \cos\theta}{4\pi\epsilon_o} \left(\frac{1}{r^2} - \frac{r}{a^3} \right) \quad (7)$$

(c) If the cavity is regarded as an equipotential at the outset, it follows from (1a) that

$$\frac{p}{4\pi\epsilon_o} \frac{1}{a^2} + Ba = 0 \Rightarrow B = -\frac{p}{4\pi\epsilon_o a^3} \quad (8)$$

in agreement with what was obtained by taking the limit, (7).

6.6.3 From (5.8.4), the potential around a perfectly conducting rod of radius R in a uniform electric field is

$$\Phi = -E_a R \left(\frac{r}{R} - \frac{R}{r} \right) \cos\phi \quad (1)$$

The potential for a two-dimensional electric dipole is given by (a) of Prob. 4.4.1.

$$\Phi = \frac{\lambda_l d \cos\phi}{2\pi\epsilon_o r} \quad (2)$$

Comparison of these expressions shows that the induced two-dimensional dipole moment is

$$\lambda_l d = (2\pi\epsilon_o R^2) E_a \quad (3)$$

The density of the rods is $(1/s^2)$ per unit area and therefore the polarization density is

$$P = (\lambda_l d / s^2) = \frac{2\pi\epsilon_o R^2}{s^2} E_a = \epsilon_o \chi_c E_a \quad (4)$$

6.6.4 The dielectric spheres have induced dipole moments that follow from (a) of Prob. 6.6.1.

$$p = 4\pi\epsilon_o E_o R^3 \frac{(\epsilon_s - \epsilon_o)}{(\epsilon_s + 2\epsilon_o)} \quad (1)$$

Using the arguments of (6.6.6)-(6.6.9), it follows that the equivalent permittivity is

$$\epsilon = 1 + 4\pi(R/s)^3 \frac{(\epsilon_s - \epsilon_o)}{(\epsilon_s + 2\epsilon_o)} \quad (2)$$

6.6.5 Writing the potential in the upper region as that of the point charge q at $y = h$ and its yet to be determined image at $y = -h$, both on the y axis, we have (Sec. 4.4)

$$\Phi = \frac{1}{4\pi\epsilon_0} \begin{cases} \frac{q}{r_+} - \frac{q_b}{r_-}; & 0 < y \\ \frac{q_a}{r_+}; & y < 0 \end{cases} \quad (1)$$

where

$$r_+ = \sqrt{x^2 + (y - h)^2 + z^2}; \quad r_- = \sqrt{x^2 + (y + h)^2 + z^2}$$

In their respective regions, these have been chosen to satisfy Poisson's equation (in the upper region) and Laplace's equation. At the interface, where $y = 0$ in (1), the potential must be continuous for all x and z

$$q - q_b = q_a \quad (2)$$

and the normal electric flux density must be continuous (there is no unpaired surface charge density)

$$\epsilon_a(q + q_b) - \epsilon_b q_a = 0 \quad (3)$$

Simultaneous solution of these expressions gives the relations for q_a and q_b summarized by (b) in the problem. To determine the force on the charge caused by the surface polarization charge it induces at the interface of the dielectric, compute the electric field at $y = h, x = 0, z = 0$ using (1) and ignoring the self field (it can produce no net force on itself) and multiply by q .

$$\mathbf{f} = \mathbf{i}_y q E_y(x = 0, y = h, z = 0) \quad (4)$$

Thus, the force is one of attraction, as given.

6.6.6 In the upper half space, the particular solution is that of a line charge. Because it has the same x dependence of its potential in the $y = 0$ plane, a homogeneous solution is added to this which is the potential of an image line charge at $(x, y) = (0, -h)$.

$$\Phi^a = -\frac{\lambda}{2\pi\epsilon_a} \ln\sqrt{x^2 + (y - h)^2} - \frac{\lambda_a}{2\pi\epsilon_a} \ln\sqrt{x^2 + (y + h)^2} \quad (1)$$

In the lower half space, the potential is taken as that due to a line charge located at $(x, y) = (0, h)$.

$$\Phi^b = -\frac{\lambda_b}{2\pi\epsilon_b} \ln\sqrt{x^2 + (y - h)^2} \quad (2)$$

The coefficients, λ_a and λ_b are now adjusted to satisfy the continuity conditions on the potential and the normal dielectric flux density in the $y = 0$ plane.

$$\frac{\lambda}{\epsilon_a} \ln\sqrt{x^2 + h^2} + \frac{\lambda_a}{\epsilon_a} \ln\sqrt{x^2 + h^2} = \frac{\lambda_b}{\epsilon_b} \sqrt{x^2 + h^2} \quad (3)$$

$$-\lambda \frac{h^2}{\sqrt{x^2 + h^2}} + \lambda_a \frac{h^2}{\sqrt{x^2 + h^2}} = -\lambda_b \frac{h^2}{\sqrt{x^2 + h^2}} \quad (4)$$

Because each of the terms in one or the other of these expressions has (by design) the same x dependence, the boundary conditions can be satisfied by adjusting the coefficients. Simultaneous solution gives

$$\lambda_a = \lambda(\epsilon_a - \epsilon_b)/(\epsilon_a + \epsilon_b) \quad (5)$$

$$\lambda_b = 2\epsilon_b\lambda/(\epsilon_a + \epsilon_b) \quad (5)$$

In the limit where $\epsilon_b \rightarrow \infty$ the field in the upper region, (5), becomes that of a line charge over a ground plane, where the image line charge is equal in magnitude and opposite in sign to that of the line charge and the field lines are perpendicular to the surface. In the opposite extreme where the upper region has a very large ϵ , the field lines in the upper region tend to have no normal component. One way to see this is to observe that in the limit $\epsilon_a \rightarrow \infty$ the image line charge becomes equal to the line charge.

- 6.6.7** (a) The uniform electric field that would exist if the permittivities were equal is written in polar coordinates as

$$\Phi = -E_o r \cos \phi \Rightarrow \mathbf{E} = E_o(\cos \phi \mathbf{i}_r - \sin \phi \mathbf{i}_\phi) \quad (1)$$

- (b) The surface polarization charge density induced by this imposed field is

$$\begin{aligned} \sigma_{sp} &= -P_r^a + P_r^b = -(\epsilon_a - \epsilon_b)E_r \\ &= \epsilon_b\left(1 - \frac{\epsilon_a}{\epsilon_b}\right)E_r = \kappa\epsilon_b E_o \cos \phi \end{aligned} \quad (2)$$

- (c) The potential induced by this surface charge density is of the form

$$\Phi = \begin{cases} A \frac{\cos \phi}{r/R}; & r > R \\ A(r/R) \cos \phi; & r < R \end{cases} \quad (3)$$

where the outer solution leaves the field as that imposed at infinity and the coefficients have been adjusted to insure continuity of the potential at the surface. The continuity condition from Gauss' law then gives

$$-\left(\epsilon_a \frac{\partial \Phi^a}{\partial r} - \epsilon_b \frac{\partial \Phi^b}{\partial r}\right) = \kappa\epsilon_b E_o \cos \phi \quad (4)$$

hence

$$A = \frac{\kappa E_o R}{2} \quad (5)$$

Substitution of this coefficient into (3) confirms the given potential.

- (d) The exact solution given by (6.6.21) and (6.6.22) is first written in terms of κ .

$$\Phi^a = -RE_o \cos \phi \left[\frac{r}{R} - \frac{R}{r} \frac{\kappa}{2 - \kappa} \right] \quad (6)$$

$$\Phi^b = -RE_o \cos \phi \left[\frac{r}{R} \frac{2(1 - \kappa)}{2 - \kappa} \right] \quad (7)$$

To linear terms in κ , note that

$$\frac{\kappa}{2 - \kappa} \rightarrow \frac{\kappa}{2}; \quad \frac{2(1 - \kappa)}{2 - \kappa} \rightarrow 1 - \frac{\kappa}{2} \quad (8)$$

Using these expressions in (6) and (7) gives the same approximate expressions for the potential as given with the problem.

- 6.6.8 (a) If the dielectric is uniform, then so is the electric field.

$$\mathbf{E} \approx \mathbf{i}_y \frac{v}{d} \equiv \mathbf{i}_y E_o \quad (1)$$

- (b) From (6.6.25), if (1) approximates the electric field then the approximate polarization surface charge density is

$$\sigma_{sp} \simeq \epsilon_o E_o \left(1 - \frac{\epsilon_a}{\epsilon_b}\right) = \epsilon_o \kappa E_o \quad (2)$$

- (c) For $\epsilon_b < \epsilon_a$, $\sigma_{sp} < 0$. Thus, the distribution of surface charge density and hence electric field is as shown in Fig. S6.6.8.

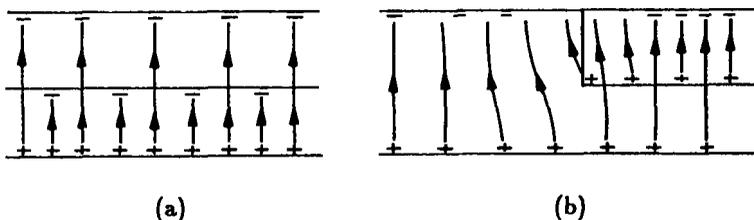


Figure S6.6.8

- (d) For the second case, the polarization surface charge density is as sketched in Fig. S6.6.8b.

- 6.6.9 (a) The potential is represented as a piece-wise continuous function. In regions (a) and (b) where the permittivity is uniform, it is expanded in solutions to Laplace's equation that have zero potential on the boundaries. To satisfy the potential boundary condition to the left, V is added to the potential in region (b).

$$\Phi = \begin{cases} \sum_{n=1}^{\infty} -A_n \frac{\sinh \left[\frac{n\pi}{d}(x-a) \right]}{\sinh(n\pi a/d)} \sin \left(\frac{n\pi}{d} y \right); & 0 < x < a \\ \sum_{n=1}^{\infty} B_n \frac{\sinh \left[\frac{n\pi}{d}(x+a) \right]}{\sinh(n\pi a/d)} \sin \left(\frac{n\pi}{d} y \right) + V; & -a < x < 0 \end{cases} \quad (1)$$

Continuity of D_x at the interface requires that

$$\epsilon_a A_n \left(\frac{n\pi}{d} \right) \coth \left(\frac{n\pi a}{d} \right) = -B_n \epsilon_b \left(\frac{n\pi}{d} \right) \coth \left(\frac{n\pi a}{d} \right) \quad (2)$$

which gives

$$B_n = -\frac{\epsilon_a}{\epsilon_b} A_n \quad (3)$$

To make the potential continuous at $x = 0$, the constant V is expanded in the same Fourier series as representing the y dependence in the other terms in (1).

$$V = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{d}\right) \quad (4)$$

Multiplication by $\sin(m\pi y/d)$ and integration on y from 0 to d then gives an expression that can be solved for the coefficients.

$$C_n = \begin{cases} \frac{4V}{n\pi}; & n \text{ odd} \\ 0; & n \text{ even} \end{cases} \quad (5)$$

The potential continuity condition is then satisfied by each term in the series.

$$A_n = B_n + \frac{4V}{n\pi}; \quad n \text{ odd} \quad (6)$$

It follows from (3) and (6) that the coefficients in (1) are $A_n = B_n = 0$ for n even and for n odd.

$$A_n = \frac{4V}{n\pi} \frac{1}{1 + \epsilon_a/\epsilon_b}; \quad B_n = -\frac{4V}{n\pi} \frac{1}{1 + \epsilon_b/\epsilon_a} \quad (7)$$

- (b) In sketching Φ and \mathbf{E} , as shown in Fig. S6.6.9a for the case where the permittivities are equal, note that the potential varies from V to 0 across the gaps. Every other point on the boundaries is either at potential V or potential 0. Thus, equipotentials all terminate and originate in the gaps. The equipotential $\Phi = V/2$ is in the $x = 0$ plane. Thus, the potential and field lines in each region are as shown in Fig. 5.5.3.
- (c) The surface charge density is given by using (6.6.25) with \mathbf{E} approximated by what it would be if the permittivities were equal.

$$\sigma_{sp} = \epsilon_o E_x \left(1 - \frac{\epsilon_a}{\epsilon_b}\right) \quad (8)$$

In the case where $\epsilon_a/\epsilon_b > 1$, $\sigma_{sp} < 0$, as illustrated by Fig. S6.6.9b. Some of the field lines originating to the left terminate in the negative σ_{sp} on the interface. Thus, the dielectric to the right tends to shield out the field. With $\epsilon_a/\epsilon_b < 1$, the surface charge density is positive, and the field tends to be shielded out of the material to the left.

- (d) With $\epsilon_a \gg \epsilon_b$, the surface becomes an equipotential and the field is concentrated in the region to the left, as shown in Fig. S6.6.9c.

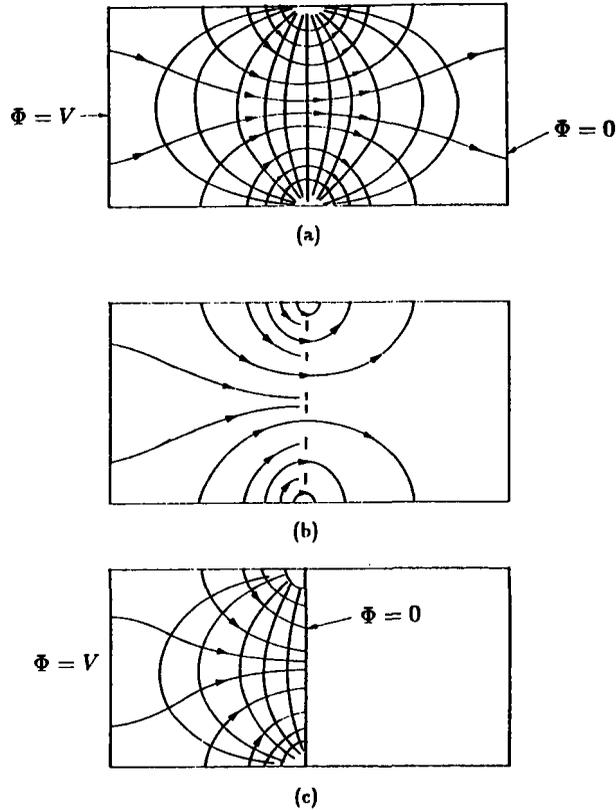


Figure S6.6.9

- (e) With $\epsilon_b \gg \epsilon_a$, the field is shielded out of the region to the left. The field looks much as in Fig. S6.6.9c except that the fields are on the right rather than the left. The equipotential $\Phi = V/2$ is in the $x = 0$ plane. Thus, the potential and field lines in each region are as shown in Fig. 5.5.3.

6.7 SMOOTHLY INHOMOGENEOUS ELECTRICALLY LINEAR DIELECTRICS

- 6.7.1 Far from the lower end, the system becomes a pair of parallel plates having the potential difference V separated by a dielectric having its permittivity gradient in the y direction. Thus, as $y \rightarrow \infty$, the potential becomes simply

$$\Phi(y \rightarrow \infty) \rightarrow V \frac{x}{a} \quad (1)$$

The product solutions throughout the region between the plates are as developed in Example 6.7.1. From those, we add to (1) those that are zero at $x = 0$ and

$x = a$ (so as not to disturb the fact that (1) already satisfies the conditions on the potential there) and that go to zero as $y \rightarrow \infty$ [again, so that the potential there becomes (1)].

$$\Phi = \sum_{n=1}^{\infty} V_n e^{[\frac{\beta}{2} - \sqrt{(\beta/2)^2 + (n\pi/a)^2}]y} \sin \frac{n\pi}{a} x + V \frac{x}{a} \quad (2)$$

To determine the coefficients, V_n , (2) is evaluated at $y = 0$ and set equal to the potential there.

$$V = \sum_{n=1}^{\infty} V_n \sin \frac{n\pi}{a} x + V \frac{x}{a} \quad (3)$$

Multiplication by $\sin(m\pi x/a)$ and integration from $x = 0$ to $x = a$ then gives the coefficients and hence the potential.

$$V_n = \frac{2}{a} \int_0^a V \left(1 - \frac{x}{a}\right) \sin \frac{n\pi}{a} x dx = (2/n\pi)V \quad (4)$$

6.7.2 The solutions to (6.7.2) for the given distributions of permittivity are as found in Example 6.7.1 with the roles of x and y interchanged. In the region to the left, $\beta \rightarrow -\beta$. Because the system extends to infinity in the $\pm x$ directions, exponential solutions are selected in each of the regions that decay to zero at infinity.

$$\Phi = \begin{cases} \sum_{n=1}^{\infty} A_n e^{[\frac{\beta}{2} - \sqrt{(\beta/2)^2 + (n\pi/a)^2}]x} \sin \frac{n\pi}{a} y; & 0 < x \\ \frac{V}{a}(a - y) + \sum_{n=1}^{\infty} B_n e^{[-\frac{\beta}{2} + \sqrt{(\beta/2)^2 + (n\pi/a)^2}]x} \sin \frac{n\pi}{a} y; & x < 0 \end{cases} \quad (1)$$

Continuity of D_x gives one condition on the coefficients.

$$[\epsilon_p e^{\beta x} E_x^b]_{x=0} = [\epsilon_p e^{-\beta x} E_x^a]_{x=0} \Rightarrow B_n = -A_n \quad (2)$$

To match the potential in the $x = 0$ plane, the first term in the solution to the left, in (1b), is expanded in the same series as the other terms.

$$\frac{V}{a}(a - y) = \sum_{n=1}^{\infty} C_n \sin \left(\frac{n\pi}{a} y\right) \quad (3)$$

The coefficient is found by multiplying this expression by $\sin(m\pi y/a)$ and integrating from $y = 0$ to $y = a$.

$$C_n = \frac{2V}{n\pi} \quad (4)$$

It follows from (2) and (4) that

$$A_n = -B_n = \frac{V}{n\pi} \quad (5)$$

6.7.3 (a) The polarization charge density is approximately

$$\begin{aligned}\rho_p &= -\nabla \cdot [(\epsilon - \epsilon_o)\mathbf{E}] \simeq -\nabla \cdot [(\epsilon - \epsilon_o)E_o\mathbf{i}_y] = -E_o\frac{\partial\epsilon}{\partial y} \\ &= E_o\epsilon_a\chi_p\frac{2y}{a^2}\exp[-(x^2 + y^2 + z^2)/a^2]\end{aligned}\quad (1)$$

(b) For $E_o\chi_p > 0$, the field induces positive and negative regions of charge density in the upper and lower regions respectively. These are centered on the y axis where the function $y\exp(-y^2/a^2)$ peaks, at $y = a/\sqrt{2}$. Thus, some of the field lines entering from below in Fig. S6.7.3 terminate on the negative charge while some leaving at the top originate on the positive charge. The field is that of a diffuse dipole.

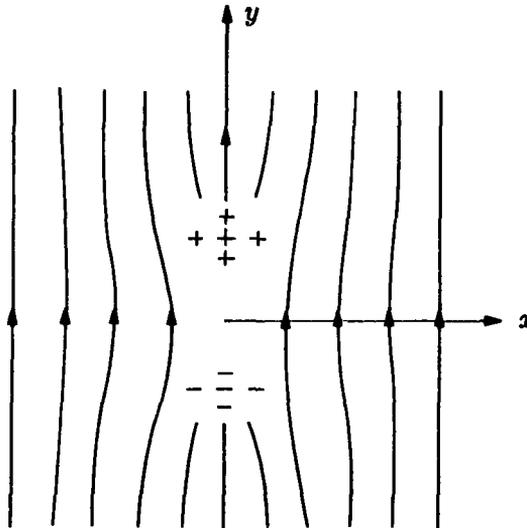


Figure S6.7.3