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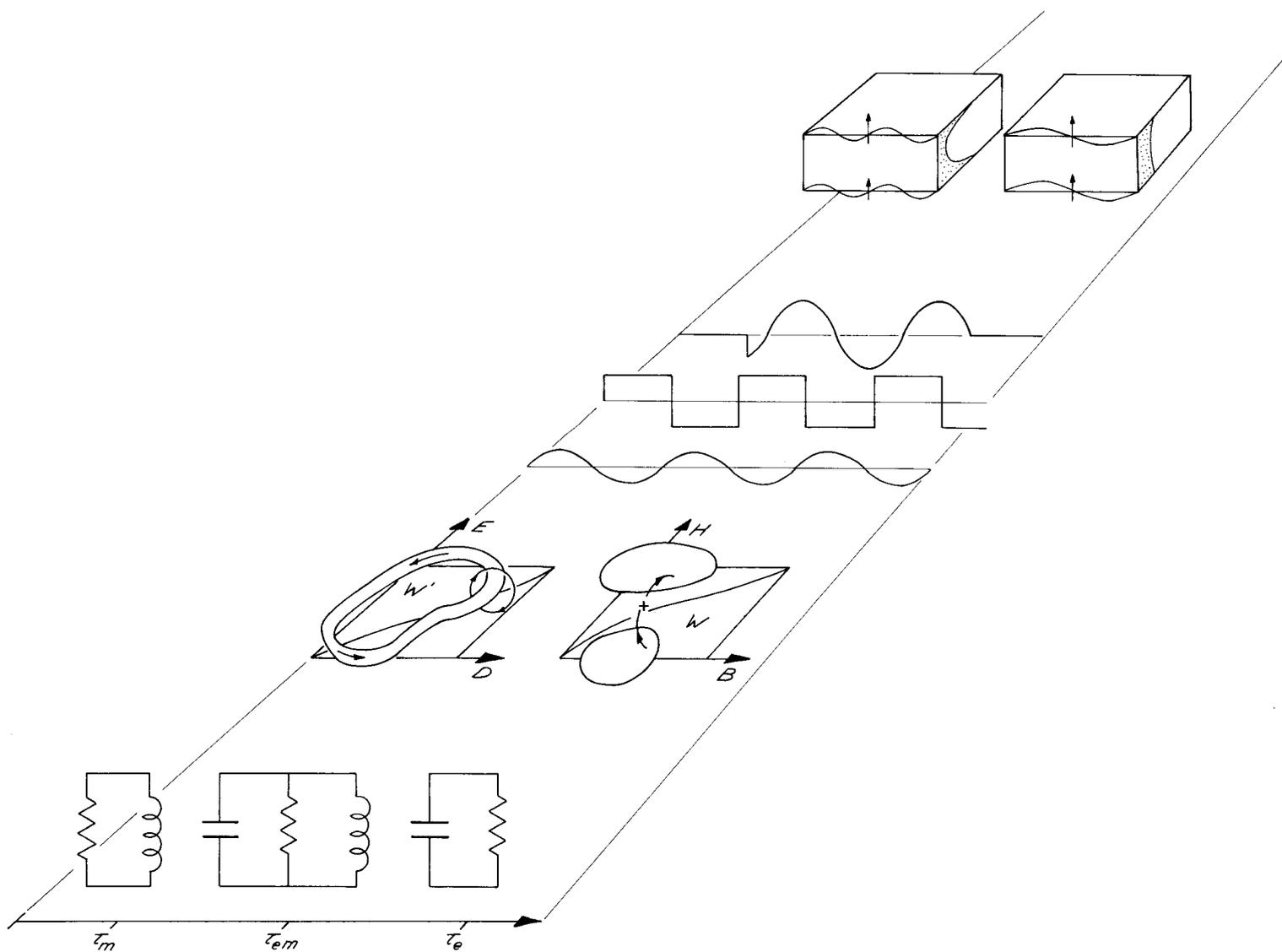
*Continuum Electromechanics*

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## Electrodynamic Laws, Approximations and Relations



## 2.1 Definitions

Continuum electromechanics brings together several disciplines, and so it is useful to summarize the definitions of electrodynamic variables and their units. Rationalized MKS units are used not only in connection with electrodynamics, but also in dealing with subjects such as fluid mechanics and heat transfer, which are often treated in English units. Unless otherwise given, basic units of meters (m), kilograms (kg), seconds (sec), and Coulombs (C) can be assumed.

Table 2.1.1. Summary of electrodynamic nomenclature.

Name	Symbol	Units
<u>Discrete Variables</u>		
Voltage or potential difference	v	[V] = volts = m <sup>2</sup> kg/C sec <sup>2</sup>
Charge	q	[C] = Coulombs = C
Current	i	[A] = Amperes = C/sec
Magnetic flux	λ	[Wb] = Weber = m <sup>2</sup> kg/C sec
Capacitance	C	[F] = Farad C <sup>2</sup> sec <sup>2</sup> /m <sup>2</sup> kg
Inductance	L	[H] = Henry = m <sup>2</sup> kg/C <sup>2</sup>
Force	f	[N] = Newtons = kg m/sec <sup>2</sup>
<u>Field Sources</u>		
Free charge density	ρ <sub>f</sub>	C/m <sup>3</sup>
Free surface charge density	σ <sub>f</sub>	C/m <sup>2</sup>
Free current density	J <sub>f</sub>	A/m <sup>2</sup>
Free surface current density	K <sub>f</sub>	A/m
<u>Fields</u> (name in quotes is often used for convenience)		
"Electric field" intensity	$\vec{E}$	V/m
"Magnetic field" intensity	$\vec{H}$	A/m
Electric displacement	$\vec{D}$	C/m <sup>2</sup>
Magnetic flux density	$\vec{B}$	Wb/m <sup>2</sup> (tesla)
Polarization density	$\vec{P}$	C/m <sup>2</sup>
Magnetization density	$\vec{M}$	A/m
Force density	$\vec{F}$	N/m <sup>3</sup>
<u>Physical Constants</u>		
Permittivity of free space	ε <sub>0</sub> = 8.854 × 10 <sup>-12</sup>	F/m
Permeability of free space	μ <sub>0</sub> = 4π × 10 <sup>-7</sup>	H/m

Although terms involving moving magnetized and polarized media may not be familiar, Maxwell's equations are summarized without prelude in the next section. The physical significance of the unfamiliar terms can best be discussed in Secs. 2.8 and 2.9 after the general laws are reduced to their quasistatic forms, and this is the objective of Sec. 2.3. Except for introducing concepts concerned with the description of continua, including integral theorems, in Secs. 2.4 and 2.6, and the discussion of Fourier amplitudes in Sec. 2.15, the remainder of the chapter is a parallel development of the consequences of these quasistatic laws. That the field transformations (Sec. 2.5), integral laws (Sec. 2.7), splicing conditions (Sec. 2.10), and energy storages are derived from the fundamental quasistatic laws, illustrates the important dictum that internal consistency be maintained within the framework of the quasistatic approximation.

The results of the sections on energy storage are used in Chap. 3 for deducing the electric and magnetic force densities on macroscopic media. The transfer relations of the last sections are an important resource throughout all of the following chapters, and give the opportunity to explore the physical significance of the quasistatic limits.

## 2.2 Differential Laws of Electrodynamics

In the Chu formulation,<sup>1</sup> with material effects on the fields accounted for by the magnetization density  $\vec{M}$  and the polarization density  $\vec{P}$  and with the material velocity denoted by  $\vec{v}$ , the laws of electrodynamics are:

### Faraday's law

$$\nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} - \mu_0 \frac{\partial \vec{M}}{\partial t} - \mu_0 \nabla \times (\vec{M} \times \vec{v}) \quad (1)$$

1. P. Penfield, Jr., and H. A. Haus, Electrodynamics of Moving Media, The M.I.T. Press, Cambridge, Massachusetts, 1967, pp. 35-40.

### Ampère's law

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} + \nabla \times (\vec{P} \times \vec{v}) + \vec{J}_f \quad (2)$$

### Gauss' law

$$\epsilon_0 \nabla \cdot \vec{E} = -\nabla \cdot \vec{P} + \rho_f \quad (3)$$

### divergence law for magnetic fields

$$\mu_0 \nabla \cdot \vec{H} = -\mu_0 \nabla \cdot \vec{M} \quad (4)$$

### and conservation of free charge

$$\nabla \cdot \vec{J}_f + \frac{\partial \rho_f}{\partial t} = 0 \quad (5)$$

This last expression is imbedded in Ampère's and Gauss' laws, as can be seen by taking the divergence of Eq. 2 and exploiting Eq. 3. In this formulation the electric displacement  $\vec{D}$  and magnetic flux density  $\vec{B}$  are defined fields:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (6)$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \quad (7)$$

## 2.3 Quasistatic Laws and the Time-Rate Expansion

With a quasistatic model, it is recognized that relevant time rates of change are sufficiently low that contributions due to a particular dynamical process are ignorable. The objective in this section is to give some formal structure to the reasoning used to deduce the quasistatic field equations from the more general Maxwell's equations. Here, quasistatics specifically means that times of interest are long compared to the time,  $\tau_{em}$ , for an electromagnetic wave to propagate through the system.

Generally, given a dynamical process characterized by some time determined by the parameters of the system, a quasistatic model can be used to exploit the comparatively long time scale for processes of interest. In this broad sense, quasistatic models abound and will be encountered in many other contexts in the chapters that follow. Specific examples are:

(a) processes slow compared to wave transit times in general; acoustic waves and the model is one of incompressible flow, Alfvén and other electromechanical waves and the model is less standard;

(b) processes slow compared to diffusion (instantaneous diffusion models). What diffuses can be magnetic field, viscous stresses, heat, molecules or hybrid electromechanical effects;

(c) processes slow compared to relaxation of continua (instantaneous relaxation or constant-potential models). Charge relaxation is an important example.

The point of making a quasistatic approximation is often to focus attention on significant dynamical processes. A quasistatic model is by no means static. Because more than one rate process is often imbedded in a given physical system, it is important to agree upon the one with respect to which the dynamics are quasistatic.

Rate processes other than those due to the transit time of electromagnetic waves enter through the dependence of the field sources on the fields and material motion. To have in view the additional characteristic times typically brought in by the field sources, in this section the free current density is postulated to have the dependence

$$\vec{J}_f = \sigma(\vec{r})\vec{E} + \vec{J}_v(\vec{v}, \rho_f, \vec{H}) \quad (1)$$

In the absence of motion,  $\vec{J}_v$  is zero. Thus, for media at rest the conduction model is ohmic, with the electrical conductivity  $\sigma$  in general a function of position. Examples of  $\vec{J}_v$  are a convection current  $\rho_f \vec{v}$ , or an ohmic motion-induced current  $\sigma(\vec{v} \times \mu_0 \vec{H})$ . With an underbar used to denote a normalized quantity, the conductivity is normalized to a typical (constant) conductivity  $\sigma_0$ :

$$\sigma = \sigma_0 \underline{\sigma}(\vec{r}, t) \quad (2)$$

To identify the hierarchy of critical time-rate parameters, the general laws are normalized. Coordinates are normalized to one typical length  $\ell$ , while  $\tau$  represents a characteristic dynamical time:

$$(x, y, z) = (\ell \underline{x}, \ell \underline{y}, \ell \underline{z}); \quad t = \tau \underline{t} \quad (3)$$

In a system sinusoidally excited at the angular frequency  $\omega$ ,  $\tau = \omega^{-1}$ .

The most convenient normalization of the fields depends on the specific system. Where electro-mechanical coupling is significant, these can usually be categorized as "electric-field dominated" and "magnetic-field dominated." Anticipating this fact, two normalizations are now developed "in parallel," the first taking  $\mathcal{E}$  as a characteristic electric field and the second taking  $\mathcal{H}$  as a characteristic magnetic field:

$$\begin{aligned} \vec{E} = \mathcal{E} \underline{\vec{E}}, \quad \vec{P} = \epsilon_0 \mathcal{E} \underline{\vec{P}}, \quad \vec{v} = (\ell/\tau) \underline{\vec{v}}, \quad \vec{J}_v = \frac{\epsilon_0 \mathcal{E}}{\tau} \underline{\vec{J}_v}, & \quad \vec{H} = \mathcal{H} \underline{\vec{H}}, \quad \vec{M} = \mathcal{H} \underline{\vec{M}}, \quad \vec{v} = (\ell/\tau) \underline{\vec{v}}, \quad \vec{J}_v = \frac{\mathcal{H}}{\ell} \underline{\vec{J}_v} \\ \rho_f = \frac{\mathcal{E} \epsilon_0}{\ell} \underline{\rho}_f, \quad \vec{H} = \frac{\epsilon_0 \ell \mathcal{E}}{\tau} \underline{\vec{H}}, \quad \vec{M} = \frac{\epsilon_0 \ell \mathcal{E}}{\tau} \underline{\vec{M}} & \quad \vec{E} = \frac{\mu_0 \ell \mathcal{H}}{\tau} \underline{\vec{E}}, \quad \rho_f = \frac{\epsilon_0 \mu_0 \mathcal{H}}{\tau} \underline{\rho}_f, \quad \vec{P} = \frac{\epsilon_0 \mu_0 \ell \mathcal{H}}{\tau} \underline{\vec{P}} \end{aligned} \quad (4) \quad \times$$

It might be appropriate with this step to recognize that the material motion introduces a characteristic (transport) time other than  $\tau$ . For simplicity, Eq. 4 takes the material velocity as being of the order of  $\ell/\tau$ .

The normalization used is arbitrary. The same quasistatic laws will be deduced regardless of the starting point, but the normalization will determine whether these laws are "zero-order" or higher order in a sense to now be defined.

The normalizations of Eq. 4 introduced into Eqs. 2.2.1-5 result in

$$\begin{aligned} \nabla \cdot \underline{\vec{E}} = -\nabla \cdot \underline{\vec{P}} + \underline{\rho}_f & \quad \nabla \cdot \underline{\vec{E}} = -\nabla \cdot \underline{\vec{P}} + \underline{\rho}_f & (5) \\ \nabla \cdot \underline{\vec{H}} = -\nabla \cdot \underline{\vec{M}} & \quad \nabla \cdot \underline{\vec{H}} = -\nabla \cdot \underline{\vec{M}} & (6) \\ \nabla \times \underline{\vec{H}} = \frac{\tau}{\tau_e} \sigma \underline{\vec{E}} + \underline{\vec{J}}_v + \frac{\partial \underline{\vec{E}}}{\partial t} + \frac{\partial \underline{\vec{P}}}{\partial t} + \nabla \times (\underline{\vec{P}} \times \underline{\vec{v}}) & \quad \nabla \times \underline{\vec{H}} = \frac{\tau_m}{\tau} \sigma \underline{\vec{E}} + \underline{\vec{J}}_v + \beta \left[ \frac{\partial \underline{\vec{E}}}{\partial t} + \frac{\partial \underline{\vec{P}}}{\partial t} + \nabla \times (\underline{\vec{P}} \times \underline{\vec{v}}) \right] & (7) \quad \times \\ \nabla \times \underline{\vec{E}} = -\beta \left[ \frac{\partial \underline{\vec{H}}}{\partial t} + \frac{\partial \underline{\vec{M}}}{\partial t} + \nabla \times (\underline{\vec{M}} \times \underline{\vec{v}}) \right] & \quad \nabla \times \underline{\vec{E}} = -\frac{\partial \underline{\vec{H}}}{\partial t} - \frac{\partial \underline{\vec{M}}}{\partial t} - \nabla \times (\underline{\vec{M}} \times \underline{\vec{v}}) & (8) \quad \times \\ \nabla \cdot \sigma \underline{\vec{E}} + \frac{\tau_e}{\tau} \left[ \nabla \cdot \underline{\vec{J}}_v + \frac{\partial \underline{\rho}_f}{\partial t} \right] = 0 & \quad \nabla \cdot \sigma \underline{\vec{E}} + \frac{\tau}{\tau_m} \nabla \cdot \underline{\vec{J}}_v + \beta \frac{\tau}{\tau_m} \frac{\partial \underline{\rho}_f}{\partial t} = 0 & (9) \end{aligned}$$

where underbars on equation numbers are used to indicate that the equations are normalized and

$$\tau_m \equiv \mu_0 \sigma_0 \ell^2, \quad \tau_e \equiv \epsilon_0 / \sigma_0$$

and

$$\beta = \left( \frac{\tau_{em}}{\tau} \right)^2; \quad \tau_{em} \equiv \sqrt{\mu_0 \epsilon_0} \ell = \ell/c \quad (10)$$

In Chap. 6,  $\tau_m$  will be identified as the magnetic diffusion time, while in Chap. 5 the role of the charge-relaxation time  $\tau_e$  is developed. The time required for an electromagnetic plane wave to propagate the distance  $\ell$  at the velocity  $c$  is  $\tau_{em}$ . Given that there is just one characteristic length, there are actually only two characteristic times, because as can be seen from Eq. 10

$$\sqrt{\tau_m \tau_e} = \tau_{em} \quad (11)$$

Unless  $\tau_e$  and  $\tau_m$ , and hence  $\tau_{em}$ , are all of the same order, there are only two possibilities for the relative magnitudes of these times, as summarized in Fig. 2.3.1.

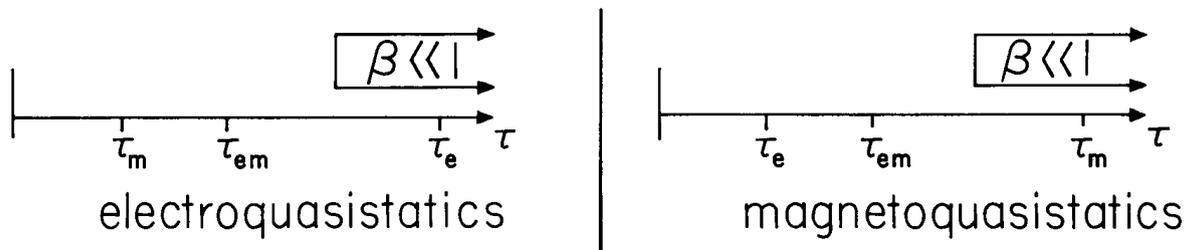


Fig. 2.3.1. Possible relations between physical time constants on a time scale  $\tau$  which typifies the dynamics of interest.

By electroquasistatic (EQS) approximation it is meant that the ordering of times is as to the left and that the parameter  $\beta = (\tau_{em}/\tau)^2$  is much less than unity. Note that  $\tau$  is still arbitrary relative to  $\tau_e$ . In the magnetoquasistatic (MQS) approximation,  $\beta$  is still small, but the ordering of characteristic times is as to the right. In this case,  $\tau$  is arbitrary relative to  $\tau_m$ .

To make a formal statement of the procedure used to find the quasistatic approximation, the normalized fields and charge density are expanded in powers of the time-rate parameter  $\beta$ .

$$\begin{aligned}\vec{E} &= \vec{E}_0 + \beta \vec{E}_1 + \beta^2 \vec{E}_2 + \dots \\ \vec{H} &= \vec{H}_0 + \beta \vec{H}_1 + \beta^2 \vec{H}_2 + \dots \\ \vec{J}_v &= (\vec{J}_v)_0 + \beta (\vec{J}_v)_1 + \beta^2 (\vec{J}_v)_2 + \dots \\ \rho_f &= (\rho_f)_0 + \beta (\rho_f)_1 + \beta^2 (\rho_f)_2 + \dots\end{aligned}\tag{12}$$

In the following, it is assumed that constitutive laws relate  $\vec{P}$  and  $\vec{M}$  to  $\vec{E}$  and  $\vec{H}$ , so that these densities are similarly expanded. The velocity  $\vec{v}$  is taken as given. Then, the series are substituted into Eqs. 5-9 and the resulting expressions arranged by factors multiplying ascending powers of  $\beta$ . The "zero order" equations are obtained by requiring that the coefficients of  $\beta^0$  vanish. These are simply Eqs. 5-9 with  $\beta = 0$ :

$\nabla \cdot \vec{E}_0 = -\nabla \cdot \vec{P}_0 + (\rho_f)_0$	$\nabla \cdot \vec{E} = -\nabla \cdot \vec{P}_0 + (\rho_f)_0 \tag{13}$
$\nabla \cdot \vec{H}_0 = -\nabla \cdot \vec{M}_0$	$\nabla \cdot \vec{H}_0 = -\nabla \cdot \vec{M}_0 \tag{14}$
$\nabla \times \vec{H}_0 = \frac{\tau}{\tau_e} \sigma \vec{E}_0 + (\vec{J}_v)_0 + \frac{\partial \vec{E}_0}{\partial t} + \frac{\partial \vec{P}_0}{\partial t} + \nabla \times (\vec{P}_0 \times \vec{v})$	$\nabla \times \vec{H}_0 = \frac{\tau_m}{\tau} \sigma \vec{E}_0 + (\vec{J}_v)_0 \tag{15}$
$\nabla \times \vec{E}_0 = 0$	$\nabla \times \vec{E}_0 = -\frac{\partial \vec{H}_0}{\partial t} - \frac{\partial \vec{M}_0}{\partial t} - \nabla \times (\vec{M}_0 \times \vec{v}) \tag{16}$
$\nabla \cdot \sigma \vec{E}_0 + \frac{\tau_e}{\tau} \left[ \nabla \cdot (\vec{J}_v)_0 + \frac{\partial (\rho_f)_0}{\partial t} \right] = 0$	$\nabla \cdot \sigma \vec{E}_0 + \frac{\tau}{\tau_m} \nabla \cdot (\vec{J}_v)_0 = 0 \tag{17}$

The zero-order solutions are found by solving these equations, augmented by appropriate boundary conditions. If the boundary conditions are themselves time dependent, normalization will turn up additional characteristic times that must be fitted into the hierarchy of Fig. 2.3.1.

Higher order contributions to the series of Eq. 12 follow from a sequential solution of the equations found by making coefficients of like powers of  $\beta$  vanish. The expressions resulting from setting the coefficients of  $\beta^n$  to zero are:

$\nabla \cdot \vec{E}_n + \nabla \cdot \vec{P}_n - (\rho_f)_n = 0$	$\nabla \cdot \vec{E}_n + \nabla \cdot \vec{P}_n - (\rho_f)_n = 0 \tag{18}$
$\nabla \cdot \vec{H}_n + \nabla \cdot \vec{M}_n = 0$	$\nabla \cdot \vec{H}_n + \nabla \cdot \vec{M}_n = 0 \tag{19}$
$\nabla \times \vec{H}_n - \frac{\tau}{\tau_e} \sigma \vec{E}_n - (\vec{J}_v)_n - \frac{\partial \vec{E}_n}{\partial t} - \frac{\partial \vec{P}_n}{\partial t} - \nabla \times (\vec{P}_n \times \vec{v}) = 0$	$\nabla \times \vec{H}_n - \frac{\tau_m}{\tau} \sigma \vec{E}_n - (\vec{J}_v)_n = \left[ \frac{\partial \vec{E}_{n-1}}{\partial t} + \frac{\partial \vec{P}_{n-1}}{\partial t} + \nabla \times (\vec{P}_{n-1} \times \vec{v}) \right] \tag{20}$
$\nabla \times \vec{E}_n = -\left[ \frac{\partial \vec{H}_{n-1}}{\partial t} + \frac{\partial \vec{M}_{n-1}}{\partial t} + \nabla \times (\vec{M}_{n-1} \times \vec{v}) \right]$	$\nabla \times \vec{E}_n + \frac{\partial \vec{H}_n}{\partial t} + \frac{\partial \vec{M}_n}{\partial t} + \nabla \times (\vec{M}_n \times \vec{v}) = 0 \tag{21}$
$\nabla \cdot \sigma \vec{E}_n + \frac{\tau_e}{\tau} \left[ \nabla \cdot (\vec{J}_v)_n + \frac{\partial (\rho_f)_n}{\partial t} \right] = 0$	$\nabla \cdot \sigma \vec{E}_n + \frac{\tau}{\tau_m} \nabla \cdot (\vec{J}_v)_n = -\frac{\tau}{\tau_m} \frac{\partial (\rho_f)_{n-1}}{\partial t} \tag{22}$

To find the first order contributions, these equations with  $n=1$  are solved with the zero order solutions making up the right-hand sides of the equations playing the role of known driving functions. Boundary conditions are satisfied by the lowest order fields. Thus higher order fields satisfy homogeneous boundary conditions.

Once the first order solutions are known, the process can be repeated with these forming the "drives" for the  $n=2$  equations.

In the absence of loss effects, there are no characteristic times to distinguish MQS and EQS systems. In that limit, which set of normalizations is used is a matter of convenience. If a situation represented by the left-hand set actually has an EQS limit, the zero order laws become the quasistatic laws. But, if these expressions are applied to a situation that is actually MQS, then first-order terms must be calculated to find the quasistatic fields. If more than the one characteristic time  $\tau_{em}$  is involved, as is the case with finite  $\tau_e$  and  $\tau_m$ , then the ordering of rate parameters can contribute to the convergence of the expansion.

In practice, a formal derivation of the quasistatic laws is seldom used. Rather, intuition and experience along with comparison of critical time constants to relevant dynamical times is used to identify one of the two sets of zero order expressions as appropriate. But, the use of normalizations to identify critical parameters, and the notion that characteristic times can be used to unscramble dynamical processes, will be used extensively in the chapters to follow.

Within the framework of quasistatic electrodynamics, the unnormalized forms of Eqs. 13-17 comprise the "exact" field laws. These equations are reordered to reflect their relative importance:

Electroquasistatic (EQS)	Magnetoquasistatic (MQS)
$\nabla \cdot \epsilon_0 \vec{E} = -\nabla \cdot \vec{P} + \rho_f$	$\nabla \times \vec{H} = \vec{J}_f$ (23)
$\nabla \times \vec{E} = 0$	$\nabla \cdot \mu_0 \vec{H} = -\nabla \cdot \mu_0 \vec{M}$ (24)
$\nabla \cdot \vec{J}_f + \frac{\partial \rho_f}{\partial t} = 0$	$\nabla \times \vec{E} = -\frac{\partial \mu_0 \vec{H}}{\partial t} - \frac{\partial \mu_0 \vec{M}}{\partial t} - \mu_0 \nabla \times (\vec{M} \times \vec{v})$ (25)
$\nabla \times \vec{H} = \vec{J}_f + \frac{\partial \epsilon_0 \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} + \nabla \times (\vec{P} \times \vec{v})$	$\nabla \cdot \vec{J}_f = 0$ (26)
$\nabla \cdot \mu_0 \vec{H} = -\nabla \cdot \mu_0 \vec{M}$	$\nabla \cdot \epsilon_0 \vec{E} = -\nabla \cdot \vec{P} + \rho_f$ (27)

The conduction current  $\vec{J}_f$  has been reintroduced to reflect the wider range of validity of these equations than might be inferred from Eq. 1. With different conduction models will come different characteristic times, exemplified in the discussions of this section by  $\tau_e$  and  $\tau_m$ . Matters are more complicated if fields and media interact electromechanically. Then,  $\vec{v}$  is determined to some extent at least by the fields themselves and must be treated on a par with the field variables. The result can be still more characteristic times.

The ordering of the quasistatic equations emphasizes the instantaneous relation between the respective dominant sources and fields. Given the charge and polarization densities in the EQS system, or given the current and magnetization densities in the MQS system, the dominant fields are known and are functions only of the sources at the given instant in time.

The dynamics enter in the EQS system with conservation of charge, and in the MQS system with Faraday's law of induction. Equations 26a and 27a are only needed if an after-the-fact determination of  $\vec{H}$  is to be made. An example where such a rare interest in  $\vec{H}$  exists is in the small magnetic field induced by electric fields and currents within the human body. The distribution of internal fields and hence currents is determined by the first three EQS equations. Given  $\vec{E}$ ,  $\vec{P}$ , and  $\vec{J}_f$ , the remaining two expressions determine  $\vec{H}$ . In the MQS system, Eq. 27b can be regarded as an expression for the after-the-fact evaluation of  $\rho_f$ , which is not usually of interest in such systems.

What makes the subject of quasistatics difficult to treat in a general way, even for a system of fixed ohmic conductivity, is the dependence of the appropriate model on considerations not conveniently represented in the differential laws. For example, a pair of perfectly conducting plates, shorted on one pair of edges and driven by a sinusoidal source at the opposite pair, will be MQS at low frequencies. The same pair of plates, open-circuited rather than shorted, will be electroquasistatic at low frequencies. The difference is in the boundary conditions.

Geometry and the inhomogeneity of the medium (insulators, perfect conductors and semiconductors) are also essential to determining the appropriate approximation. Most systems require more than one

characteristic dimension and perhaps conductivity for their description, with the result that more than two time constants are often involved. Thus, the two possibilities identified in Fig. 2.3.1 can in principle become many possibilities. Even so, for a wide range of practical problems, the appropriate field laws are either clearly electroquasistatic or magnetoquasistatic.

Problems accompanying this section help to make the significance of the quasistatic limits more substantive by considering cases that can also be solved exactly.

#### 2.4 Continuum Coordinates and the Convective Derivative

There are two commonly used representations of continuum variables. One of these is familiar from classical mechanics, while the other is universally used in electrodynamics. Because electro-mechanics involves both of these subjects, attention is now drawn to the salient features of the two representations.

Consider first the "Lagrangian representation." The position of a material particle is a natural example and is depicted by Fig. 2.4.1a. When the time  $t$  is zero, a particle is found at the position  $\vec{r}_0$ . The position of the particle at some subsequent time is  $\vec{\xi}$ . To let  $\vec{\xi}$  represent the displacement of a continuum of particles, the position variable  $\vec{r}_0$  is used to distinguish particles. In this sense, the displacement  $\vec{\xi}$  then also becomes a continuum variable capable of representing the relative displacements of an infinitude of particles.

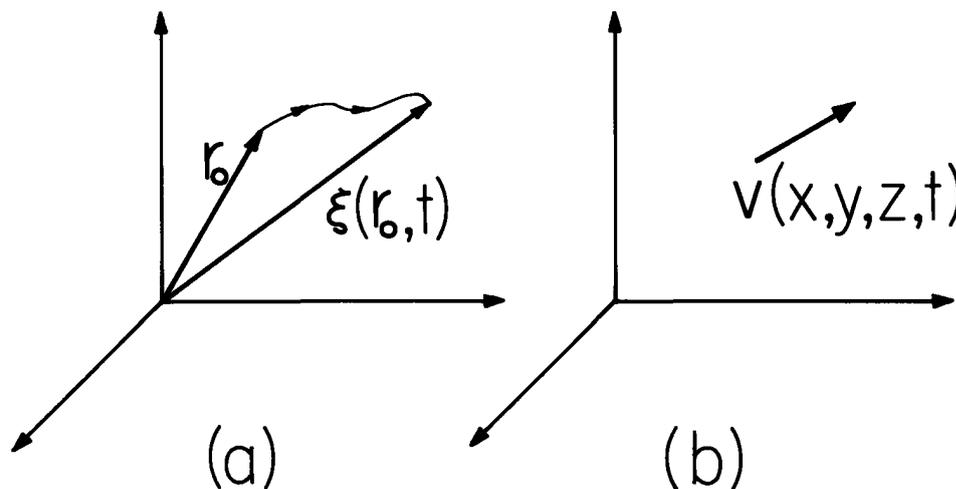


Fig. 2.4.1. Particle motions represented in terms of (a) Lagrangian coordinates, where the initial particle coordinate  $\vec{r}_0$  designates the particle of interest, and (b) Eulerian coordinates, where  $(x,y,z)$  designates the spatial position of interest.

In a Lagrangian representation, the velocity of the particle is simply

$$\vec{v} = \frac{\partial \vec{\xi}}{\partial t} \quad (1)$$

If concern is with only one particle, there is no point in writing the derivative as a partial derivative. However, it is understood that, when the derivative is taken, it is a particular particle which is being considered. So, it is understood that  $\vec{r}_0$  is fixed. Using the same line of reasoning, the acceleration of a particle is given by

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} \quad (2)$$

The idea of representing continuum variables in terms of the coordinates  $(x,y,z)$  connected with the space itself is familiar from electromagnetic theory. But what does it mean if the variable is mechanical rather than electrical? We could represent the velocity of the continuum of particles filling the space of interest by a vector function  $\vec{v}(x,y,z,t) = \vec{v}(\vec{r},t)$ . The velocity of particles having the position  $(x,y,z)$  at a given time  $t$  is determined by evaluating the function  $\vec{v}(\vec{r},t)$ . The velocity appearing in Sec. 2.2 is an example. As suggested by Fig. 2.4.1b, if the function is the velocity evaluated at a given position in space, it describes whichever particle is at that point at the time of interest. Generally, there is a continuous stream of particles through the point  $(x,y,z)$ .

Computation of the particle acceleration makes evident the contrast between Eulerian and Lagrangian representations. By definition, the acceleration is the rate of change of the velocity computed for a given particle of matter. A particle having the position  $(x,y,z)$  at time  $t$  will be found an instant  $\Delta t$  later at the position  $(x + v_x \Delta t, y + v_y \Delta t, z + v_z \Delta t)$ . Hence the acceleration is

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\vec{v}(x + v_x \Delta t, y + v_y \Delta t, z + v_z \Delta t, t + \Delta t) - \vec{v}(x, y, z, t)}{\Delta t} \quad (3)$$

Expansion of the first term in Eq. 3 about the initial coordinates of the particle gives the convective derivative of  $\vec{v}$ :

$$\vec{a} = \frac{\partial \vec{v}}{\partial t} + v_x \frac{\partial \vec{v}}{\partial x} + v_y \frac{\partial \vec{v}}{\partial y} + v_z \frac{\partial \vec{v}}{\partial z} \equiv \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} \quad (4)$$

The difference between Eq. 2 and Eq. 4 is resolved by recognizing the difference in the significance of the partial derivatives. In Eq. 2, it is understood that the coordinates being held fixed are the initial coordinates of the particle of interest. In Eq. 4, the partial derivative is taken, holding fixed the particular point of interest in space.

The same steps show that the rate of change of any vector variable  $\vec{A}$ , as viewed from a particle having the velocity  $\vec{v}$ , is

$$\frac{D\vec{A}}{Dt} \equiv \frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \nabla) \vec{A}; \quad \vec{A} = \vec{A}(x, y, z, t) \quad (5)$$

The time rate of change of any scalar variable for an observer moving with the velocity  $\vec{v}$  is obtained from Eq. 5 by considering the particular case in which  $\vec{A}$  has only one component, say  $\vec{A} = f(x, y, z, t) \hat{i}_x$ . Then Eq. 5 becomes

$$\frac{Df}{Dt} \equiv \frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f \quad (6)$$

Reference 3 of Appendix C is a film useful in understanding this section.

## 2.5 Transformations between Inertial Frames

In extending empirically determined conduction, polarization and magnetization laws to include material motion, it is often necessary to relate field variables evaluated in different reference frames. A given point in space can be designated either in terms of the coordinate  $\vec{r}$  or of the coordinate  $\vec{r}'$  of Fig. 2.5.1. By "inertial reference frames," it is meant that the relative velocity between these two frames is constant, designated by  $\vec{u}$ . The positions in the two coordinate systems are related by the Galilean transformation:

$$\vec{r}' = \vec{r} - \vec{u}t; \quad t' = t \quad (1)$$

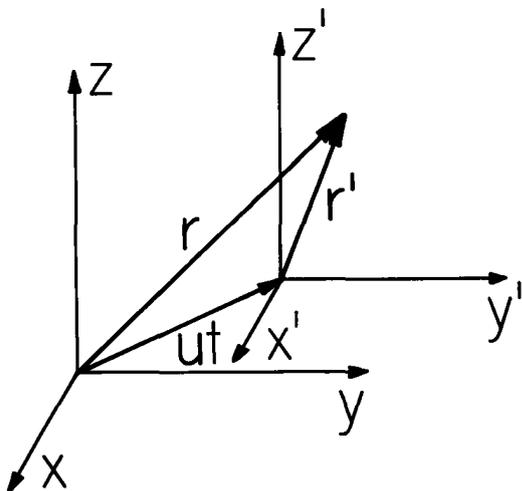


Fig. 2.5.1

Reference frames have constant relative velocity  $\vec{u}$ . The coordinates  $\vec{r} = (x, y, z)$  and  $\vec{r}' = (x', y', z')$  designate the same position.

It is a familiar fact that variables describing a given physical situation in one reference frame will not be the same as those in the other. An example is material velocity, which, if measured in one frame, will differ from that in the other frame by the relative velocity  $\vec{u}$ .

There are two objectives in this section: one is to show that the quasistatic laws are invariant when subject to a Galilean transformation between inertial reference frames. But, of more use is the relationship between electromagnetic variables in the two frames of reference that follows from this

proof. The approach is as follows. First, the postulate is made that the quasistatic equations take the same form in the primed and unprimed inertial reference frames. But, in writing the laws in the primed frame, the spatial and temporal derivatives must be taken with respect to the coordinates of that reference frame, and the dependent field variables are then fields defined in that reference frame. In general, these must be designated by primes, since their relation to the variables in the unprimed frame is not known.

For the purpose of writing the primed equations of electrodynamics in terms of the unprimed coordinates, recognize that

$$\begin{aligned}\nabla' &\rightarrow \nabla \\ \frac{\partial \vec{A}}{\partial t'} &\rightarrow \left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \vec{A} \equiv \frac{\partial \vec{A}}{\partial t} + \vec{u} \nabla \cdot \vec{A} - \nabla \times (\vec{u} \times \vec{A}) \\ \frac{\partial \psi}{\partial t'} &\rightarrow \left( \frac{\partial}{\partial t} + \vec{u} \cdot \nabla \right) \psi \equiv \frac{\partial \psi}{\partial t} + \nabla \cdot \vec{u} \psi\end{aligned}\quad (2)$$

The left relations follow by using the chain rule of differentiation and the transformation of Eq. 1. That the spatial derivatives taken with respect to one frame must be the same as those with respect to the other frame physically means that a single "snapshot" of the physical process would be all required to evaluate the spatial derivatives in either frame. There would be no way of telling which frame was the one from which the snapshot was taken. By contrast, the time rate of change for an observer in the primed frame is, by definition, taken with the primed spatial coordinates held fixed. In terms of the fixed frame coordinates, this is the convective derivative defined with Eqs. 2.4.5 and 2.4.6. However,  $\vec{v}$  in these equations is in general a function of space and time. In the context of this section it is specialized to the constant  $\vec{u}$ . Thus, in rewriting the convective derivatives of Eq. 2 the constancy of  $\vec{u}$  and a vector identity (Eq. 16, Appendix B) have been used.

So far, what has been said in this section is a matter of coordinates. Now, a physically motivated postulate is made concerning the electromagnetic laws. Imagine one electromagnetic experiment that is to be described from the two different reference frames. The postulate is that provided each of these frames is inertial, the governing laws must take the same form. Thus, Eqs. 23-27 apply with  $[\nabla \rightarrow \nabla', \partial(\ )/\partial t \rightarrow \partial(\ )/\partial t']$  and all dependent variables primed. By way of comparing these laws to those expressed in the fixed-frame, Eqs. 2 are used to rewrite these expressions in terms of the unprimed independent variables. Also, the moving-frame material velocity is rewritten in terms of the unprimed frame velocity using the relation

$$\vec{v}' = \vec{v} - \vec{u} \quad (3)$$

Thus, the laws originally expressed in the primed frame of reference become

$\nabla \cdot \epsilon_0 \vec{E}' = -\nabla \cdot \vec{P}' + \rho_f'$	$\nabla \times \vec{H}' = \vec{J}_f'$
$\nabla \times \vec{E}' = 0$	$\nabla \cdot \mu_0 \vec{H}' = -\nabla \cdot \mu_0 \vec{M}'$
$\nabla \cdot (\vec{J}_f' + \vec{u} \rho_f') + \frac{\partial \rho_f'}{\partial t} = 0$	$\nabla \times (\vec{E}' - \vec{u} \times \mu_0 \vec{H}') = -\frac{\partial \mu_0 \vec{H}'}{\partial t} - \frac{\partial \mu_0 \vec{M}'}{\partial t} - \mu_0 \nabla \times (\vec{M}' \times \vec{v})$
$\nabla \times (\vec{H}' + \vec{u} \times \epsilon_0 \vec{E}') = (\vec{J}_f' + \vec{u} \rho_f')$ $+ \frac{\partial \epsilon_0 \vec{E}'}{\partial t} + \frac{\partial \vec{P}'}{\partial t} + \nabla \times (\vec{P}' \times \vec{v})$	$\nabla \cdot \vec{J}_f' = 0$
$\nabla \cdot \mu_0 \vec{H}' = -\nabla \cdot \mu_0 \vec{M}'$	$\nabla \cdot \epsilon_0 \vec{E}' = -\nabla \cdot \vec{P}' + \rho_f'$

In writing Eq. 7a, Eq. 4a is used. Similarly, Eq. 5b is used to write Eq. 6b. For the one experiment under consideration, these equations will predict the same behavior as the fixed frame laws, Eqs. 2.3.23-27, if the identification is made:

## EQS

$$\vec{E}' = \vec{E}$$

$$\vec{P}' = \vec{P}$$

$$\rho_f' = \rho_f$$

$$\vec{J}'_f = \vec{J}_f - \vec{u}\rho_f$$

$$\vec{H}' = \vec{H} - \vec{u} \times \epsilon_0 \vec{E}$$

and hence, from Eq. 2.2.6

$$\vec{D}' = \vec{D}$$

## MQS

$$\vec{H}' = \vec{H} \quad (9)$$

$$\vec{M}' = \vec{M} \quad (10)$$

$$\vec{J}'_f = \vec{J}_f \quad (11)$$

$$\vec{E}' = \vec{E} + \vec{u} \times \mu_0 \vec{H} \quad (12)$$

and hence, from Eq. 2.2.7

$$\vec{B}' = \vec{B} \quad (14)$$

The primary fields are the same whether viewed from one frame or the other. Thus, the EQS electric field polarization density and charge density are the same in both frames, as are the MQS magnetic field, magnetization density and current density. The respective dynamic laws can be associated with those field transformations that involve the relative velocity. That the free current density is altered by the relative motion of the net free charge in the EQS system is not surprising. But, it is the contribution of this same convection current to Ampère's law that generates the velocity dependent contribution to the EQS magnetic field measured in the moving frame of reference. Similarly, the velocity dependent contribution to the MQS electric field transformation is a direct consequence of Faraday's law.

The transformations, like the quasistatic laws from which they originate, are approximate. It would require Lorentz transformations to carry out a similar procedure for the exact electrodynamic laws of Sec. 2.2. The general laws are not invariant in form to a Galilean transformation, and therein is the origin of special relativity. Built in from the start in the quasistatic field laws is a self-consistency with other Galilean invariant laws describing mechanical continua that will be brought in in later chapters.

## 2.6 Integral Theorems

Several integral theorems prove useful, not only in the description of electromagnetic fields but also in dealing with continuum mechanics and electromechanics. These theorems will be stated here without proof.

If it is recognized that the gradient operator is defined such that its line integral between two endpoints (a) and (b) is simply the scalar function evaluated at the endpoints, then<sup>1</sup>

$$\int_a^b \nabla\psi \cdot d\vec{\ell} = \psi(\vec{b}) - \psi(\vec{a}) \quad (1)$$

Two more familiar theorems<sup>1</sup> are useful in dealing with vector functions. For a closed surface S, enclosing the volume V, Gauss' theorem states that

$$\int_V \nabla \cdot \vec{A} dV = \oint_S \vec{A} \cdot \vec{n} da \quad (2)$$

while Stokes' theorem pertains to an open surface S with the contour C as its periphery:

$$\int_S \nabla \times \vec{A} \cdot \vec{n} da = \oint_C \vec{A} \cdot d\vec{\ell} \quad (3)$$

In stating these theorems, the normal vector is defined as being outward from the enclosed volume for Gauss' theorem, and the contour is taken as positive in a direction such that  $d\vec{\ell}$  is related to  $\vec{n}$  by the right-hand rule. Contours, surfaces, and volumes are sketched in Fig. 2.6.1.

A possibly less familiar theorem is the generalized Leibnitz rule.<sup>2</sup> In those cases where the surface is itself a function of time, it tells how to take the derivative with respect to time of the integral over an open surface of a vector function:

1. Markus Zahn, Electromagnetic Field Theory, a problem solving approach, John Wiley & Sons, New York, 1979, pp. 18-36.
2. H. H. Woodson and J. R. Melcher, Electromechanical Dynamics, Vol. 1. John Wiley & Sons, New York, 1968, pp. B32-B36. (See Prob. 2.6.2 for the derivation of this theorem.)

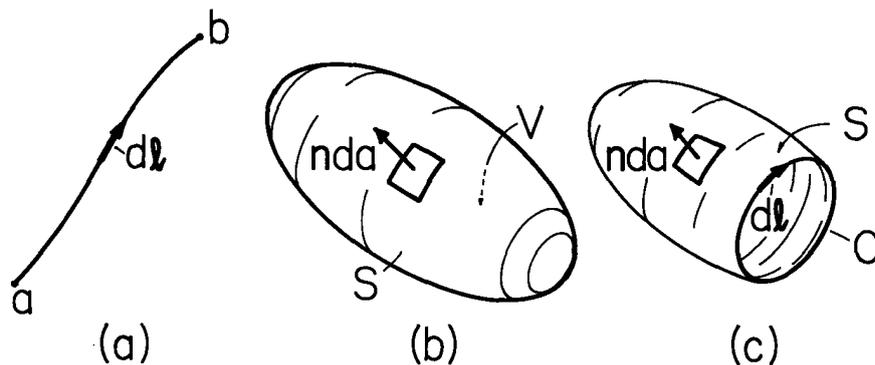


Fig. 2.6.1. Arbitrary contours, volumes and surfaces: (a) open contour C; (b) closed surface S, enclosing volume V; (c) open surface S with boundary contour C.

$$\frac{d}{dt} \int_S \vec{A} \cdot \vec{n} da = \int_S \left[ \frac{\partial \vec{A}}{\partial t} + (\nabla \cdot \vec{A}) \vec{v}_s \right] \cdot \vec{n} da + \oint_C (\vec{A} \times \vec{v}_s) \cdot d\vec{l} \quad (4)$$

Again, C is the contour which is the periphery of the open surface S. The velocity  $\vec{v}_s$  is the velocity of the surface and the contour. Unless given a physical significance, its meaning is purely geometrical.

A limiting form of the generalized Leibnitz rule will be handy in dealing with closed surfaces. Let the contour C of Eq. 4 shrink to zero, so that the surface S becomes a closed one. This process can be readily visualized in terms of the surface and contour sketch in Fig. 2.6.1c if the contour C is pictured as the draw-string on a bag. Then, if  $\zeta \equiv \nabla \cdot \vec{A}$ , and use is made of Gauss' theorem (Eq. 2), Eq. 4 becomes a statement of how to take the time derivative of a volume integral when the volume is a function of time:

$$\frac{d}{dt} \int_V \zeta dV = \int_V \frac{\partial \zeta}{\partial t} dV + \oint_S \zeta \vec{v}_s \cdot \vec{n} da \quad (5)$$

Again,  $\vec{v}_s$  is the velocity of the surface enclosing the volume V.

## 2.7 Quasistatic Integral Laws

There are at least three reasons for desiring Maxwell's equations in integral form. First, the integral equations are convenient for establishing jump conditions implied by the differential equations. Second, they are the basis for defining lumped parameter variables such as the voltage, charge, current, and flux. Third, they are useful in understanding (as opposed to predicting) physical processes. Since Maxwell's equations have already been divided into the two quasistatic systems, it is now possible to proceed in a straightforward way to write the integral laws for contours, surfaces, and volumes which are distorting, i.e., that are functions of time. The velocity of a surface S is  $\vec{v}_s$ .

To obtain the integral laws implied by the laws of Eqs. 2.3.23-27, each equation is either (i) integrated over an open surface S with Stokes's theorem used where the integrand is a curl operator to convert to a line integration on C and Eq. 2.6.4 used to bring the time derivative outside the integral, or (ii) integrated over a closed volume V with Gauss' theorem used to convert integrations of a divergence operator to integrals over closed surfaces S and Eq. 2.6.5 used to bring the time derivative outside the integration:

$$\oint_S (\epsilon_0 \vec{E} + \vec{P}) \cdot \vec{n} da = \int_V \rho_f dV$$

$$\oint_C \vec{E} \cdot d\vec{l} = 0$$

$$\oint_S \vec{J}_f \cdot \vec{n} da + \frac{d}{dt} \int_V \rho_f dV = 0$$

$$\oint_C \vec{H} \cdot d\vec{l} = \int_S \vec{J}_f \cdot \vec{n} da \quad (1)$$

$$\oint_S \mu_0 (\vec{H} + \vec{M}) \cdot \vec{n} da = 0 \quad (2)$$

$$\begin{aligned} \oint_C \vec{E} \cdot d\vec{l} = & - \frac{d}{dt} \int_S \mu_0 (\vec{H} + \vec{M}) \cdot \vec{n} da \\ & - \oint_C \mu_0 \vec{M} \times (\vec{v} - \vec{v}_s) \cdot d\vec{l} \end{aligned} \quad (3)$$

$$\oint_C \vec{H}' \cdot d\vec{\ell} = \int_S \vec{J}'_f \cdot \vec{n} da + \frac{d}{dt} \int_S (\epsilon_0 \vec{E} + \vec{P}) \cdot \vec{n} da \quad \left| \quad \oint_S \vec{J}'_f \cdot \vec{n} da = 0 \right. \quad (4)$$

$$+ \oint_C \vec{P} \times (\vec{v} - \vec{v}_s) \cdot d\vec{\ell}$$

$$\oint_S \mu_0 (\vec{H} + \vec{M}) \cdot \vec{n} da = 0 \quad \left| \quad \oint_S (\epsilon_0 \vec{E} + \vec{P}) \cdot \vec{n} da = \int_V \rho_f dV \right. \quad (5)$$

where

$$\vec{J}'_f = \vec{J}_f - \vec{v}_s \rho_f$$

$$\vec{H}' = \vec{H} - \vec{v}_s \times \epsilon_0 \vec{E}$$

where

$$\vec{E}' = \vec{E} + \vec{v}_s \times \mu_0 \vec{H}$$

The primed variables are simply summaries of the variables found in deducing these equations. However, these definitions are consistent with the transform relationships found in Sec. 2.5, and the velocity of these surfaces and contours,  $\vec{v}_s$ , can be identified with the velocity of an inertial frame instantaneously attached to the surface or contour at the point in question. Approximations implicit to the original differential quasistatic laws are now implicit to these integral laws.

### 2.8 Polarization of Moving Media

Effects of polarization and magnetization are included in the formulation of electrodynamics postulated in Sec. 2.2. In this and the next section a review is made of the underlying models.

Consider the electroquasistatic systems, where the dominant field source is the charge density. Not all of this charge is externally accessible, in the sense that it cannot all be brought to some position through a conduction process. If an initially neutral dielectric medium is stressed by an electric field, the constituent molecules and domains become polarized. Even though the material retains its charge neutrality, there can be a local accrual or loss of charge because of the polarization. The first order of business is to deduce the relation of such polarization charge to the polarization density.

For conceptual purposes, the polarization of a material is pictured as shown in Fig. 2.8.1.

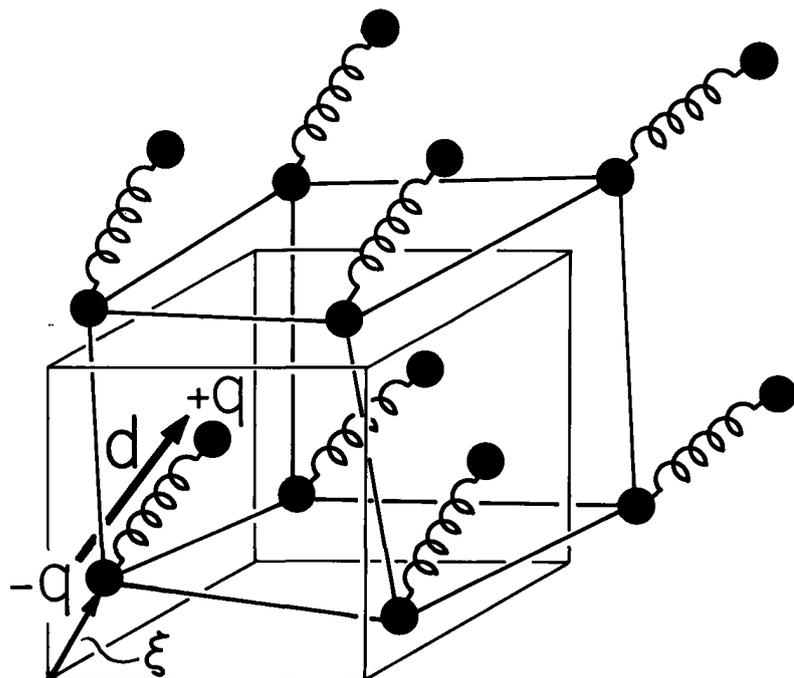


Fig. 2.8.1. Model for dipoles fixed to deformable material. The model pictures the negative charges as fixed to the material, and then the positive halves of the dipoles fixed to the negative charges through internal constraints.

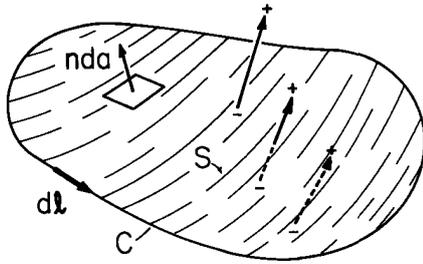


Fig. 2.8.2

Polarization results in net charges passing through a surface.

The molecules or domains are represented by dipoles composed of positive and negative charges  $\pm q$ , separated by the vector distance  $\vec{d}$ . The dipole moment is then  $\vec{p} = q\vec{d}$ , and if the particles have a number density  $n$ , the polarization density is defined as

$$\vec{P} = nq\vec{d} \quad (1)$$

In the most common dielectrics, the polarization results because of the application of an external electric field. In that case, the internal constraints (represented by the springs in Fig. 2.8.1) make the charges essentially coincident in the absence of an electric field, so that, on the average, the material is (macroscopically) neutral. Then, with the application of the electric field, there is a separation of the charges in some direction which might be coincident with the applied electric field intensity. The effect of the dipoles on the average electric field distribution is equivalent to that of the medium they model.

To see how the polarization charge density is related to the polarization density, consider the motion of charges through the arbitrary surface  $S$  shown in Fig. 2.8.2. For the moment, consider the surface as being closed, so that the contour enclosing the surface shown is shrunk to zero. Because polarization results in motion of the positive charge, leaving behind the negative image charge, the net polarization charge within the volume  $V$  enclosed by the surface  $S$  is equal to the negative of the net charge having left the volume across the surface  $S$ . Thus,

$$\int \rho_p dV = - \oint_S nq\vec{d} \cdot \vec{n} da = - \oint_S \vec{P} \cdot \vec{n} da \quad (2)$$

Gauss' theorem, Eq. 2.6.2, converts the surface integral to one over the arbitrary volume  $V$ . It follows that the integrand must vanish so that

$$\rho_p = - \nabla \cdot \vec{P} \quad (3)$$

This polarization charge density is now added to the free charge density as a source of the electric field intensity in Gauss' law:

$$\nabla \cdot \epsilon_0 \vec{E} = \rho_f + \rho_p \quad (4)$$

and Eqs. 3 and 4 comprise the postulated form of Gauss' law, Eq. 2.3.23a.

By definition, polarization charge is conserved, independent of the free charge. Hence, the polarization current  $\vec{J}_p$  is defined such that it satisfies the conservation equation

$$\nabla \cdot \vec{J}_p + \frac{\partial \rho_p}{\partial t} = 0 \quad (5)$$

To establish the way in which  $\vec{J}_p$  transforms between inertial reference frames, observe that in a primed frame of reference, by dint of Eq. 2.5.2c, the conservation of polarization charge equation becomes

$$\nabla \cdot [\vec{J}'_p + \vec{u}\rho'_p] + \frac{\partial \rho'_p}{\partial t} = 0 \quad (6)$$

It has been shown that  $\vec{P}$ , and hence  $\rho_p$ , are the same in both frames (Eq. 2.5.10a). It follows that the required transformation law is

$$\vec{J}'_p = \vec{J}_p - \vec{u}\rho_p \quad (7)$$

If the dipoles are attached to a moving medium, so that the negative charges move with the same velocity  $\vec{v}$  as the moving material, the motion gives rise to a current which should be included in Ampère's law as a source of magnetic field. Even if the material is fixed, but the applied field is

time-varying so as to induce a time-varying polarization density, a given surface is crossed by a net charge and there is a current caused by a time-varying polarization density. The following steps determine the current density  $\vec{J}_p$  in terms of the polarization density and the material velocity.

The starting point is the statement

$$\int_S \vec{J}'_p \cdot \vec{n} da = \frac{d}{dt} \int_S \vec{P} \cdot \vec{n} da \quad (8)$$

The surface S, depicted by Fig. 2.8.2, is attached to the material itself. It moves with the negative charges of the dipoles. Integrated over this deforming surface of fixed identity, the polarization current density evaluated in the frame of reference of the material is equal to the rate of change with respect to time of the net charge penetrating that surface.

With the surface velocity identified with the material velocity, Eq. 2.6.4 and Eq. 3 convert Eq. 8 to

$$\int_S \vec{J}'_p \cdot \vec{n} da = \int_S \left( \frac{\partial \vec{P}}{\partial t} - \rho_p \vec{v} \right) \cdot \vec{n} da + \oint_C \vec{P} \times \vec{v} \cdot d\vec{\ell} \quad (9)$$

On the left,  $\vec{J}'_p$  is replaced by Eq. 7 evaluated with  $\vec{u} = \vec{v}$ , while on the right Stokes's theorem, Eq. 2.6.3, is used to convert the line integral to a surface integral. The result is an equation in surface integrals alone. Although fixed to the deforming material, the surface S is otherwise arbitrary and so it follows that the required relation between  $\vec{J}_p$  and  $\vec{P}$  for the moving material is

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t} + \nabla \times (\vec{P} \times \vec{v}) \quad (10)$$

It is this current density that has been added to the right-hand side of Ampère's law, Eq. 2.3.26a, to complete the formulation of polarization effects in the electroquasistatic system.

## 2.9 Magnetization of Moving Media

It is natural to use polarization charge to represent the effect of macroscopic media on the macroscopic electric field. Actually, this is one of two alternatives for representing polarization. That such a choice has been made becomes clear when the analogous question is asked for magnetization. In the absence of magnetization, the free current density is the source of the magnetic field, and it is therefore natural to represent the macroscopic effects of magnetizable media on  $\vec{H}$  through an equivalent magnetization current density. Indeed, this viewpoint is often used and supported by the contention that what is modeled at the atomic level is really a system of currents (the electrons in their orbits). It is important to understand that the use of equivalent currents, or of equivalent magnetic charge as used here, if carried out self-consistently, results in the same predictions of physical processes. The choice of models in no way hinges on the microscopic processes accounting for the magnetization. Moreover, the magnetization is often dominated by dynamical processes that have more to do with the behavior of domains than with individual atoms, and these are most realistically pictured as small magnets (dipoles). With the Chu formulation postulated in Sec. 2.2, the dipole model for representing magnetization has been adopted.

An advantage of the Chu formulation is that magnetization is developed in analogy to polarization. But rather than starting with a magnetic charge density, and deducing its relation to the polarization density, think of the magnetic material as influencing the macroscopic fields through an intrinsic flux density  $\mu_0 \vec{M}$  that might be given, or might be itself induced by the macroscopic  $\vec{H}$ . For lack of evidence to support the existence of "free" magnetic monopoles, the total flux density due to all macroscopic fields must be solenoidal. Hence, the intrinsic flux density  $\mu_0 \vec{M}$ , added to the flux density in free space  $\mu_0 \vec{H}$ , must have no divergence:

$$\nabla \cdot \mu_0 (\vec{H} + \vec{M}) = 0 \quad (1)$$

This is Eq. 2.3.24b. It is profitable to think of  $-\nabla \cdot \mu_0 \vec{M}$  as a source of  $\vec{H}$ . That is, Eq. 1 can be written to make it look like Gauss' law for the electric field:

$$\nabla \cdot \mu_0 \vec{H} = \rho_m; \quad \rho_m = -\nabla \cdot \mu_0 \vec{M} \quad (2)$$

The magnetic charge density  $\rho_m$  is in this sense the source of the magnetic field intensity.

Faraday's law of induction must be revised if magnetization is present. If  $\mu_0 \vec{M}$  is a magnetic flux density, then, through magnetic induction, its rate of change is capable of producing an induced electric field intensity. Also, if Faraday's law of induction were to remain valid without alteration, then its divergence must be consistent with Eq. 1; obviously, it is not.

To generalize the law of induction to include magnetization, it is stated in integral form for a contour C enclosing a surface S fixed to the material in which the magnetized entities are imbedded. Then, because  $\mu_0(\vec{H} + \vec{M})$  is the total flux density,

$$\oint_C \vec{E}' \cdot d\vec{\ell} = - \frac{d}{dt} \int_S \mu_0(\vec{H} + \vec{M}) \cdot \vec{n} da \quad (3)$$

The electric field  $\vec{E}'$  is evaluated in the frame of reference of the moving contour. With the time derivative taken inside the temporally varying surface integrals (Eq. 2.6.4) and because of Eq. 1,

$$\oint_C \vec{E}' \cdot d\vec{\ell} = - \int_S \frac{\partial}{\partial t} [\mu_0(\vec{H} + \vec{M})] \cdot \vec{n} da + \int_S \nabla \times [\vec{v} \times \mu_0(\vec{H} + \vec{M})] \cdot \vec{n} da \quad (4)$$

The transformation law for  $\vec{E}$  (Eq. 2.5.12b with  $\vec{u} = \vec{v}$ ) is now used to evaluate  $\vec{E}'$ , and Stokes's theorem, Eq. 2.6.3, used to convert the line integral to a surface integral. Because S is arbitrary, it then follows that the integrand must vanish:

$$\nabla \times \vec{E} = - \frac{\partial}{\partial t} [\mu_0(\vec{H} + \vec{M})] + \nabla \times (\vec{v} \times \mu_0 \vec{M}) \quad (5)$$

This generalization of Faraday's law is the postulated equation, Eq. 2.3.25b.

### 2.10 Jump Conditions

Systems having nonuniform properties are often modeled by regions of uniform properties, separated by boundaries across which these properties change abruptly. Fields are similarly often given a piecewise representation with jump conditions used to "splice" them together at the discontinuities. These conditions, derived here for reference, are implied by the integral laws. They guarantee that the associated differential laws are satisfied through the singular region of the discontinuity.

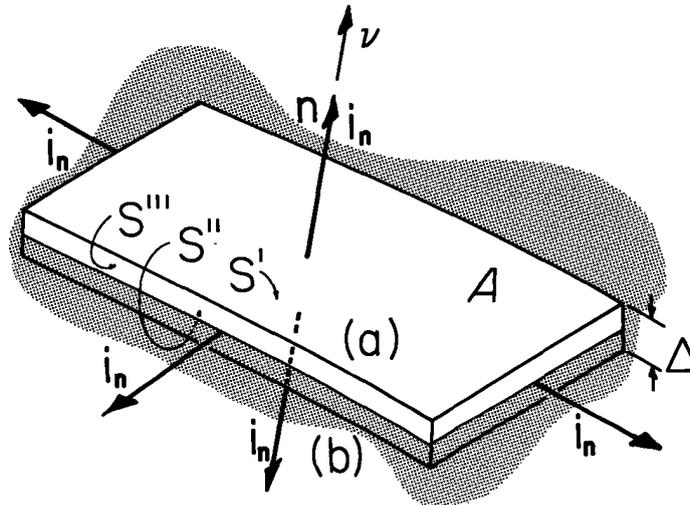


Fig. 2.10.1. Volume element enclosing a boundary. Dimensions of area A are much greater than  $\Delta$ .

**Electroquasistatic Jump Conditions:** A section of the boundary can be enclosed by a volume element having the thickness  $\Delta$  and cross-sectional area A, as depicted by Fig. 2.10.1. The linear dimensions of the cross-sectional area A are, by definition, much greater than the thickness  $\Delta$ . Implicit to this statement is the assumption that, although the surface can be curvilinear, its radius of curvature must be much greater than a characteristic thickness over which variations in the properties and fields take place.

The normal vector  $\vec{n}$  used in this section is a unit vector perpendicular to the boundary and direct from region b to region a, as shown in Fig. 2.10.1. Since this same symbol is used in connection with integral theorems and laws to denote a normal vector to surfaces of integration, these latter vectors are denoted by  $\vec{i}_n$ .

First, consider the boundary conditions implied by Gauss' law, Eq. 2.3.23a, with Eq. 2.8.3 used to introduce  $\rho_p$ . This law is first multiplied by  $v^m$  and then integrated over the volume V:

$$\int_V v^m \nabla \cdot \epsilon_0 \vec{E} dV = \int_V v^m \rho_f dV + \int_V v^m \rho_p dV \quad (1)$$

Here,  $v$  is a coordinate (like  $x, y$ , or  $z$ ) perpendicular to the boundary and hence in the direction of  $\vec{n}$ , as shown in Fig. 2.10.1.

First, consider the particular case of Eq. 1 with  $m = 0$ . Then, the integration gives

$$\vec{n} \cdot \llbracket \epsilon_0 \vec{E} \rrbracket = \sigma_f + \sigma_p \quad (2)$$

where  $\llbracket \vec{A} \rrbracket \equiv \vec{A}^a - \vec{A}^b$  and  $\llbracket \psi \rrbracket \equiv \psi^a - \psi^b$  and the free surface charge density  $\sigma_f$  and polarization surface charge density  $\sigma_p$  have been defined as

$$\sigma_f = \lim_{A \rightarrow 0} \frac{1}{A} \int \rho_f dV, \quad \sigma_p = \lim_{A \rightarrow 0} \frac{1}{A} \int \rho_p dV \quad (3)$$

The relationship between the surface charge and the electric field intensity normal to the boundary can be pictured as shown in Fig. 2.10.2b.

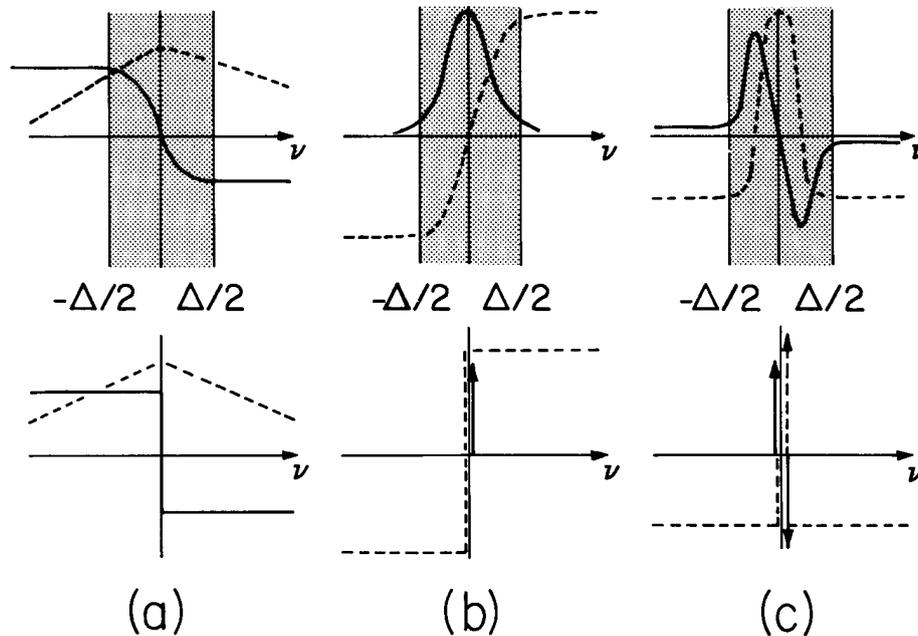


Fig. 2.10.2. Sketches of the charge distribution represented by the solid lines, and the electric field intensity normal to the boundary represented by broken lines. Sketches at the top represent actual distributions, while those below represent idealizations appropriate if the thickness  $\Delta$  of the region over which the electric field intensity makes its transition is small compared to other dimensions of interest: (a) volume charge density to either side of interface but no surface charge; (b) surface charge; (c) double layer.

In view of Eq. 2, the normal electric field intensity is continuous at the interface unless there is a singularity in charge. Thus, with volume charges to either side of the interface, there is an abrupt change in the rate of change of the electric field intensity normal to the boundary, but the field is itself continuous. On the other hand, as illustrated by the sketches of Fig. 2.10.2b, if there is an appreciable charge per unit area within the boundary, the electric field intensity is discontinuous, and undergoes a step discontinuity.

A somewhat less familiar situation is that of Fig. 2.10.2c. Within the boundary there are regions of large positive and negative charge concentrations with an associated intense electric field between. In the limit where the boundary becomes very thin, a component of the surface charge density becomes a doublet, and the electric field becomes an impulse.

The double layer can be pictured as being positive surface charges disposed on one side of the boundary, and negative surface charges distributed on the other, with an internal component of the electric field originating on the positive charges and terminating on the negative ones. The magnitude of the double layer is equal to the product of the positive surface charge density and the distance between these layers,  $\Delta$ . In the limit where the layer thickness becomes infinitely thin while the double-layer magnitude remains constant, the electric field within the double layer must approach infinity. Thus, associated with the doublet of charge density, there is an impulse in the electric field intensity, as sketched in Fig. 2.10.2c.

The boundary condition to be used in connection with a double layer is found from Eq. 1 by letting  $m = 1$ . The left-hand side of Eq. 1 can be integrated by parts, so that it becomes

$$\int_V \nabla \cdot (\epsilon_0 \nabla \vec{E}) dV - \int_V \epsilon_0 \vec{E} \cdot \nabla \nu dV = \int_V \nu (\rho_f + \rho_p) dV \quad (4)$$

For the incremental volume, the surface double layer density is defined as

$$p_\Sigma = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \int \nu (\rho_f + \rho_p) dV = \int_{\nu_-}^{\nu_+} \nu (\rho_f + \rho_p) d\nu \quad (5)$$

and so the right-hand side of Eq. 4 is  $A p_\Sigma$ . The origin of the  $\Delta$  axis remains to be defined but  $\Delta \equiv \nu_+ - \nu_-$ . To glean a jump condition from the equation, the second EQS law is incorporated. That  $\vec{E}$  is irrotational Eq. 2.3.24a, is represented by defining the electric potential

$$\vec{E} = -\nabla \phi \quad (6)$$

Thus, the second term on the left in Eq. 4 becomes

$$\int_V \epsilon_0 \vec{E} \cdot \nabla \nu dV = - \int_V \epsilon_0 \nabla \phi \cdot \nabla \nu dV = - \int_V \epsilon_0 \nabla \cdot (\phi \nabla \nu) dV + \int_V \epsilon_0 \phi \nabla^2 \nu dV \quad (7)$$

Evaluation of  $\nabla^2 \nu$  gives nothing because  $\nu$  is defined as a local Cartesian coordinate. The last integral vanishes, and with the application of Gauss' theorem, Eq. 2.6.2, it follows that Eq. 4 becomes

$$\oint_S \epsilon_0 \nabla \vec{E} \cdot \vec{i}_n da + \oint_S \epsilon_0 \phi \nabla \nu \cdot \vec{i}_n da = A p_\Sigma \quad (8)$$

Provided that within the layer,  $\vec{E}$  parallel to the interface and  $\phi$  are finite (not impulses in the limit  $\Delta \rightarrow 0$ ), Eq. 8 only has contributions to the surface integrals from the regions to either side of the interface. Thus,

$$A \epsilon_0 (\nu_+ \vec{E}^a - \nu_- \vec{E}^b) \cdot \vec{n} + A \epsilon_0 [\phi] = A p_\Sigma \quad (9)$$

The origin of the  $\nu$  axis is adjusted to make the first term vanish. The required boundary condition to be associated with Eqs. 2.3.23a and 2.3.23b is

$$\epsilon_0 [\phi] = p_\Sigma \quad (10)$$

The gradient of Eq. 10 within the plane of the interface converts the jump condition to one in terms of the electric field:

$$\epsilon_0 [\vec{E}_t] = -\nabla_\Sigma p_\Sigma \quad (11)$$

Here  $\nabla_\Sigma$  is the surface gradient and t denotes components tangential to the interfacial plane.

In the absence of a double-layer surface density, these last two boundary conditions are the familiar statement that the tangential electric field intensity at a boundary must be continuous. The statement given in Eq. 10 that the potential must be continuous at a boundary is another way of stating

this requirement on the tangential electric field intensity. With a double layer, the tangential electric field intensity is discontinuous, as is also the potential.

Equations 10 and 11 could also be derived using the condition that the line integral of the electric field intensity around a closed loop intersecting the boundary vanish. Usually, the tangential electric field is continuous because there is no contribution to this line integral from those segments of the contour passing through the boundary. However, with the double layer, the electric field intensity within the boundary is infinite; so, even though the segments of the line integral across the boundary vanish as  $\Delta \rightarrow 0$ , there is a net contribution from these segments of the integration.

It is clear that higher order singularities could also be handled by considering values of  $m$  in Eq. 1 greater than unity. However, the doublet is as singular a charge distribution as of interest physically.

There are two reasons for wishing to include the doublet charge distribution, one mathematical and one physical. Just as the surface charge density is a singularity in the volume charge density which can be used to terminate a normal electric field intensity at a boundary, the double layer is a termination of a tangential electric field. On the physical side, there are many situations in which a double layer actually exists within a very thin region of material. Double layers abound at interfaces between liquids and metals and between metals. The double-layer concept is useful for modeling electromechanical coupling involving these interfacial regions.

So far, those EQS laws have been considered that do not explicitly involve time rates of change. Conservation of charge does involve a dynamic term. Its associated boundary conditions can therefore be derived only by making further stipulations as to the nature of the boundary. It is now admitted that the boundary can, in general, be one which is deforming. Because time did not appear explicitly in the previous derivations of this section, the conditions derived are automatically appropriate, even if the boundary is moving.

The integral form of charge conservation, Eq. 2.7.3a, is written for a volume  $V$  and surface  $S$  tied to the material itself. Thus, with  $\vec{v}_s \rightarrow \vec{v}$ ,

$$\oint_S (\vec{J}_f - \rho_f \vec{v}) \cdot \vec{i}_n da = - \frac{d}{dt} \int_V \rho_f dV \quad (12)$$

As seen in Fig. 2.10.1, the volume of integration always encloses material of fixed identity and intersects the boundary. Implicit to this statement is the assumption that the boundary is one of demarcation between material regions. The material velocity is presumed to at most have a step singularity across the boundary. (It is important to recognize that there are other types of boundaries. For example, the boundary could be a shock front, with a gas moving through from one side of the interface to the other. In that case, the boundary conditions thus far derived would remain correct, because no mention has yet been made of the physical nature of the boundary.)

The left-hand side of Eq. 12 can be handled in a manner similar to that already illustrated, since it does not involve time rates of change. The integration is divided into two parts: one over the upper and lower surfaces of the volume, the other over the parts of the surface which intersect the boundary. The contributions to a current flow through these side surfaces comes from a surface current. It follows by using a two-dimensional form of Gauss' theorem, Eq. 2.6.2, that the left-hand side of Eq. 12 is

$$\int_{S'+S''} (\vec{J}_f - \rho_f \vec{v}) \cdot \vec{i}_n da + \int_{S'''} (\vec{J}_f - \rho_f \vec{v}) \cdot \vec{i}_n da = A \{ \vec{n} \cdot [ \vec{J}_f - \vec{v} \rho_f ] + \nabla_{\Sigma} \cdot (\vec{k}_f - \sigma_f \vec{v}_t) \} \quad (13)$$

Here,  $A$  is the area of intersection between the volume element and the boundary. The right-hand side of Eq. 12 is, by the definition of Eq. 3,

$$\frac{d}{dt} \int_V \rho_f dV = \frac{d}{dt} \int_A \sigma_f da \quad (14)$$

Note that, if the volume of integration  $V$ , and hence the area of integration  $A$ , is one always fixed to the material, then the area  $A$  is time-varying. The surface charge density is a function only of the two dimensions within the plane of the interface. Thus, the term on the right in Eq. 14 is a time derivative of a two-dimensional integral. This is a two-dimensional special case of the situation described by the generalized Leibnitz rule, Eq. 2.6.5, which stated how the time derivative of a volume integral could be represented, even if the volume of integration were time-varying. Thus, Eq. 14 becomes

$$\frac{d}{dt} \int_V \rho_f dV = A \left[ \frac{\partial \sigma_f}{\partial t} + \nabla_{\Sigma} \cdot (\vec{v}_t \sigma_f) \right] \quad (15)$$

Finally, with the use of Eqs. 13 and 15, Eq. 12 becomes the required jump condition representing charge conservation:

$$\vec{n} \cdot \llbracket \vec{J}_f - \rho_f \vec{v} \rrbracket + \nabla_{\Sigma} \cdot \vec{K}_f = - \frac{\partial \sigma_f}{\partial t} \quad (16)$$

By contrast with Eqs. 10 and 11, the expression is specialized to interfaces that do not support charge distributions so singular as a double layer. In using Eq. 16, note that a partial derivative with respect to time is usually defined as one taken holding the spatial coordinates constant. A review of the derivation of Eq. 16 will make it clear that such is not the significance of the partial derivative on the right in Eq. 16. The surface charge density is not defined throughout the three-dimensional space. Thus, this derivative means the partial derivative with respect to time, holding the coordinates within the plane of the interface constant.

The component of current normal to the boundary represented by the first term in Eq. 16 will be recognized as the free current density in a frame of reference moving with the boundary. A good question would be, "why is it that the normal current density appears in Eq. 16 evaluated in the primed frame of reference, while the surface free current density is not?" The answer points to the physical situation for which Eq. 16 is appropriate. As the material boundary moves in the normal direction, the material ahead and behind carries a charge distribution along, but one that never reaches the boundary. By contrast, materials can flow in and out within the surface of the volume of interest, and carry with them a surface charge density of a convective nature. Thus, the surface divergence appearing in the second term of Eq. 16 can include both a conduction surface current and a convection surface current.

Magnetoquasistatic Jump Conditions: The integral forms of Ampère's law and Gauss' law for magnetic fields incorporate no time rates of change. Hence, the jump conditions implied by these laws are familiar from elementary electrodynamics. Ampère's law, Eq. 2.7.1b, is integrated over the surface S and around the contour C enclosing the boundary, as sketched in Fig. 2.10.3, to obtain

$$\vec{n} \times \llbracket \vec{H} \rrbracket = \vec{K}_f \quad (17)$$

where  $\vec{K}_f$  is the surface current density. Although it is entirely possible to consider a doublet of current density as a model, this impulsive singularity in the distribution of free current density is of as high an order as necessary to model MQS electromechanical situations of general interest.

From Gauss' law for magnetic fields, Eq. 2.7.2b, applied to the incremental volume enclosing the interface, Fig. 2.10.1, the jump condition is

$$\vec{n} \cdot \llbracket \mu_0 (\vec{H} + \vec{M}) \rrbracket = 0 \quad (18)$$

Faraday's law of induction brings into play the time rate of change, and it is expected that motion of the boundary leads to an addition to the jump condition not found for stationary media. According to Eq. 2.7.3b, the integral form of Faraday's law, for a contour fixed to the material (of fixed identity) so that  $\vec{v}_S \rightarrow \vec{v}$ , is

$$\oint_C (\vec{E} + \vec{v} \times \mu_0 \vec{H}) \cdot d\vec{l} = - \frac{d}{dt} \int_S \mu_0 (\vec{H} + \vec{M}) \cdot \vec{n} da \quad (19)$$

With Eq. 19, it has already been assumed that the boundary is a material one. Consistent with Eq. 17 is the assumption that it can be carrying a surface current with it as it deforms. If the surface S were not one of fixed identity, this would mean that the surface integral on the right could be a step function of time as the boundary passed through the surface of integration. The result would be a temporal impulse on the right which would make a contribution to the boundary condition even in the limit where the surface S becomes vanishingly small. By contrast, because the surface S is one of fixed identity, in the limit where the surface area vanishes, the right-hand side of Eq. 19 makes no contribution.

With the assumption that fields and velocity are at most step functions across the boundary, the integral on the left in Eq. 19 gives

$$\vec{n} \times \llbracket \vec{E} + \vec{v} \times \mu_0 \vec{H} \rrbracket = 0 \quad (20)$$

This expression is what would be expected, in view of the transformation law for the electric field in

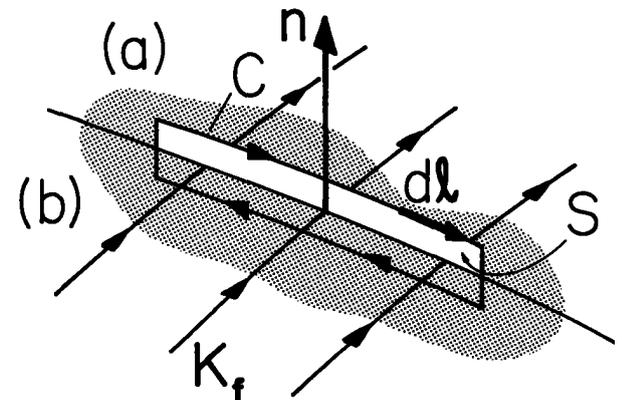


Fig. 2.10.3. Contour of integration C enclosing a surface S that intersects the boundary between regions (a) and (b).

the MQS system. It states that  $\vec{E}_t'$  is continuous across the interface.

Summary of Electroquasistatic and Magnetoquasistatic Conditions: Table 2.10.1 summarizes the jump conditions.

Table 2.10.1. Quasistatic jump conditions;  $\llbracket \vec{A} \rrbracket \equiv \vec{A}^a - \vec{A}^b$ .

EQS	MQS
$\vec{n} \cdot \llbracket \epsilon_0 \vec{E} + \vec{P} \rrbracket = \sigma_f$ $\vec{n} \cdot \llbracket \vec{P} \rrbracket = -\sigma_p$	$\vec{n} \times \llbracket \vec{H} \rrbracket = \vec{K}_f \quad (21)$
$\epsilon_0 \llbracket \Phi \rrbracket = \sigma_d$ $\epsilon_0 \llbracket \vec{E}_t \rrbracket = -\nabla_{\Sigma} \sigma_d$	$\vec{n} \cdot \mu_0 \llbracket \vec{H} + \vec{M} \rrbracket = 0 \quad (22)$ $\vec{n} \cdot \mu_0 \llbracket \vec{M} \rrbracket = -\sigma_m$
$\vec{n} \cdot \llbracket \vec{J}_f - \rho_f \vec{v} \rrbracket + \nabla_{\Sigma} \cdot \vec{K}_f = -\frac{\partial \sigma_f}{\partial t}$	$\vec{n} \times \llbracket \vec{E} + \vec{v} \times \mu_0 \vec{H} \rrbracket = 0 \quad (23)$
$\vec{n} \times \llbracket \vec{H} - \vec{v} \times \epsilon_0 \vec{E} \rrbracket = \vec{K}_f - \sigma_f \vec{v}_t$	$\vec{n} \cdot \llbracket \vec{J}_f \rrbracket = 0 \quad (24)$

Included in the summary are several that are either rarely used, are matters of definition, or are obvious. That the surface polarization charge and surface magnetic charge are related to  $\vec{P}$  and  $\vec{M}$  respectively follows from Eqs. 2.8.3 and 2.9.2 used in conjunction with Gauss' theorem and the elemental volume of Fig. 2.10.1. Similarly, Eq. 24b follows from the solenoidal nature of the MQS current density. Finally, Eq. 24a follows from the EQS form of Ampère's law, integrated over the surface S of Fig. 2.10.3, following the line of reasoning used in connection with Eq. 20.

### 2.11 Lumped Parameter Electroquasistatic Elements

Lumped parameter electromechanical models are sufficiently practical that they warrant detailed examination.<sup>1</sup> Even though the electromechanical coupling may be of a definitely continuum and distributed nature, it is most often the case that interest is in inputs and outputs at discrete terminal pairs. This section reviews the definition of energy storage elements in EQS systems.

An abstract representation of a system of perfectly conducting electrodes, each having a potential  $v_i$  relative to a reference electrode, is shown in Fig. 2.11.1. Not only are the electrodes and their connecting leads perfectly conducting, but the environment surrounding them is perfectly insulating.

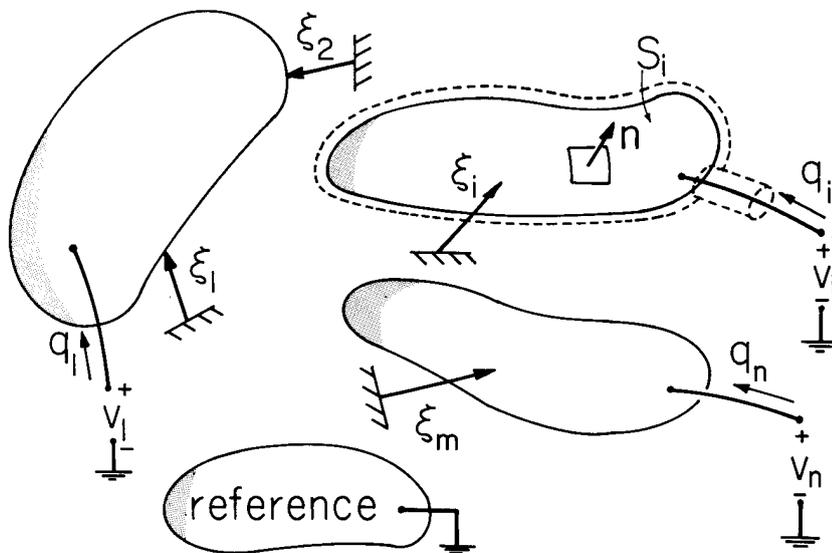


Fig. 2.11.1

Schematic view of an electrode system consisting of  $n$  electrodes composed of perfect conductors and immersed in a perfectly insulating medium.

1. H. H. Woodson and J. R. Melcher, Electromechanical Dynamics, Vol. I, John Wiley & Sons, New York, 1968.

The charge on each of the  $n$  electrodes is the free charge density integrated over a volume enclosing the electrode:

$$q_i \equiv \int_{V_i} \rho_f dV = \oint_{S_i} \vec{D} \cdot \vec{n} da \quad (1)$$

The total charge on an electrode is indicated by an arrow pointing toward the electrode from the terminal pair attached to that electrode. The associated voltage is defined in terms of the electric field and potential by

$$v_i = - \int_{\text{ref}}^{(i)} \vec{E} \cdot \vec{d}\ell = \phi_i - \phi_{\text{ref}} = \phi_i \quad (2)$$

This relation is justified because the electric field is irrotational and hence the negative gradient of  $\phi$ .

Given the geometry of the electrodes at a certain instant in time, displacements  $\xi_1 \cdots \xi_j \cdots \xi_m$  are known, and the condition that the field be irrotational and satisfy Gauss' law leads to equations that can in principle be used to determine the charges on the individual electrodes at a given instant:

$$q_i = q_i(v_1 \cdots v_n, \xi_1 \cdots \xi_m) \quad (3)$$

If the dielectrics are electrically linear in the sense that  $\vec{D} = \epsilon \vec{E}$ , where  $\epsilon$  is a function of position but not of time or the field, then it is useful to define a capacitance

$$C_{ij} = \left. \frac{q_i}{v_j} \right|_{v_{i \neq j} = 0} = \frac{\oint_{S_i} \epsilon \vec{E} \cdot \vec{n} da}{-\int_{\text{ref}}^j \vec{E} \cdot \vec{d}\ell} \quad (4)$$

The capacitance of the  $i$ th electrode relative to the  $j$ th electrode is the charge on the  $i$ th electrode per unit voltage on the  $j$ th electrode, with all other electrodes held at zero voltage. The capacitance is useful as a parameter because the charge on an electrode in a linear dielectric is proportional to the voltage itself; hence, the capacitance is purely a function of the electrical properties of the system and the geometry:

$$q_i = \sum_{j=1}^n C_{ij} v_j, \quad C_{ij} = C_{ij}(\xi_1 \cdots \xi_m) \quad (5)$$

To define the capacitance as with Eqs. 4 and 5, no reference is required to the time rate of change. In these relations  $q_i$ ,  $v_i$ , and  $\xi_i$  can all be functions of time. The dynamics enter by virtue of conservation of charge, which can be written for a volume including the  $i$ th electrode as (Eq. 2.7.3a):

$$\oint_{S_i} \vec{J}_f' \cdot \vec{n} da = - \frac{d}{dt} \int_{V_i} \rho_f dV \quad (6)$$

The quantity on the right in this expression is the negative of the time rate of change of the total free charge on the  $i$ th electrode. The only free current density normal to a surface enclosing the electrode is that through the wire itself. Note that the normal vector is defined as outward from this surface, while a positive current through the wire flows inward. Hence, the left-hand side of Eq. 6 becomes the negative of the total current at the  $i$ th electrical terminal pair:

$$i_i = \frac{dq_i}{dt} \quad (7)$$

With the charge given as a function of the voltages and the geometry by Eq. 3, or in particular by Eq. 5, Eq. 7 can be used to compute the current flowing into a given terminal of the electrode system.

## 2.12 Lumped Parameter Magnetoquasistatic Elements

An extremely practical idealization of lumped parameter magnetoquasistatic systems is sketched schematically in Fig. 2.12.1. Perfectly conducting coils are excited at their terminals by currents  $i_1$  and, in general, coupled together by the induced magnetic flux. The surrounding medium is magnetizable

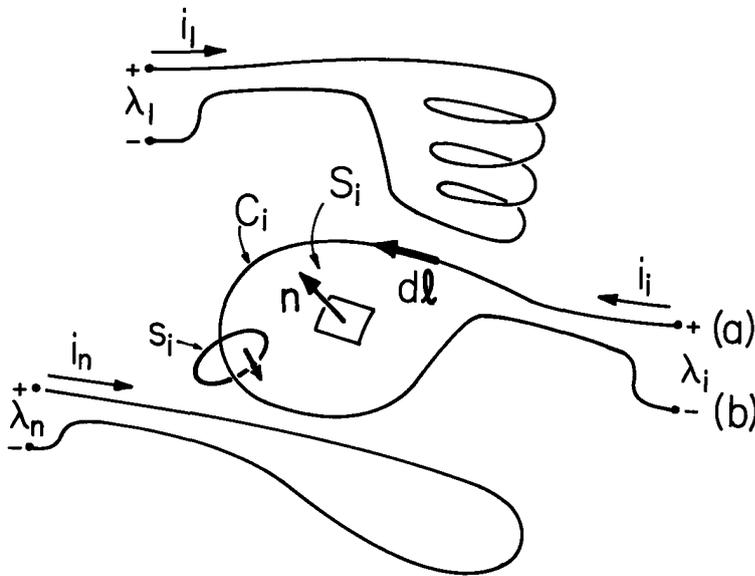


Fig. 2.12.1

Schematic representation of a system of perfectly conducting coils. The  $i$ th coil is shown with the wire assuming the contour  $C_i$  enclosing a surface  $S_i$ . There is a total of  $n$  coils in the system.

but free of electrical losses. The total flux  $\lambda_i$  linked by the  $i$ th coil is a terminal variable, defined such that

$$\lambda_i = \int_{S_i} \vec{B} \cdot \vec{n} da \quad (1)$$

A positive  $\lambda$  is determined by first assigning the direction of a positive current  $i_i$ . Then, the direction of the normal vector (and hence the positive flux) to the surface  $S_i$ , enclosed by the contour  $C_i$  followed by the current  $i_i$ , has a direction consistent with the right-hand rule, as Fig. 2.12.1 illustrates.

Because the MQS current density is solenoidal, the same current flows through the cross section of the wire at any point. Thus, the terminal current is defined by

$$i_i = \int_{s_i} \vec{J}_f \cdot \vec{i}_n da \quad (2)$$

where the surface  $s_i$  intersects all of the cross section of the wire at any point, as illustrated in the figure.

The first two MQS equations are sufficient to determine the flux linkages as a function of the current excitations and the geometry of the coil. Thus, Ampère's law and the condition that the magnetic flux density be solenoidal are solved to obtain relations having the form

$$\lambda_i = \lambda_i(i_1 \dots i_n, \xi_1 \dots \xi_m) \quad (3)$$

If the materials involved are magnetically linear, so that  $\vec{B} = \mu \vec{H}$ , where  $\mu$  is a function of position but not of time or the fields, then it is convenient to define inductance parameters which depend only on the geometry:

$$L_{ij} = \left. \frac{\lambda_i}{i_j} \right|_{i_{i \neq j} = 0} = \frac{\int_{S_i} \mu \vec{H} \cdot \vec{n} da}{\int_{s_j} \vec{J}_f \cdot \vec{i}_n da} \quad (4)$$

The inductance  $L_{ij}$  is the flux linked by the  $i$ th coil per unit current in the  $j$ th coil, with all other currents zero. For the particular cases in which an inductance can be defined, Eq. 3 becomes

$$\lambda_i = \sum_{j=1}^n L_{ij} i_j, \quad L_{ij} = L_{ij}(\xi_1 \dots \xi_m) \quad (5)$$

The dynamics of a lumped parameter system arise through Faraday's integral law of induction, Eq. 2.7.3b, which can be written for the  $i$ th coil as

$$\oint_{C_1} \vec{E}' \cdot d\vec{\ell} = - \frac{d}{dt} \int_{S_1} \vec{B} \cdot \vec{n} da \quad (6)$$

Here the contour is one attached to the wire and so  $\vec{v}_s = \vec{v}$  in Eq. 2.7.3b. The line integration can be broken into two parts, one of which follows the wire from the positive terminal at (a) to (b), while the other follows a path from (b) to (a) in the insulating region outside the wire

$$\oint_{C_1} \vec{E}' \cdot d\vec{\ell} = \int_a^b \vec{E}' \cdot d\vec{\ell} + \int_b^a \vec{E}' \cdot d\vec{\ell} \quad (7)$$

Even though the wire is in general deforming and moving, because it is perfectly conducting, the electric field intensity  $\vec{E}'$  must vanish in the conductor, and so the first integral called for on the right in Eq. 7 must vanish. By contrast with the EQS fields, the electric field here is not irrotational. This means that the remaining integration of the electric field intensity between the terminals must be carefully defined. Usually, the terminals are located in a region in which the magnetic field is sufficiently small to take the electric field intensity as being irrotational, and therefore definable in terms of the gradient of the potential. With the assumption that such is the case, the remaining integral of Eq. 7 is written as

$$\int_b^a \vec{E}' \cdot d\vec{\ell} = - \int_b^a \nabla \phi \cdot d\vec{\ell} = -(\phi_a - \phi_b) \equiv -v_1 \quad (8)$$

Thus it follows from Eq. 6, combined with Eqs. 1 and 8, that the voltage at the coil terminals is the time rate of change of the associated flux linked:

$$v_1 = \frac{d\lambda_1}{dt} \quad (9)$$

With  $\lambda_1$  given by Eq. 3 or Eq. 5, the terminal voltage follows from Eq. 9.

### 2.13 Conservation of Electroquasistatic Energy

This and the next section develop a field picture of electromagnetic energy storage from fundamental definitions and principles. Results are a first step in the derivation of macroscopic force densities in Chap. 3. Energy storage in a conservative EQS system is considered first, followed by a statement of power flow. In this and the next section the macroscopic medium is at rest.

Thermodynamics: Whether in electric or magnetic form, energy storage follows from the definition of the electric field as a force per unit charge. The work required to transport an element of charge,  $\delta q$ , from a reference position to a position p in the presence of the electric field intensity is

$$\delta w = - \int_{ref}^p \delta q \vec{E} \cdot d\vec{\ell} \quad (1)$$

The integral is the work done by the external force on the electric subsystem in placing the charge at p. If this process can be reversed, it can be said that the work done results in a stored energy equal to Eq. 1. In an electroquasistatic system, the electric field is irrotational. Hence,  $\vec{E} = -\nabla\phi$ . Then, if  $\phi_{ref}$  is defined as zero, it follows that Eq. 1 becomes

$$\delta w = \int_{ref}^p \delta q \nabla \phi \cdot d\vec{\ell} = \delta q \phi \quad (2)$$

where use has been made of the gradient integral theorem, Eq. 2.6.1. Consider now energy storage in the system abstractly represented by Fig. 2.13.1. The system is perfectly insulating, except for the perfectly conducting electrodes introduced into the volume of interest, as in Sec. 2.11. It will be termed an "electroquasistatic thermodynamic subsystem."

The electrodes have terminal variables as defined in Sec. 2.11; voltages  $v_1$  and total charges  $q_1$ . But, in addition, the volume between the electrodes supports a free charge density  $\rho_f$ . By definition, the energy stored in assembling these charges is equal to the work required to carry the charges from a reference position to the positions of interest. Thus, the incremental energy storage associated with incremental changes in the electrode charges,  $\delta q_1$ , or in the charge density,  $\delta \rho_f$ , in a given neighborhood on the insulator, is

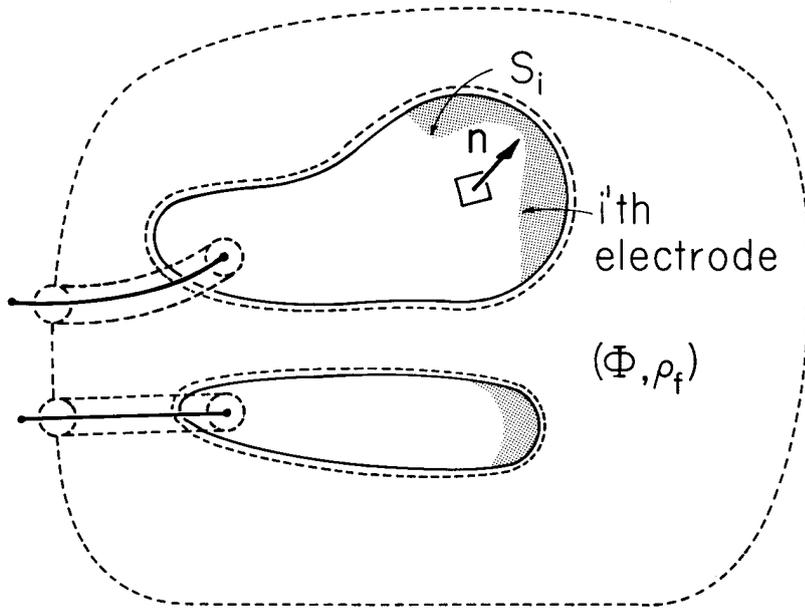


Fig. 2.13.1

Schematic representation of electroquasistatic system composed of perfectly conducting electrodes imbedded in a perfectly insulating dielectric medium.

$$\delta w = \sum_{i=1}^n v_i \delta q_i + \int_{V'} \Phi \delta \rho_f dV \quad (3)$$

The volume  $V'$  is the volume excluded by the electrodes. Note that the reference electrode is not included in the summation, because the electric potential on that electrode is, by definition, zero. The work required to place a free charge at its final position correctly accounts for the polarization, because the polarization charges induced in carrying the free charges to their final position are reflected in the potential.

Consider now the field representation of the electroquasistatic stored energy. From Gauss' law (Eq. 2.3.23a), the contribution of the summation in Eq. 3 can be represented in terms of an integral over the surfaces  $S_i$  of the electrodes:

$$\delta w = \sum_{i=1}^n \oint_{S_i} \Phi_i \delta \vec{D} \cdot \vec{n} da + \int_{V'} \Phi \delta \rho_f dV \quad (4)$$

Here,  $\Phi_i$  is the potential on the surface  $S_i$ . The surfaces enclosing the electrodes can be joined together at infinity, as shown in Fig. 2.13.1. The resulting simply connected surface encloses all of the electrodes, the wires as they extend to infinity, with the surface completed by a closure at infinity. Thus, the surface integration called for with the first term on the right in Eq. 4 can be represented by an integration over a closed surface. Gauss' theorem is then used to convert this surface integral to a volume integration. However, note that the normal vector used in Eq. 4 points into the volume  $V'$  excluded by the electrodes and included by the surface at infinity. Thus, in using Gauss' theorem, a minus sign is introduced and Eq. 4 becomes

$$\delta w = - \int_{V'} \nabla \cdot (\Phi \delta \vec{D}) dV + \int_{V'} \Phi \delta \rho_f dV = \int_{V'} [-\nabla \cdot \delta \vec{D} - \delta \vec{D} \cdot \nabla \Phi + \Phi \delta \rho_f] dV \quad (5)$$

In rewriting the integral, the identity  $\nabla \cdot \psi \vec{C} = \vec{C} \cdot \nabla \psi + \psi \nabla \cdot \vec{C}$  has been used.

From Gauss' law,  $\delta \rho_f = \delta \nabla \cdot \vec{D} = \nabla \cdot \delta \vec{D}$ . It follows that the first and last terms in Eq. 5 cancel. Also, the electric field is irrotational ( $\vec{E} = -\nabla \Phi$ ). So Eq. 5 becomes

$$\delta w = \int_V \vec{E} \cdot \delta \vec{D} dV \quad (6)$$

There is no  $\vec{E}$  inside the electrodes, so the integration is now over all of the volume  $V$ .

The integrand in Eq. 6 is an energy density, and it is therefore appropriate to define the incremental change in electric energy density as

$$\delta W = \vec{E} \cdot \delta \vec{D} \quad (7)$$

X

The field representation of the energy, as given by Eqs. 6 and 7, should be compared to that for lumped parameters. Suppose all of the charge resided on electrodes. Then, the second term in Eq. 3 would be zero, and the incremental change in energy would be given by the first term:

$$\delta w = \sum_{i=1}^n v_i \delta q_i \quad (8)$$

Comparison of Eqs. 6 and 8 suggests that the electric field plays a role analogous to the terminal voltage while the displacement vector is the analog of the charge on the electrodes. If the relationship between the variables  $\vec{E}$  and  $\vec{D}$ , or  $v$  and  $q$ , is single-valued, then the energy density and the total energy in the continuum and lumped parameter systems can be viewed, respectively, as integrals or areas under curves as sketched in Fig. 2.13.2.

If it is more convenient to have all of the voltages, rather than the charges, as independent variables, then Legendre's dual transformation can be used. That is, with the observation that

$$v_i \delta q_i = \delta v_i q_i - q_i \delta v_i \quad (9)$$

Eq. 8 becomes

$$\delta w' = \sum_{i=1}^n q_i \delta v_i; \quad w' \equiv \sum_{i=1}^n (v_i q_i - w) \quad (10)$$

with  $w'$  defined as the coenergy function.

In an analogous manner, a coenergy density,  $W'$ , is defined by writing  $\vec{E} \cdot \delta \vec{D} = \delta(\vec{E} \cdot \vec{D}) - \vec{D} \cdot \delta \vec{E}$  and thus defining

$$\delta W' = \vec{D} \cdot \delta \vec{E}; \quad W' \equiv \vec{E} \cdot \vec{D} - W \quad (11)$$

The coenergy and coenergy density functions have the geometric relationship to the energy and energy density functions, respectively, sketched in Fig. 2.13.2. In those systems in which there is no distribution of charge other than on perfectly conducting electrodes, Eqs. 6 and 8 can be regarded as equivalent ways of computing the same incremental change in electroquasistatic energy. If the charge is distributed throughout the volume, Eq. 6 remains valid.

With the notion of electrical energy storage goes the concept of a conservative subsystem. In the process of building up free charges on perfectly conducting electrodes or slowly conducting charge to the bulk positions (one mechanism for carrying out the process pictured abstractly by Eq. 3), the work is stored much as it would be in cocking a spring. The electrical energy, like that of the spring, can later be released (discharged). Included in the subsystem is storage in the polarization. For work done on polarizable entities to be stored, this polarization process must also be reversible. Here, it is profitable to think of the dipoles as internally constrained by spring-like nondissipative elements, capable of releasing energy when the polarizing field is turned off. Mathematically, this restriction on the nature of the polarization is brought in by requiring that  $\vec{P}$  and hence  $\vec{D}$  be a single-valued function of the instantaneous  $\vec{E}$ , or that  $\vec{E} = \vec{E}(\vec{D})$ . In lumped parameter systems, this is tantamount to  $q = q(v)$  or  $v = v(q)$ .

**Power Flow:** The electric and polarization energy storage subsystem is the field theory generalization of a capacitor. Just as practical circuits involve a capacitor interconnected with resistors and other types of elements, in any actual physical system the ideal energy storage subsystem is imbedded with and coupled to other subsystems. The field equations, like Kirchhoff's laws in circuit theory, encompass all of these subsystems. The following discussion is based on forming quadratic expressions from the field laws, and hence relate to the energy balance between subsystems.

For a geometrical part of the  $i$ th subsystem, having the volume  $V$  enclosed by the surface  $S$ , a statement of power flow takes the integral form

$$\oint_S \vec{S}_i \cdot \vec{n} da + \int_V \frac{\partial w_i}{\partial t} dV = \int_V \phi_i dV \quad (12)$$

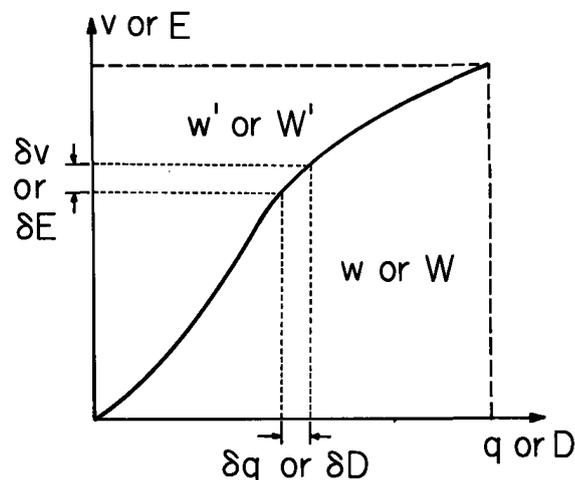


Fig. 2.13.2. Geometric representation of energy  $w$ , coenergy  $w'$ , energy density  $W$ , and coenergy density  $W'$  for electric field systems.

Here,  $S_i$  is the power flux density,  $W_i$  is the energy density, and  $\phi_i$  is the dissipation density.

Different subsystems can occupy the same volume  $V$ . In Eq. 12,  $V$  is arbitrary, while  $i$  distinguishes the particular physical processes considered. The differential form of Eq. 12 follows by applying Gauss' theorem to the first term and (because  $V$  is arbitrary) setting the integrand to zero:

$$\nabla \cdot \vec{S}_i + \frac{\partial W_i}{\partial t} = \phi_i \quad (13)$$

This is a canonical form which will be used to describe various subsystems. In a given region,  $W_i$  can increase with time either because of the volumetric source  $\phi_i$  or because of a power flux  $-\vec{n} \cdot \vec{S}_i$  into the region across its bordering surfaces.

For an electrical lumped parameter terminal pair, power is the product of voltage and current. This serves as a clue for finding a statement of power flow from the basic laws. The generalization of the voltage is the potential, while conservation of charge as expressed by Eq. 2.3.25a brings in the free current density. So, the sum of Eqs. 2.3.25a and the conservation of polarization charge equation, Eq. 2.8.5, is multiplied by  $\Phi$  to obtain

$$\Phi[\nabla \cdot (\vec{J}_f + \vec{J}_p) + \frac{\partial}{\partial t} (\rho_f + \rho_p)] = 0 \quad (14)$$

With the objective an expression having the form of Eq. 13, a vector identity (Eq. 15, Appendix B) and Gauss' law, Eq. 2.3.23a, convert Eq. 14 to

$$\nabla \cdot [\Phi(\vec{J}_f + \vec{J}_p)] + \vec{E} \cdot (\vec{J}_f + \vec{J}_p) + \Phi \frac{\partial}{\partial t} \nabla \cdot \epsilon_0 \vec{E} = 0 \quad (15)$$

In the last term the time derivative and divergence are interchanged and the vector identity used again to obtain the expression

$$\nabla \cdot \vec{S}_e + \frac{\partial W_e}{\partial t} = \phi_e \quad (16)$$

where, with Eq. 2.8.10 used for  $\vec{J}_p$ ,

$$\vec{S}_e \equiv \Phi \left( \vec{J}_f + \vec{J}_p + \frac{\partial \epsilon_0 \vec{E}}{\partial t} \right) = \Phi \left[ \vec{J}_f + \frac{\partial \vec{D}}{\partial t} + \nabla \times (\vec{P} \times \vec{v}) \right]$$

$$W_e \equiv \frac{1}{2} \epsilon_0 \vec{E} \cdot \vec{E}$$

$$\phi_e \equiv -\vec{E} \cdot (\vec{J}_f + \vec{J}_p) = -\vec{E} \cdot \left[ \vec{J}_f + \frac{\partial \vec{P}}{\partial t} + \nabla \times (\vec{P} \times \vec{v}) \right]$$

Which terms appear where in this expression is a matter of what part of a physical system (which subsystem) is being described. Note that  $W_e$  does not include energy stored by polarizing the medium. Also, it can be shown that  $\nabla \cdot \vec{S}_e = \nabla \cdot (\vec{E} \times \vec{H})$ , so that  $\vec{S}_e$  is the Poynting vector familiar from conventional classical electrodynamics. In the dissipation density,  $\vec{E} \cdot \vec{J}_f$  can represent work done on an external mechanical system due to polarization forces or, if the polarization process involves dissipation, heat energy given up to a thermal subsystem.

The polarization terms in  $\phi_e$  can also represent energy storage in the polarization. This is illustrated by specializing Eq. 16 to describe a subsystem in which  $\vec{P}$  is a single-valued function of the instantaneous  $\vec{E}$ , the free current density is purely ohmic,  $\vec{J}_f = \sigma \vec{E}$ , and the medium is at rest. Then, the polarization term from  $\phi_e$  can be lumped with the energy density term to describe power flow in a subsystem that includes energy storage in the polarization:

$$\nabla \cdot \vec{S}_E + \frac{\partial W_E}{\partial t} = \phi_E \quad (17)$$

where

$$\vec{S}_E \equiv \Phi \left[ \sigma \vec{E} + \frac{\partial \vec{D}}{\partial t} \right]; \quad W_E \equiv \int_0^{\vec{D}} \vec{E} \cdot \delta \vec{D}; \quad \phi_E \equiv -\sigma \vec{E} \cdot \vec{E}$$

Note that the integral defining the energy density  $W_E$ , which is consistent with Eq. 7, involves an integrand  $\vec{E}$  which is time dependent only through the time dependence of  $\vec{D}$ :  $\vec{E} = \vec{E}[\vec{D}(t)]$ . Thus,  $\partial W_E / \partial t = \vec{E} \cdot (\partial \vec{D} / \partial t)$ .

With the power flux density placed on the right, Eq. 17 states that the energy density decreases because of electrical losses (note that  $\phi_E < 0$ ) and because of the divergence of the power density.

## 2.14 Conservation of Magnetoquasistatic Energy

Fundamentally, the energy stored in a magnetic field involves the same work done by moving a test charge from a reference position to the position of interest as was the starting point in Sec. 2.13. But, the same starting point leads to an entirely different form of energy storage. In a magnetoquasistatic system, the net free charge is a quantity evaluated after the fact. A self-consistent representation of the fields is built upon a statement of current continuity, Eq. 2.3.26b, in which the free charge density is ignored altogether. Yet, the energy stored in a magnetic field is energy stored in charges transported against an electric field intensity. The apparent discrepancy in these statements is resolved by recognizing that the charges of interest in a magnetoquasistatic system are at least of two species, with the charge density of one species alone far outweighing the net charge density.

Thermodynamics: Because the free current density is solenoidal, a current "tube" can be defined as shown in Fig. 2.14.1. This tube is defined with a cross section having a normal  $\vec{i}_n$  in the direction of the local current density, and a surrounding surface having a normal perpendicular to the local current density. An example of a current tube is a wire surrounded by insulation and hence carrying a total current  $i$  which is the same at one cross section as at another.

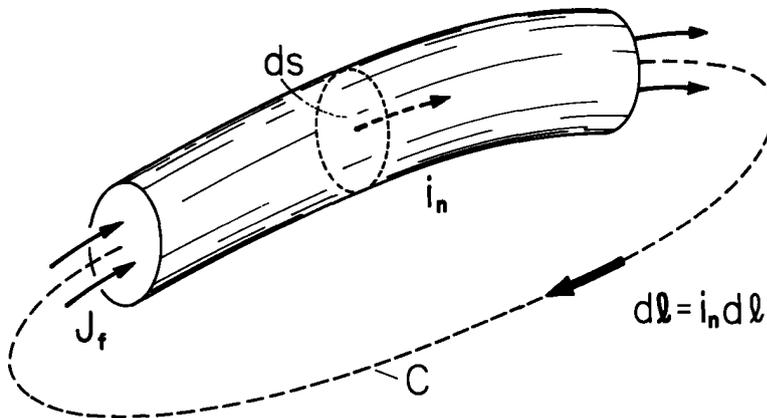


Fig. 2.14.1

Current tube defined as having cross-sectional area  $ds$  perpendicular to the local current density, and an outside surface with a normal vector perpendicular to the current density.

For bipolar conduction, and a stationary medium, the current density within the tube is related to the charge density by the expression

$$\vec{J}_f = \rho_+ \vec{v}_+ - \rho_- \vec{v}_- \quad (1)$$

Here the conduction process is visualized as involving two types of carriers, one positive, with a charge density  $\rho_+$ , and the other negative, with a magnitude  $\rho_-$ . The carriers then have velocities which are, respectively,  $\vec{v}_+$  and  $\vec{v}_-$ . Even though there is a current density, in the magnetoquasistatic system there is essentially no net charge:  $\rho_f = \rho_+ - \rho_- \approx 0$ . In an increment of time  $\delta t$ , the product of the respective charge densities and net displacements is  $\rho_+ \vec{v}_+ \delta t$  and  $-\rho_- \vec{v}_- \delta t$ . The work done on the charges as they undergo these displacements is the energy stored in magnetic form. This work is computed by recognizing that the force on each of the charged species is the product of the charge density and the electric field intensity. Hence, the energy stored in the field by a length of the current tube  $d\ell$  is to first order in differentials  $d\ell$  and  $ds$ ,

$$-(\rho_+ \vec{v}_+ - \rho_- \vec{v}_-) \cdot \vec{E} \delta t ds d\ell = -\vec{J}_f \cdot \vec{E} \delta t ds d\ell \quad (2)$$

The expression for the free current density, Eq. 1, is used on the right to restate the energy stored in the increment of time  $\delta t$ . The unit vector  $\vec{i}_n$  is defined to be the direction of  $\vec{J}_f$ . Thus,  $\vec{J}_f = (\vec{J}_f \cdot \vec{i}_n) \vec{i}_n$ . Because the current density is solenoidal, it follows closed paths. The product  $\vec{J}_f \cdot \vec{i}_n ds$  is, by definition, constant along one of these paths, and if  $\vec{i}_n d\ell$  is defined as an increment of the line integral, it then follows from Eq. 2 that the energy stored in a single current tube is

$$-\vec{J}_f \cdot \vec{i}_n ds \left( \oint_C \vec{E} \cdot \vec{i}_n d\ell \right) \delta t \quad (3)$$

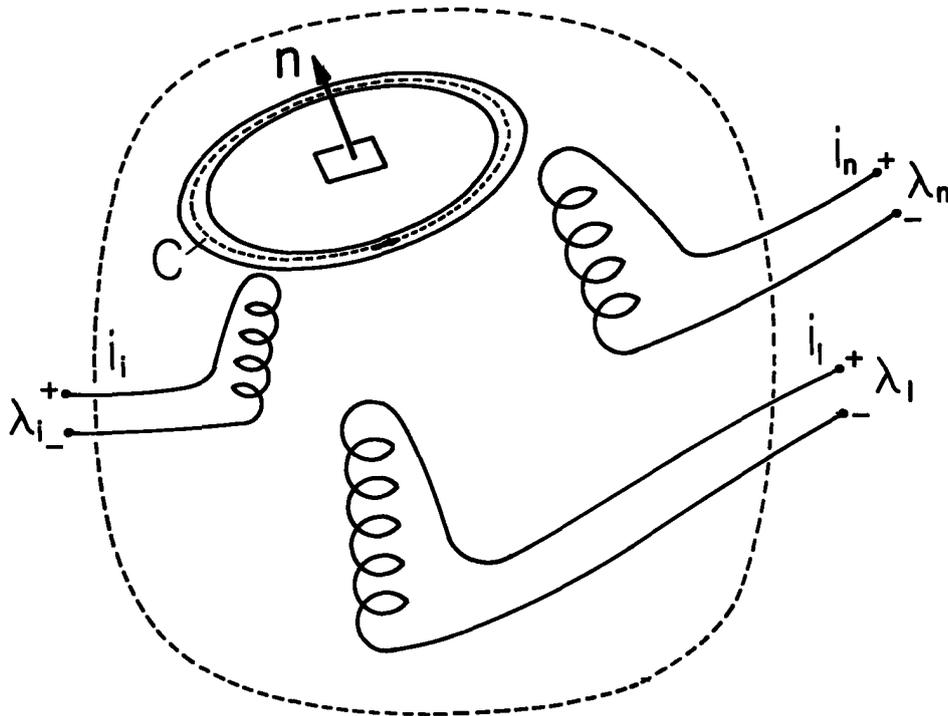


Fig. 2.14.2

Schematic representation of a magnetoquasistatic energy storage system. Currents are either distributed in current loops throughout the volume of interest, or confined to one of  $n$  possible contours connected to the discrete terminal pairs.

By contrast with the electroquasistatic system, in which the electric field intensity is induced by the charge density (Gauss' law), the electric field intensity in Eq. 3 is clearly rotational. This emphasizes the essential role played by Faraday's law of magnetic induction.

It is helpful to have in mind at least the abstraction of a physical system. Figure 2.14.2 shows a volume of interest in which the currents are either distributed throughout the volume or confined to particular contours (coils), the latter case having been discussed in Sec. 2.12.

First, consider the energy stored in the current paths defined by coils having cross-sectional area  $ds$ . From Eq. 3, this contribution to the total energy is conveniently written as

$$-\vec{J}_f \cdot \vec{i}_n ds \oint_{C_i} \vec{E} \cdot \vec{i}_n d\ell \delta t = i_1 \delta \lambda_1 \quad (4)$$

Faraday's law and the definition of flux linkage, Eqs. 2.12.1 and 2.12.6, are the basis for representing the line integral as a change in the flux linkage.

Because the free current density is solenoidal, the distribution of free currents within the volume  $V$  excluded by the discrete coils can be represented as the superposition of current tubes. From Eq. 4 and the integral form of Faraday's law, Eq. 2.7.3b with  $\vec{v}_s = \vec{v} = 0$  (the medium is fixed), it follows that the energy stored in a current tube is

$$\delta w_{\text{current tube}} = \vec{J}_f \cdot \vec{i}_n ds \int_{S_{\text{tube}}} \delta \vec{B} \cdot \vec{n} da \quad (5)$$

The magnetic flux density is also solenoidal, and for this reason it is convenient to introduce the magnetic vector potential  $\vec{A}$ , defined such that  $\vec{B} = \nabla \times \vec{A}$ , so that the magnetic flux density is automatically solenoidal. With this representation of the flux density in terms of the vector potential, Stokes's theorem, Eq. 2.6.3, converts Eq. 5 to

$$\vec{J}_f \cdot \vec{i}_n ds \oint_{C_{\text{tube}}} \delta \vec{A} \cdot \vec{i}_n d\ell = \oint_{C_{\text{tube}}} (\vec{J}_f \cdot \delta \vec{A}) ds d\ell = \int_{V_{\text{tube}}} \vec{J}_f \cdot \delta \vec{A} dV \quad (6)$$

Here,  $\vec{J}_f$  is by definition in the direction of  $\vec{i}_n$ , so that  $\vec{J}_f \cdot \delta \vec{A}$  takes the component of  $\delta \vec{A}$  in the  $\vec{i}_n$  direction. The second equality is based upon recognition that the product  $ds \cdot d\ell$  is a volume element of the current tube, and the line integration constitutes an integration over the volume,  $V_{\text{tube}}$ , of the tube.

To include all of the energy stored in the distributed current loops, it is necessary only that

the integral on the right in Eq. 6 be extended over all of the volume occupied by the tubes. The combination of the incremental energy stored in the discrete loops, Eq. 4, and that from the distributed current loops, Eq. 6, is the incremental total energy of the system

$$\delta w = \sum_{i=1}^n i_i \delta \lambda_i + \int_V \vec{J}_f \cdot \delta \vec{A} dV \quad (7)$$

In this expression,  $V$  is the volume excluded by the discrete current paths. This incremental magnetic energy storage is analogous to that for the electric field storage represented by Eq. 2.13.3.

In retrospect, it is apparent from the derivation that the division into discrete and distributed current paths, represented by the two terms in Eq. 7, is a matter of convenience. In representing the incremental energy in terms of the magnetic fields alone, it is handy to extend the volume  $V$  over all of the currents within the volume of interest, including those that might be represented by discrete terminal pairs. With this understanding, the incremental change in energy, Eq. 7, is the last term only, with  $V$  extended over the total volume. Moreover, Ampere's law represents the current density in terms of the magnetic field intensity, and, in turn, the integrand can be rewritten by use of a vector identity (Eq. 8, Appendix B):

$$\delta w = \int_V \nabla \times \vec{H} \cdot \delta \vec{A} dV = \int_V [\vec{H} \cdot \nabla \times \delta \vec{A} + \nabla \cdot (\vec{H} \times \delta \vec{A})] dV \quad (8)$$

The last term in Eq. 8 can be converted to a surface integral by using Gauss' theorem. With the understanding that the system is closed in the sense that the fields fall off rapidly enough at infinity so that the surface integration can be ignored, the remaining volume integration on the right in Eq. 8 can be used to obtain a field representation of the incremental energy change. With the curl of the vector potential converted back to a flux density, Eq. 8 becomes

$$\delta w = \int_V \vec{H} \cdot \delta \vec{B} dV \quad (9)$$

The integrand of Eq. 9 is defined as an incremental magnetic energy density

$$\delta W = \vec{H} \cdot \delta \vec{B} \quad (10)$$

It is helpful to note the clear analogy between this energy density and the incremental total energy represented by lumped parameters. In the absence of volume free current densities that cannot be represented by discrete terminal pairs, Eq. 7 reduces to the lumped parameter form

$$\delta w = \sum_{i=1}^n i_i \delta \lambda_i \quad (11)$$

The magnetic field intensity plays the continuum role of the discrete terminal currents, and the magnetic flux density is the continuum analog of the lumped parameter flux linkages. The situation in this magnetic case is, of course, analogous to the electrical incremental energy storages in continuum and in lumped parameter cases, as discussed with Eqs. 7 and 8 of Sec. 2.13.

Just as it is often convenient in dealing with electrical lumped parameters to use the voltage as an independent variable, so also in magnetic field systems it is helpful to use the terminal currents as independent variables. In that case, the coenergy function  $w'$  is conveniently introduced as an energy function

$$\delta w' = \sum_{i=1}^n \lambda_i \delta i_i \quad (12)$$

In an analogous way, the co-energy density,  $w'$ , is defined such that

$$\delta W' = \vec{B} \cdot \delta \vec{H}; \quad W' = \vec{H} \cdot \vec{B} - W \quad (13)$$

Power Flow: Thus far, the storage of energy in magnetic form has been examined. The postulate has been that all work done in moving the charges against an electric field is stored. In any system as a whole this is not likely to be the case. The general magnetoquasistatic laws enable a deduction of an equation representing the flow of power, and the rate of change of the stored energy. This places

the energy storage in the context of a more general system.

A clue as to how an energy conservation statement might be constructed from the differential magnetoquasistatic laws is obtained from Eq. 2, which makes it clear that the product of the free current density and the electric field intensity are closely connected with the statement of conservation of energy. The dot product of the electric field and Ampère's law, Eq. 2.3.23b, is

$$\vec{E} \cdot [\nabla \times \vec{H} - \vec{J}_f] = 0 \quad (14)$$

Use of a vector identity (Eq. 8, Appendix B) makes it possible to rewrite this expression as

$$\vec{H} \cdot \nabla \times \vec{E} - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \vec{J}_f \quad (15)$$

With the additional use of Faraday's law to represent  $\nabla \times \vec{E}$ , Eq. 15 takes the form of Eq. 2.13.16, with

$$\begin{aligned} S_e &\equiv \vec{E} \times \vec{H} \\ W_e &\equiv \frac{1}{2} \mu_o \vec{H} \cdot \vec{H} \\ \phi_e &\equiv -\vec{E} \cdot \vec{J}_f - \vec{H} \cdot \frac{\partial \mu_o \vec{M}}{\partial t} - \vec{H} \cdot \nabla \times (\mu_o \vec{M} \times \vec{v}) \end{aligned} \quad (16)$$

These quantities have much the same physical significances discussed in connection with Eq. 2.13.16.

To place the magnetic energy storage identified with the thermodynamic arguments in the context of an actual system, consider a material which is ohmic and fixed so that  $\vec{v} = 0$  and  $\vec{J}_f = \sigma \vec{E}$ . Then the second term on the right in Eq. 16c is in the form of a time rate of change of magnetization energy density. Hence, the power flow equation assumes the form of Eq. 2.13.17, with

$$\begin{aligned} \vec{S}_E &= \vec{E} \times \vec{H} \\ W_E &= \int_0^{\vec{B}} \vec{H} \cdot \delta \vec{B} \\ \phi_E &= -\sigma \vec{E} \cdot \vec{E} \end{aligned} \quad (17)$$

Implicit is the assumption that  $\vec{H}$  is a single-valued function of the instantaneous  $\vec{B}$ . The resulting energy density includes magnetization energy and is consistent with Eq. 2.14.10.

## 2.15 Complex Amplitudes; Fourier Amplitudes and Fourier Transforms

The notion of a continuum network is introduced for the first time in the next section. The associated transfer relations illustrated there are a theme throughout the chapters which follow. Among several reasons for their use is the organization they lend to the representation of complicated, largely linear, systems. In this chapter, the continuum networks represent electromagnetic fields. Later, they represent fluid and (to some degree) solid mechanics, heat and mass transfer, and electromechanical continua in general. These networks make it possible to set aside one part of a given problem, derive the associated relations once and for all and accumulate these for later use. Such relations will be picked up over and over in solving different problems and, properly understood, are a useful reference.

**Complex Amplitudes:** In many practical situations, excitations are periodic in one or two spatial directions, in time or in space and time. The complex amplitude representation of fields, useful in dealing with these situations, is illustrated by considering the function  $\Phi(z,t)$  which has dependence on  $z$  given explicitly by

$$\Phi(z,t) = \text{Re} \{ \tilde{\Phi}(t) e^{-jkz} \} \quad (1)$$

With the wavenumber  $k$  real, the spatial distribution is periodic with wavelength  $\lambda = 2\pi/k$  and spatial phase determined by the complex amplitude  $\tilde{\Phi}$ . For example, if  $\tilde{\Phi} = \tilde{\Phi}_o(t)$  is real and  $k$  is real, then  $\Phi(z,t) = \tilde{\Phi}_o(t) \cos kz$ .

The spatial derivative of  $\Phi$  follows from Eq. 1 as

$$\frac{\partial \Phi}{\partial z} = \text{Re} \{ -jk \tilde{\Phi}(t) e^{-jkz} \} \quad (2)$$

The following identifications can therefore be made:

$$[\Phi(z,t), \frac{\partial \Phi}{\partial z}(z,t)] \longleftrightarrow [\tilde{\Phi}(t), -jk\tilde{\Phi}(t)] \quad (3)$$

with it being understood that even though complex amplitudes are being used, the temporal dependence is arbitrary. There will be occasions where the time dependence is specified, but the space dependence is not. For example, complex amplitudes will take the form

$$\Phi(z,t) = \text{Re} \hat{\Phi}(z) e^{j\omega t} \quad (4)$$

where  $\hat{\Phi}(z)$  is itself perhaps expressed as a Fourier series or transform (see Sec. 5.16).

Most often, complex amplitudes will be used to represent both temporal and spatial dependences:

$$\Phi(z,t) = \text{Re} \hat{\Phi} e^{j(\omega t - kz)} \quad (5)$$

The (angular) frequency  $\omega$  can in general be complex. If  $\Phi$  is periodic in time with period  $T$ , then  $T = 2\pi/\omega$ . For complex amplitudes  $\hat{\Phi}$ , the identifications are:

$$[\Phi(z,t), \frac{\partial \Phi}{\partial z}(z,t), \frac{\partial \Phi}{\partial t}(z,t)] \longleftrightarrow [\hat{\Phi}, -jk\hat{\Phi}, j\omega\hat{\Phi}] \quad (6)$$

If  $\omega$  and  $k$  are real, Eq. 5 represents a traveling wave. At any instant, its wavelength is  $2\pi/k$ , at any position its frequency is  $\omega$  and points of constant phase propagate in the  $+z$  direction with the phase velocity  $\omega/k$ .

**Fourier Amplitudes and Transforms:** The relations between complex amplitudes are identical to those between Fourier amplitudes or between Fourier transforms provided that these are suitably defined. For a wide range of physical situations it is the spatially periodic response or the temporal sinusoidal steady state that is of interest. Simple combinations of solutions represented by the complex amplitudes then suffice, and there is no need to introduce Fourier concepts. Even so, it is important to recognize at the outset that the spatial information required for analysis of excitations with arbitrary spatial distributions is inherent to the transfer relations based on single-complex-amplitude solutions.

The Fourier series represents an arbitrary function periodic in  $z$  with fundamental periodicity length  $\ell$  by a superposition of complex exponentials. In terms of complex Fourier coefficients  $\tilde{\Phi}_n(t)$ , such a series is

$$\Phi(z,t) = \sum_{n=-\infty}^{\infty} \tilde{\Phi}_n(t) e^{-jk_n z}; \quad k_n \equiv 2n\pi/\ell; \quad \tilde{\Phi}_n^* = \tilde{\Phi}_{-n} \quad (7)$$

where the condition on  $\tilde{\Phi}_n$  insures that  $\Phi$  is real. Thus, with the identification  $\tilde{\Phi} \rightarrow \tilde{\Phi}_n$  and  $k \rightarrow k_n$ , each complex exponential solution of the form of Eq. 1 can be taken as one term in the Fourier series. The  $m$ th Fourier amplitude  $\tilde{\Phi}_m$  follows by multiplying Eq. 7 by the complex conjugate function  $\exp(jk_m z)$  and integrating over the length  $\ell$  to obtain only one term on the right. This expression can then be solved for  $\tilde{\Phi}_m$  to obtain the inverse relation

$$\tilde{\Phi}_m = \frac{1}{\ell} \int_z^{z+\ell} \Phi(z,t) e^{jk_m z} dz \quad (8)$$

If the temporal dependence is also periodic, with fundamental period  $T$ , the Fourier series can also be used to represent the time dependence in Eq. 7:

$$\Phi(z,t) = \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \hat{\Phi}_{mn} e^{j(\omega_m t - k_n z)}; \quad \hat{\Phi}_{mn}^* = \hat{\Phi}_{-m-n} \quad (9)$$

where the condition on the amplitudes insures that  $\Phi(z,t)$  is real. One component out of this double summation is the traveling-wave solution represented by the complex amplitude form, Eq. 5. The rules given by Eqs. 3 and 6 pertain either to the complex amplitudes or the Fourier coefficients.

The Fourier transform is convenient if the dependence is not periodic. With the Fourier transform  $\tilde{\Phi}(k,t)$  given by

$$\tilde{\Phi}(k,t) = \int_{-\infty}^{+\infty} \Phi(z,t) e^{jkz} dz$$

the functional dependence on  $z$  is a superposition of the complex exponentials

$$\Phi(z, t) = \int_{-\infty}^{+\infty} \tilde{\Phi}(k, t) e^{-jkz} \frac{dk}{2\pi} \quad (11)$$

The relation between the transform and the transform of the derivative can be found by taking the transform of  $\partial\Phi/\partial z$  using Eq. 11 and integrating by parts. Recall that  $\int v du = uv - \int u dv$  and identify  $du \rightarrow \partial\Phi/\partial z dz$  and  $v \rightarrow \exp jkz$ , and it follows that

$$\int_{-\infty}^{+\infty} \frac{\partial\Phi}{\partial z} e^{jkz} dz = \Phi e^{jkz} \Big|_{-\infty}^{+\infty} - jk \int_{-\infty}^{+\infty} \Phi e^{jkz} dz \quad (12)$$

For properly bounded functions the first term on the right vanishes and the second is  $-jk\tilde{\Phi}(k, t)$ . The transform of  $\partial\Phi/\partial z$  is simply  $-jk\tilde{\Phi}$  and thus the Fourier transform also follows the rules given with Eq. 3.

Extension of the Fourier transform to a second dimension results in the transform pair

$$\begin{aligned} \Phi(z, t) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \hat{\Phi}(k, \omega) e^{j(\omega t - kz)} \frac{dk}{2\pi} \frac{d\omega}{2\pi} \\ \hat{\Phi}(k, \omega) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Phi(z, t) e^{-j(\omega t - kz)} dt dz \end{aligned} \quad (13)$$

which illustrates how the traveling-wave solution of Eq. 5 can be viewed as a component of a complicated function. Again, relations between complex amplitudes are governed by the same rules, Eq. 6, as are the Fourier amplitudes  $\hat{\Phi}(k, \omega)$ .

If relationships are found among quantities  $\tilde{\Phi}(t)$ , then the same relations hold with  $\tilde{\Phi} \rightarrow \hat{\Phi}$  and  $\partial(\ )/\partial t \rightarrow j\omega$ , because the time dependence  $\exp(j\omega t)$  is a particular case of the more general form  $\tilde{\Phi}(t)$ .

Averages of Periodic Functions: An identity often used to evaluate temporal or spatial averages of complex-amplitude expressions is

$$\left\langle \text{Re } \tilde{A} e^{-jkz} \text{Re } \tilde{B} e^{-jkz} \right\rangle_z = \frac{1}{2} \text{Re } \tilde{A} \tilde{B}^* \quad (14)$$

where  $\langle \rangle_z$  signifies an average over the length  $2\pi/k$  and it is assumed that  $k$  is real. This relation follows by letting

$$\text{Re } \tilde{A} e^{-jkz} \text{Re } \tilde{B} e^{-jkz} = \frac{1}{2} \left[ \tilde{A} e^{-jkz} + \tilde{A}^* e^{jkz} \right] \frac{1}{2} \left[ \tilde{B} e^{-jkz} + \tilde{B}^* e^{jkz} \right] \quad (15)$$

and multiplying out the right-hand side to obtain

$$\frac{1}{4} \left[ \tilde{A} \tilde{B} e^{-2jkz} + \tilde{A}^* \tilde{B}^* e^{2jkz} \right] + \frac{1}{4} \left[ \tilde{A} \tilde{B}^* + \tilde{A}^* \tilde{B} \right] \quad (16)$$

The first term is a linear combination of  $\cos 2kz$  and  $\sin 2kz$  and hence averages to zero. The second term is constant and identical to the right-hand side of Eq. 14.

A similar theorem simplifies evaluation of the average of two periodic functions expressed in the form of Eq. 7:

$$\begin{aligned} \left\langle \tilde{A} \tilde{B} \right\rangle_z &= \left\langle \sum_{n=-\infty}^{+\infty} \tilde{A}_n(t) e^{-jk_n z} \sum_{m=-\infty}^{+\infty} \tilde{B}_m(t) e^{-jk_m z} \right\rangle_z \\ &= \sum_{n=-\infty}^{+\infty} \tilde{A}_n \tilde{B}_{-n} = \sum_{n=-\infty}^{+\infty} \tilde{A}_n \tilde{B}_n^* \end{aligned} \quad (17)$$

Of course, either the complex amplitude theorem of Eq. 14 or the Fourier amplitude theorem of Eq. 17 applies to time averages with  $kz \rightarrow -\omega t$ .

## 2.16 Flux-Potential Transfer Relations for Laplacian Fields

It is often convenient in the modeling of a physical system to divide the volume of interest into regions having uniform properties. Surfaces enclosing these regions are often planar, cylindrical or spherical, with the volume then taking the form of a planar layer, a cylindrical annulus or a spherical shell. Such volumes and bounding surfaces are illustrated in Tables 2.16.1-3. The question answered in this section is: given the potential on the bounding surfaces, what are the associated normal flux densities? Of immediate interest is the relation of the electric potentials to the normal displacement vectors. But also treated in this section is the relation of the magnetic potential to the normal magnetic flux densities. First the electroquasistatic fields are considered, and then the magnetoquasistatic relations follow by analogy.

Electric Fields: If any one of the regions shown in Tables 2.16.1-3 is filled with insulating charge-free ( $\rho_f = 0$ ) material of uniform permittivity  $\epsilon$ ,

$$\vec{P} = (\epsilon - \epsilon_0)\vec{E}, \quad \vec{D} = \epsilon\vec{E} \quad (1)$$

the governing field equations are Gauss' law, Eq. 2.3.23a,

$$\nabla \cdot \vec{D} = 0 \quad (2)$$

and the condition that  $\vec{E}$  be irrotational, Eq. 2.3.24a. The latter is equivalent to

$$\vec{E} = -\nabla\phi \quad (3)$$

Thus, the potential distribution within a volume is described by Laplace's equation

$$\nabla^2\phi = 0 \quad (4)$$

In terms of  $\phi$ ,

$$\vec{D} = -\epsilon\nabla\phi \quad (5)$$

Magnetic Fields: For magnetoquasistatic fields in an insulating region ( $J_f = 0$ ) of uniform permeability

$$\vec{M} = \left(\frac{\mu}{\mu_0} - 1\right)\vec{H}; \quad \vec{B} = \mu\vec{H} \quad (6)$$

Thus, from Ampère's law, Eq. 2.3.23b,  $\vec{H}$  is irrotational and it is appropriate to define a magnetic potential  $\Psi$ :

$$\vec{H} = -\nabla\Psi \quad (7)$$

In addition, there is Eq. 2.3.24b:

$$\nabla \cdot \vec{B} = 0 \quad (8)$$

Thus, the potential again satisfies Laplace's equation

$$\nabla^2\Psi = 0 \quad (9)$$

and in terms of  $\Psi$ , the magnetic flux density is

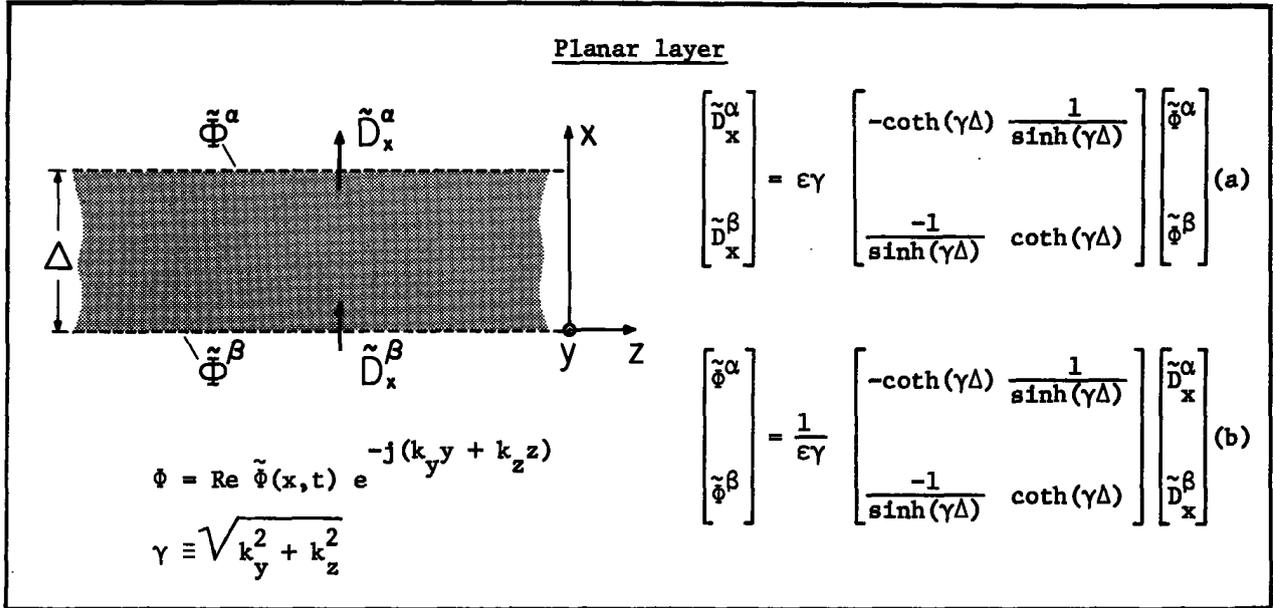
$$\vec{B} = -\mu\nabla\Psi \quad (10)$$

Comparison of the last two relations to Eqs. 4 and 5 shows that relations now derived for the electric fields can be carried over to describe the magnetic fields by making the identification  $(\phi, \vec{D}, \epsilon) \rightarrow (\Psi, \vec{B}, \mu)$ .

Planar Layer: Bounding surfaces at  $x = \Delta$  and  $x = 0$ , respectively denoted by  $\alpha$  and  $\beta$ , are shown in Table 2.16.1. So far as developments in this section are concerned, these are not physical boundaries. They are simply surfaces at which the potentials are respectively

$$\phi(\Delta, y, z, t) = \text{Re } \tilde{\phi}^\alpha(t) \exp[-j(k_y y + k_z z)]; \quad \phi(0, y, z, t) = \text{Re } \tilde{\phi}^\beta(t) \exp[-j(k_y y + k_z z)] \quad (11)$$

Table 2.16.1. Flux-potential transfer relations for planar layer in terms of electric potential and normal displacement ( $\Phi, D_x$ ). To obtain magnetic relations, substitute  $(\Phi, D_x, \epsilon) \rightarrow (\Psi, B_x, \mu)$ .



These will be recognized as generalizations of the complex amplitudes introduced with Eq. 2.15.1. That the potentials at the  $\alpha$  and  $\beta$  surfaces can be quite general follows from the discussion of Sec. 2.15, which shows that the following arguments apply when  $\tilde{\Phi}$  is a spatial Fourier amplitude or a Fourier transform.

In view of the surface potential distributions, solutions to Eq. 4 are assumed to take the form

$$\Phi = \text{Re } \tilde{\Phi}(x, t) \exp[-j(k_y y + k_z z)] \quad (12)$$

Substitution shows that

$$\frac{d^2 \tilde{\Phi}}{dx^2} - \gamma^2 \tilde{\Phi} = 0; \quad \gamma = \sqrt{k_y^2 + k_z^2} \quad (13)$$

Solutions of this equation are linear combinations of  $e^{\pm \gamma x}$  or alternatively of  $\sinh \gamma x$  and  $\cosh \gamma x$ . With  $\tilde{\Phi}_1$  and  $\tilde{\Phi}_2$  arbitrary functions of time, the solution takes the form

$$\tilde{\Phi} = \tilde{\Phi}_1 \sinh \gamma x + \tilde{\Phi}_2 \cosh \gamma x \quad (14)$$

The two coefficients are determined by requiring that the conditions of Eq. 11 be satisfied. For the simple situation at hand, an instructive alternative to performing the algebra necessary to evaluate  $(\tilde{\Phi}_1, \tilde{\Phi}_2)$  consists in recognizing that a linear combination of the two solutions in Eq. 14 is  $\sinh \gamma(x - \Delta)$ . Thus, the solution can be written as the sum of solutions that are individually zero on one or the other of the bounding surfaces. By inspection, it follows that

$$\tilde{\Phi} = \tilde{\Phi}^\alpha \frac{\sinh \gamma x}{\sinh \gamma \Delta} - \tilde{\Phi}^\beta \frac{\sinh \gamma(x - \Delta)}{\sinh \gamma \Delta} \quad (15)$$

From Eqs. 5 and 15,  $\tilde{D}$  can be determined:

$$\tilde{D}_x = -\epsilon \frac{\partial \Phi}{\partial x} = -\epsilon \text{Re } \gamma \left[ \tilde{\Phi}^\alpha \frac{\cosh \gamma x}{\sinh \gamma \Delta} - \tilde{\Phi}^\beta \frac{\cosh \gamma(x - \Delta)}{\sinh \gamma \Delta} \right] e^{-j(k_y y + k_z z)} \quad (16)$$

Evaluation of this equation at  $x = \Delta$  gives the displacement vector normal to the  $\alpha$  surface, with complex amplitude  $\tilde{D}_x^\alpha$ . Similarly, evaluated at  $x = 0$ , Eq. 16 gives  $\tilde{D}_x^\beta$ . The components of the "flux" ( $\tilde{D}_x^\alpha, \tilde{D}_x^\beta$ ) are now determined, given the respective potentials ( $\tilde{\Phi}^\alpha, \tilde{\Phi}^\beta$ ). The transfer relations, Eq. (a) of Table 2.16.1, summarize what is found. These relations can be solved for any pair of variables as a function of the remaining pair. The inverse transfer relations are also summarized for reference in Table 2.16.1, Eq. (b)

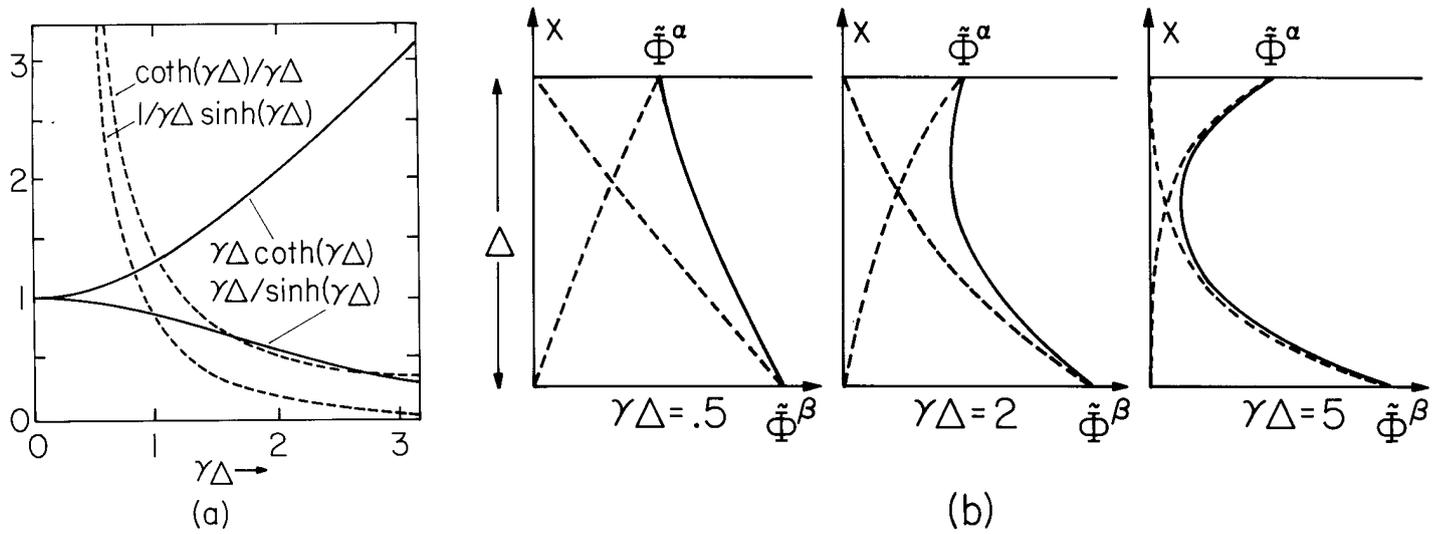


Fig. 2.16.1. (a) Transfer coefficients as a function of  $\Delta\gamma \equiv \Delta \sqrt{k_y^2 + k_z^2}$ .  
 (b) Distribution of  $\Phi$  across layer.

That the layer is essentially a distributed capacitance (inductance) is emphasized by drawing attention to the analogy between the transfer relations and constitutive laws for a system of linear capacitors (inductors). For a two-terminal-pair system, Eq. 2.11.5 comprises two terminal charges ( $q_1, q_2$ ) expressed as linear functions of the terminal voltages ( $v_1, v_2$ ). Analogously, the ( $D_x^\alpha, D_x^\beta$ ) (which have units of charge per unit area and an arbitrary time dependence) are given as linear functions of the potentials by Eq. (a) of Table 2.16.1. A similar analogy exists between Eq. 2.12.5, expressing ( $\lambda_1, \lambda_2$ ) as functions of ( $i_1, i_2$ ), and the transfer relations between ( $B_x^\alpha, B_x^\beta$ ) (units of flux per unit area) and the magnetic potentials ( $\Psi^\alpha, \Psi^\beta$ ).

According to Eq. (a) of Table 2.16.1,  $D_x$  is induced by a "self term" (proportional to the potential at the same surface) and a "mutual term." The coefficients which express this self- and mutual-coupling have a dependence on  $\Delta\gamma$  ( $2\pi/\gamma$  the wavelength in the y-z plane) shown in Fig. 2.16.1a. Written in the form of Eq. 15, the potential has components, excited at each surface, that decay to zero, as shown in Fig. 2.16.1b, at a rate that is proportional to how rapidly the fields vary in the y-z plane. For long waves the decay is relatively slow, as depicted by the case  $\Delta\gamma = 0.5$ , and the mutual-field is almost as great as the self field. But as the wavelength is shortened relative to  $\Delta$  ( $\Delta\gamma$  increased), the surfaces couple less and less.

In this discussion it is assumed that  $\gamma$  is real, which it is if  $k_y$  and  $k_z$  are real. In fact, the transfer relations are valid and useful for complex values of ( $k_y, k_z$ ). If these numbers are purely imaginary, the field distributions over the layer cross section are periodic. Such solutions are needed to satisfy boundary conditions imposed in an x-y plane.

**Cylindrical Annulus:** With the bounding surfaces coaxial cylinders having radii  $\alpha$  and  $\beta$ , it is natural to use cylindrical coordinates ( $r, \theta, z$ ). A cross section of this prototype region and the coordinates are shown in Table 2.16.2. On the outer and inner surfaces, the potential has the respective forms

$$\Phi(\alpha, \theta, z, t) = \text{Re } \tilde{\Phi}^\alpha(t) e^{-j(m\theta + kz)}; \quad \Phi(\beta, \theta, z, t) = \text{Re } \tilde{\Phi}^\beta(t) e^{-j(m\theta + kz)} \quad (17)$$

Hence, it is appropriate to assume a bulk potential

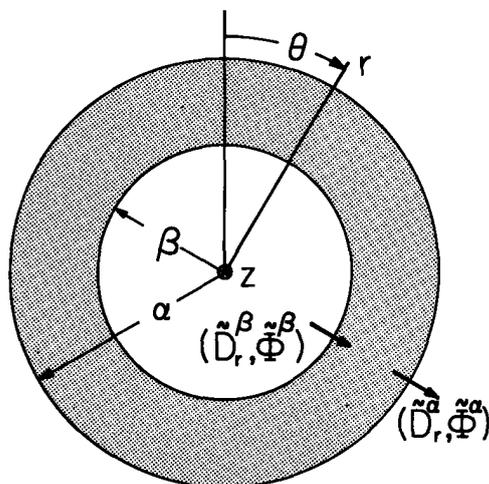
$$\Phi = \text{Re } \tilde{\Phi}(r, t) e^{-j(m\theta + kz)} \quad (18)$$

Substitution in Laplace's equation (see Appendix A for operations in cylindrical coordinates), Eq. 4, then shows that

$$\frac{d^2 \tilde{\Phi}}{dr^2} + \frac{1}{r} \frac{d\tilde{\Phi}}{dr} - (k^2 + \frac{m^2}{r^2}) \tilde{\Phi} = 0 \quad (19)$$

By contrast with Eq. 13, this one has space-varying coefficients. It is convenient to categorize the solutions according to the values of ( $m, k$ ). With  $m = 0$  and  $k = 0$ , the remaining terms are a perfect differential which can be integrated twice to give the solutions familiar from the problem of the field

Table 2.16.2. Flux-potential relations for cylindrical annulus in terms of electric potential and normal displacement ( $\Phi, D_r$ ). To obtain magnetic relations, substitute  $(\Phi, D_r, \epsilon) \rightarrow (\Psi, B_r, \mu)$ .



$$\Phi = \text{Re } \tilde{\Phi}(r, z) e^{-j(m\theta + kz)}$$

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \epsilon \begin{bmatrix} f_m(\beta, \alpha) & g_m(\alpha, \beta) \\ g_m(\beta, \alpha) & f_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad (a)$$

$$k = 0, m = 0$$

$$f_0(x, y) = \frac{1}{y} \ln\left(\frac{x}{y}\right); \quad g_0(x, y) = \frac{1}{x} \ln\left(\frac{x}{y}\right)$$

$$k = 0, m = 1, 2, \dots$$

$$f_m(x, y) = \frac{m}{y} \frac{\left[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m\right]}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$g_m(x, y) = \frac{2m}{x} \frac{1}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$k \neq 0, m = 0, 1, 2, \dots^*$$

$$f_m(x, y) = \frac{jk[H_m(jkx)J'_m(jky) - J_m(jkx)H'_m(jky)]}{[J_m(jkx)H'_m(jky) - J_m(jky)H'_m(jkx)]}$$

$$g_m(x, y) = \frac{-2j}{\pi x [J_m(jkx)H'_m(jky) - J_m(jky)H'_m(jkx)]}$$

$$f_m(x, y) = \frac{k[K'_m(kx)I'_m(ky) - I'_m(kx)K'_m(ky)]}{[I'_m(kx)K'_m(ky) - I'_m(ky)K'_m(kx)]}$$

$$g_m(x, y) = \frac{1}{x [I'_m(kx)K'_m(ky) - I'_m(ky)K'_m(kx)]}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} F_m(\beta, \alpha) & G_m(\alpha, \beta) \\ G_m(\beta, \alpha) & F_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad (b)$$

$$k = 0, m = 0$$

No inverse

$$k = 0, m = 1, 2, \dots$$

$$F_m(x, y) = \frac{y}{m} \frac{\left[\left(\frac{x}{y}\right)^m + \left(\frac{y}{x}\right)^m\right]}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$G_m(x, y) = \frac{2y}{m} \frac{1}{\left[\left(\frac{x}{y}\right)^m - \left(\frac{y}{x}\right)^m\right]}$$

$$k \neq 0, m = 0, 1, 2, \dots^*$$

$$F_m(x, y) = \frac{1}{jk} \frac{[J'_m(jkx)H'_m(jky) - H'_m(jkx)J'_m(jky)]}{[J'_m(jky)H'_m(jkx) - J'_m(jkx)H'_m(jky)]}$$

$$G_m(x, y) = \frac{-2}{j\pi k(kx) [J'_m(jky)H'_m(jkx) - J'_m(jkx)H'_m(jky)]}$$

$$F_m(x, y) = \frac{1}{k} \frac{[I'_m(kx)K'_m(ky) - K'_m(kx)I'_m(ky)]}{[I'_m(ky)K'_m(kx) - I'_m(kx)K'_m(ky)]}$$

$$G_m(x, y) = \frac{1}{k(kx) [I'_m(ky)K'_m(kx) - I'_m(kx)K'_m(ky)]}$$

$$\beta \rightarrow 0$$



$$\tilde{D}_r^\alpha = \epsilon f_m(0, \alpha) \tilde{\Phi}^\alpha; \quad f_m(0, \alpha) = -\frac{kI'_m(k\alpha)}{I_m(k\alpha)} \quad (c)$$

$$\alpha \rightarrow \infty$$



$$\tilde{D}_r^\beta = \epsilon f_m(\infty, \beta) \tilde{\Phi}^\beta; \quad f_m(\infty, \beta) = -\frac{kK'_m(k\beta)}{K_m(k\beta)} \quad (d)$$

\* See Prob. 2.17.2 for proof that  $H_m(jkx)J'_m(jkx) - J_m(jkx)H'_m(jkx) = -2/(\pi kx)$  and  $K_m(kx)I'_m(kx) - I_m(kx)K'_m(kx) = 1/kx$  incorporated into  $g_m$  and  $G_m$ .

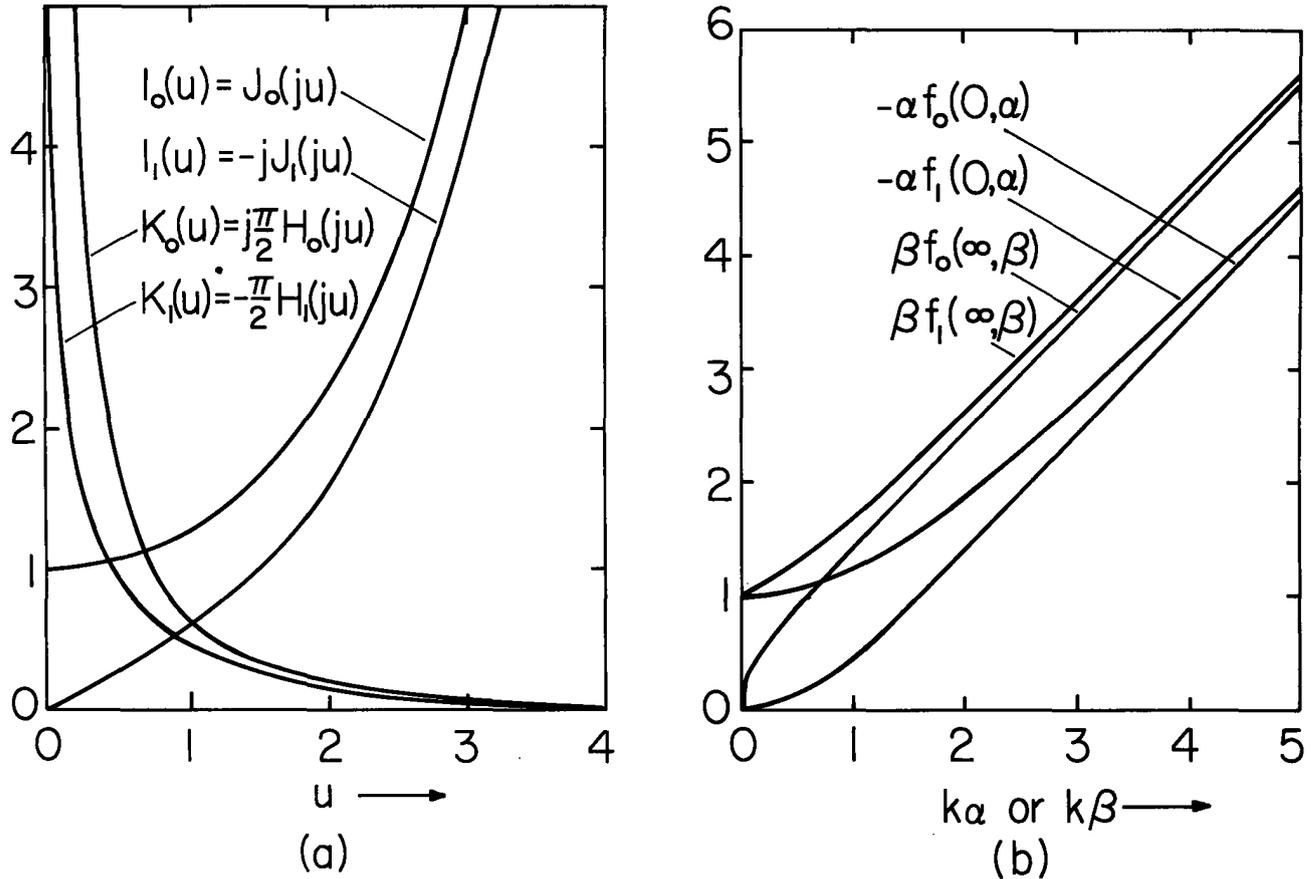


Fig. 2.16.2. (a) Modified Bessel functions. (b) Self-field coefficients of cylindrical transfer relations in limits where surfaces do not interact.

between coaxial circular conductors. In view of the boundary conditions at  $r = \alpha$  and  $r = \beta$ ,

$$\tilde{\phi} = \tilde{\phi}^\alpha \frac{\ln \left(\frac{r}{\beta}\right)}{\ln \left(\frac{\alpha}{\beta}\right)} - \tilde{\phi}^\beta \frac{\ln \left(\frac{r}{\alpha}\right)}{\ln \left(\frac{\alpha}{\beta}\right)} = \tilde{\phi}^\alpha + (\tilde{\phi}^\beta - \tilde{\phi}^\alpha) \frac{\ln \left(\frac{r}{\alpha}\right)}{\ln \left(\frac{\beta}{\alpha}\right)}; \quad (m, k) = (0, 0) \quad (20)$$

For situations that depend on  $\theta$ , but not on  $z$  (polar coordinates) so that  $k = 0$ , substitution shows the solutions to Eq. 19 are  $r^{\pm m}$ . By inspection or algebraic manipulation, the linear combination of these that satisfies the conditions of Eq. 17 is

$$\tilde{\phi} = \tilde{\phi}^\alpha \frac{\left[\left(\frac{\beta}{r}\right)^m - \left(\frac{r}{\beta}\right)^m\right]}{\left[\left(\frac{\beta}{\alpha}\right)^m - \left(\frac{\alpha}{\beta}\right)^m\right]} + \tilde{\phi}^\beta \frac{\left[\left(\frac{r}{\alpha}\right)^m - \left(\frac{\alpha}{r}\right)^m\right]}{\left[\left(\frac{\beta}{\alpha}\right)^m - \left(\frac{\alpha}{\beta}\right)^m\right]}; \quad (m, k) = (m, 0) \quad (21)$$

For  $k$  finite, the solutions to Eq. 19 are the modified Bessel functions  $I_m(kr)$  and  $K_m(kr)$ . These play a role in the circular geometry analogous to  $\exp(\pm\gamma x)$  in Cartesian geometry. The radial dependences of the functions of order  $m = 0$  and  $m = 1$  are shown in Fig. 2.16.2a. Note that  $I_m$  and  $K_m$  are respectively singular at infinity and the origin.

Just as the exponential solutions could be determined from Eq. 13 by assuming a power series in  $x$ , the Bessel functions are determined from an infinite series solution to Eq. 19. Like  $\gamma$ ,  $k$  can in general be complex. If it is, it is customary to define two new functions which, in the special case where  $k$  is real, have imaginary arguments:

$$J_m(jkr) \equiv j^m I_m(kr), \quad H_m(jkr) = \frac{2}{\pi} j^{-(m+1)} K_m(kr) \quad (22)$$

These are respectively the Bessel and Hankel functions of first kind. For real arguments,  $I_m$  and  $K_m$  are real, and hence  $J_m$  and  $H_m$  can be either purely real or imaginary, depending on the order.

Large real-argument limits of the functions  $I_m$  and  $K_m$  reinforce the analogy to the Cartesian

exponential solutions:

$$\lim_{u \rightarrow \infty} I_m(u) = \frac{1}{\sqrt{2\pi u}} \exp(u); \lim_{u \rightarrow \infty} K_m(u) = \sqrt{\frac{\pi}{2u}} \exp(-u) \quad (23)$$

Useful relations in the opposite extreme of small arguments are

$$\lim_{u \rightarrow 0} jH_0(ju) = \frac{2}{\pi} \ln \left( \frac{2}{1.781072u} \right); \lim_{u \rightarrow 0} J_m(ju) = \frac{(ju)^m}{m!2^m} \quad (24)$$

$$\lim_{u \rightarrow 0} H_m(ju) = \frac{(m-1)!2^m}{j\pi(ju)^m}; m \neq 0$$

By inspection or algebraic manipulation, the linear combination of  $J_m$  and  $H_m$  satisfying the boundary conditions of Eq. 17 is

$$\tilde{\phi} = \tilde{\phi}^\alpha \frac{[H_m(jk\beta)J_m(jkr) - J_m(jk\beta)H_m(jkr)]}{[H_m(jk\beta)J_m(jk\alpha) - J_m(jk\beta)H_m(jk\alpha)]} + \tilde{\phi}^\beta \frac{[J_m(jk\alpha)H_m(jkr) - H_m(jk\alpha)J_m(jkr)]}{[J_m(jk\alpha)H_m(jk\beta) - H_m(jk\alpha)J_m(jk\beta)]} \quad (25)$$

The evaluation of the surface displacements ( $D_r^\alpha, D_r^\beta$ ) using Eqs. 20, 21, or 25 is now accomplished using the same steps as for the planar layer. The resulting transfer relations are summarized by Eq. (a) in Table 2.16.2. Inversion of these relations, to give the surface potentials as functions of the surface displacements, results in the relations summarized by Eq. (b) of that table. Primes denote derivatives with respect to the entire specified argument of the function. Useful identities are:

$$\begin{aligned} uI'_m(u) &= mI_m(u) + uI_{m+1}(u); uI'_m(u) = -mI_m(u) + uI_{m-1}(u) \\ uK'_m(u) &= mK_m(u) - uK_{m+1}(u) \\ R'_0(u) &= -R_1(u) \\ uR'_m(u) &= -mR_m(u) + uR_{m-1}(u); uR'_m(u) = mR_m(u) - uR_{m+1}(u) \end{aligned} \quad (26)$$

where  $R_m$  can be  $J_m$ ,  $H_m$ , or the function  $N_m$  to be defined with Eq. 29.

Two useful limits of the transfer relations are given by Eqs. (c) and (d) of Table 2.16.2. In the first, the inner surface is absent, while in the second the outer surface is removed many wavelengths  $2\pi/k$ . The self-field coefficients  $f_m(0, \alpha)$  and  $f_m(\infty, \beta)$  are sketched for  $m=0$  and  $m=1$  in Fig. 2.16.2b. Again, it is useful to note the analogy to the planar layer case where the appropriate limit is  $k\Delta \rightarrow \infty$ . In fact, for  $k\alpha$  or  $k\beta$  reasonably large, the  $k$  dependence and the signs are the same as for the planar geometry:

$$\lim_{k\alpha \rightarrow \infty} \alpha f_m(0, \alpha) \rightarrow -k\alpha; \lim_{k\beta \rightarrow \infty} \beta f_m(\infty, \beta) \rightarrow k\beta \quad (27)$$

For small arguments, these functions become

$$\lim_{k\alpha \rightarrow 0} \alpha f_0(0, \alpha) \rightarrow -\frac{(k\alpha)^2}{2}; \lim_{k\beta \rightarrow 0} \beta f_0(\infty, \beta) \rightarrow \frac{1}{\ln \left[ \frac{2}{1.781072k\beta} \right]} \quad (28)$$

$$\lim_{k\alpha \rightarrow 0} \alpha f_m(0, \alpha) \rightarrow -m \text{ for } m \neq 0; \lim_{k\beta \rightarrow 0} \beta f_m(\infty, \beta) \rightarrow m \text{ for } m \neq 0$$

In general,  $k$  can be complex. In fact the most familiar form for Bessel functions is with  $k$  purely imaginary. In that case,  $J_m$  is real but  $H_m$  is complex. By convention

$$H_m(u) \equiv J_m(u) + jN_m(u) \quad (29)$$

where, if  $u$  is real,  $J_m$  and  $N_m$  are real and Bessel functions of first and second kind. As might be expected from the planar analogue, the radial dependence becomes periodic if  $k$  is imaginary. Plots of the functions in this case are given in Fig. 2.16.3.

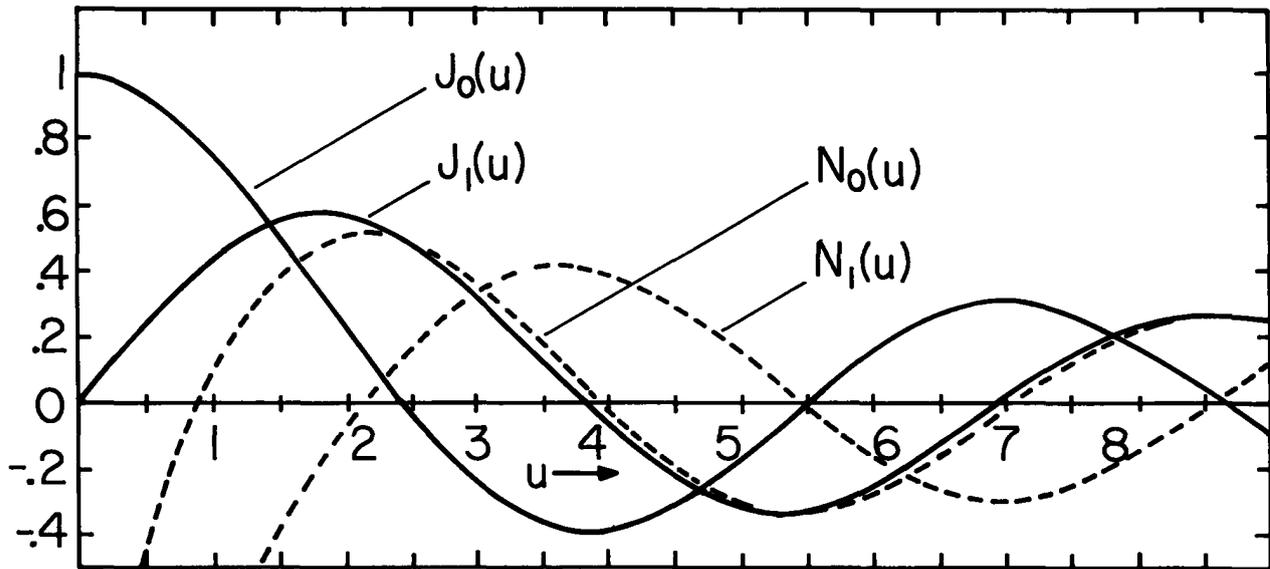


Fig. 2.16.3. Bessel functions of first and second kind and real arguments. References for the Bessel and related functions should be consulted for more details concerning their properties and numerical values.<sup>1-4</sup>

**Spherical Shell:** A region between spherical surfaces having outer and inner radii  $\alpha$  and  $\beta$ , respectively, is shown in the figure of Table 2.16.3. In the volume, the potential conveniently takes the variable separable form

$$\phi = \text{Re } \tilde{\Phi}(r, t) \Theta(\theta) e^{-jm\phi} \quad (30)$$

where  $(r, \theta, \phi)$  are spherical coordinates as defined in the figure. Substitution of Eq. 30 into Laplace's equation, Eq. 4, shows that the  $\phi$  dependence is correctly assumed and that the  $(r, \theta)$  dependence is determined from the equations

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left[ \sin \theta \frac{d\Theta}{d\theta} \right] - \frac{m^2}{\sin^2 \theta} = -K^2 \quad (31)$$

$$\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{d\tilde{\Phi}}{dr} \right) = K^2$$

where the separation coefficient  $K^2$  is independent of  $(r, \theta)$ . With the substitutions

$$u = \cos \theta, \sqrt{1 - u^2} = \sin \theta \quad (32)$$

Eq. 31a is converted to

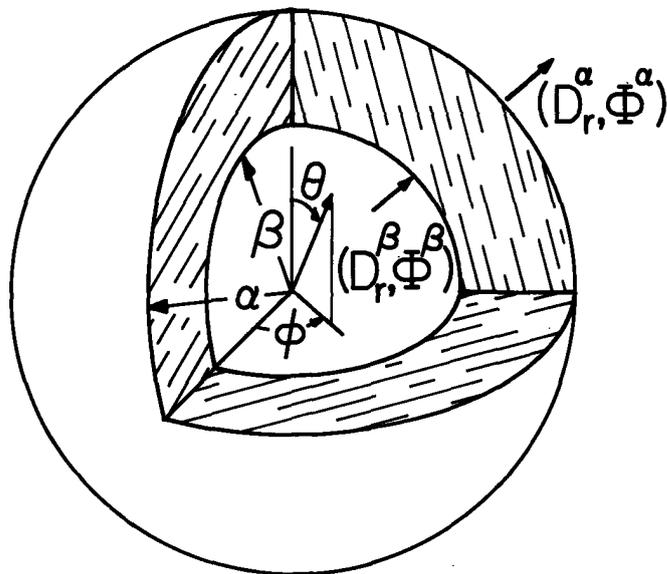
$$(1 - u^2) \frac{d^2\Theta}{du^2} - 2u \frac{d\Theta}{du} + \left( K^2 - \frac{m^2}{1-u^2} \right) \Theta = 0 \quad (33)$$

For  $K^2 = n(n+1)$  and  $n$  an integer, solutions to Eq. 33 are

$$\Theta = P_n^m(u) \quad (34)$$

1. F. B. Hildebrand, Advanced Calculus for Applications, Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 142-165.
2. S. Ramo, J. R. Whinnery and T. Van Duzer, Fields and Waves in Communication Electronics, John Wiley and Sons, New York, 1965, pp. 207-218.
3. M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, National Bureau of Standards, Applied Mathematics Series 55, U.S. Government Printing Office, Washington D. C. 20402, 1964, pp. 355-494.
4. E. Jahnke and F. Emde, Table of Functions with Formulae and Curves, Dover Publications, New York, 1945, pp. 128-210.

Table 2.16.3. Flux-potential transfer relations for spherical shell in terms of electric potential and normal displacement ( $\Phi, D_r$ ). To obtain magnetic relations, substitute  $(\Phi, D_r, \epsilon) \rightarrow (\Psi, B_r, \mu)$ .



$$\Phi = \text{Re } \tilde{\Phi}(r, t) P_n^m(\cos \theta) e^{-jm\phi}$$

$$P_n^m = (1-x^2)^{m/2} \frac{d^m P_n}{dx^m}$$

$$P_0 = 1, P_1 = x, P_2 = \frac{1}{2}(3x^2 - 1)$$

$$P_3 = \frac{1}{2}(5x^3 - 3x)$$

$$P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

m	$P_0^m$	$P_1^m$	$P_1^m \cos m\phi$	$P_2^m$	$P_2^m \cos m\phi$	$P_3^m$	$P_3^m \cos m\phi$
0	1	$\cos \theta$	$\begin{array}{ c } \hline + \\ \hline - \\ \hline \end{array}$	$\frac{1}{2}(3 \cos^2 \theta - 1)$	$\begin{array}{ c } \hline + \\ \hline - \\ \hline + \\ \hline \end{array}$	$\frac{1}{2}(5 \cos^3 \theta - 3 \cos \theta)$	$\begin{array}{ c } \hline + \\ \hline - \\ \hline + \\ \hline - \\ \hline \end{array}$
1	0	$\sin \theta$	$\begin{array}{ c c c } \hline + & - & + \\ \hline \end{array}$	$3 \sin \theta \cos \theta$	$\begin{array}{ c c c } \hline + & - & + \\ \hline - & + & - \\ \hline \end{array}$	$\frac{3}{2} \sin \theta (5 \cos^2 \theta - 1)$	$\begin{array}{ c c c } \hline + & - & + \\ \hline - & + & - \\ \hline + & - & + \\ \hline \end{array}$
2	0	0		$3 \sin^2 \theta$	$\begin{array}{ c c c c } \hline + & - & + & - \\ \hline \end{array}$	$15 \sin^2 \theta \cos \theta$	$\begin{array}{ c c c c } \hline + & - & + & - \\ \hline - & + & - & + \\ \hline \end{array}$
3	0	0		0		$15 \sin^3 \theta$	$\begin{array}{ c c c c c } \hline + & - & + & - & + \\ \hline \end{array}$

$$\begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} = \epsilon \begin{bmatrix} f_n(\beta, \alpha) & g_n(\alpha, \beta) \\ g_n(\beta, \alpha) & f_n(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} \quad (a)$$

$$f_n(x, y) = \frac{[n(\frac{y}{x})^n + (n+1)(\frac{x}{y})^{n+1}]}{[x(\frac{x}{y})^n - y(\frac{y}{x})^n]}$$

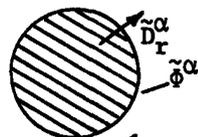
$$g_n(x, y) = \frac{(2n+1)}{x^2 [\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$$\begin{bmatrix} \tilde{\Phi}^\alpha \\ \tilde{\Phi}^\beta \end{bmatrix} = \frac{1}{\epsilon} \begin{bmatrix} F_n(\beta, \alpha) & G_n(\alpha, \beta) \\ G_n(\beta, \alpha) & F_n(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{D}_r^\alpha \\ \tilde{D}_r^\beta \end{bmatrix} \quad (b)$$

$$F_n(x, y) = \frac{y}{x} \frac{[\frac{1}{n}(\frac{y}{x})^n + \frac{1}{n+1}(\frac{x}{y})^{n+1}]}{[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

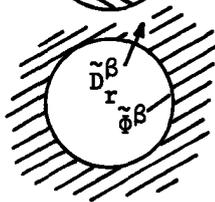
$$G_n(x, y) = \frac{y}{x} \frac{(2n+1)}{n(n+1)} \frac{1}{[\frac{1}{y}(\frac{x}{y})^n - \frac{1}{x}(\frac{y}{x})^n]}$$

$\beta \rightarrow 0$



$$\tilde{D}_r^\alpha = -\frac{\epsilon n}{\alpha} \tilde{\Phi}^\alpha \quad (c)$$

$\alpha \rightarrow \infty$



$$\tilde{D}_r^\beta = \frac{\epsilon(n+1)}{\beta} \tilde{\Phi}^\beta \quad (d)$$

where  $P_n^m$  are the associated Legendre functions of the first kind, order  $n$  and degree  $m$ . In terms of the Legendre polynomials  $P_n$ , these functions are summarized in Table 2.16.3. Note that these solutions are closed. They do not require infinite series for their representation.

To the second order differential equation, Eq. 33, there must be a second set of solutions  $Q_n^m$ . Because these are singular in the interval  $0 \leq \theta \leq \pi$ , and situations of interest here include the entire spherical surface at any given radius, these solutions are not included. The functions  $P_n^m$  play the role of  $\exp(jkz)$  (say) in cylindrical geometry, while  $\exp(jm\phi)$  is analogous to  $\exp(jm\theta)$ . The radial dependence, which is much of the bother in cylindrical coordinates, is actually quite simple in spherical coordinates. From Eq. 31b it is seen that solutions are a linear combination of  $r^n$  and  $r^{-(n+1)}$ . With the assumption that surface potentials respectively have the form

$$\Phi_{\beta}^{\alpha}(\theta, \phi, t) = \text{Re } \tilde{\Phi}_{\beta}^{\alpha}(t) P_n^m(\cos\theta) \exp(jm\phi) \quad (35)$$

it follows that the appropriate linear combination is

$$\tilde{\Phi} = \tilde{\Phi}^{\alpha} \frac{[(\frac{r}{\beta})^n - (\frac{\beta}{r})^{n+1}]}{[(\frac{\alpha}{\beta})^n - (\frac{\beta}{\alpha})^{n+1}]} + \tilde{\Phi}^{\beta} \frac{[(\frac{r}{\alpha})^n - (\frac{\alpha}{r})^{n+1}]}{[(\frac{\beta}{\alpha})^n - (\frac{\alpha}{\beta})^{n+1}]} \quad (36)$$

The complex amplitudes  $(\tilde{\Phi}^{\alpha}, \tilde{\Phi}^{\beta})$  determine the combination of  $\cos m\phi$  and  $\sin m\phi$ , constituting the distribution of  $\Phi$  with longitudinal distance. For a real amplitude, the distribution is proportional to  $\cos m\phi$ . In the summary of Table 2.16.3, the lowest orders of  $P_n^m(\cos\theta)$  are tabulated, together with diagrams showing the zones that are positive and negative relative to each other. In the rectangular plots, the ordinate is  $0 \leq \theta \leq \pi$ , while the abscissa is  $0 \leq \phi \leq 2\pi$ . Thus, the top and bottom lines are the north and south poles while the lines within are nodes. The horizontal register of each diagram is determined by the complex amplitude, which determines the phase of  $\exp(jm\phi)$ .

Evaluation of the transfer relations given in Table 2.16.3 by Eqs. (a) and (b) is now carried out following the same procedure as for the planar layer. From these relations follow the limiting situations of a solid spherical region or one where the outer surface is well removed from the region of interest summarized for reference by Eqs. (c) and (d) of Table 2.16.3.

Further useful aspects of solutions to Laplace's equation in spherical coordinates, including orthogonality relations that permit Fourier-like expansions and evaluation of averages, are given in standard references.<sup>5</sup>

## 2.17 Energy Conservation and Quasistatic Transfer Relations

Applied to one of the three regions considered in Sec. 2.16, the incremental total electric energy given by Eq. 2.13.6, can be written as

$$\delta w = - \int_V \nabla\Phi \cdot \delta\vec{D} dV = - \int_V \nabla \cdot (\Phi\delta\vec{D}) dV + \int_V \Phi \nabla \cdot \delta\vec{D} dV \quad (1)$$

Because  $\rho_f = 0$ , the last integral is zero. The remaining integral is converted to a surface integral by Gauss' theorem, and the equation reduces to

$$\delta w = - \oint_S \Phi \delta\vec{D} \cdot \vec{n} da \quad (2)$$

Similar arguments apply in the magnetic cases. Because there is no volume free current density,  $\vec{H} = -\nabla\Psi$  and Eq. 2.14.9 becomes

$$\delta w = - \oint_S \Psi \delta\vec{B} \cdot \vec{n} da \quad (3)$$

Consider now the implications of these last two expressions for the transfer relations derived in Sec. 2.16. Discussion is in terms of the electrical relations, but the analogy made in Sec. 2.16 clearly pertains as well to Eqs. 2 and 3, so that the arguments also apply to the magnetic transfer relations.

Suppose that the increment of energy  $\delta w$  is introduced through  $S$  to a volume bounded by sections of the  $\alpha$  and  $\beta$  surfaces extending one "wavelength" in the surface dimensions. In Cartesian coordinates,

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5. F. B. Hildebrand, loc. cit., pp. 159-165.

this volume is bounded by (y,z) surfaces extending one wavelength in the y and z directions. In cylindrical coordinates, the volume is a pie-shaped cylinder subtended by outside and inside surfaces having length  $2\pi/k$  in the z direction and  $2\pi\alpha/m$  and  $2\pi\beta/m$  respectively in the azimuthal direction. In spherical coordinates, the volume is a sector from a sphere with  $\theta = 2\pi/m$  radians along the equator,  $\theta$  extending from  $0 \rightarrow \pi$  and the surfaces at  $r = \alpha$  and  $r = \beta$ . In any of these cases, conservation of energy, as expressed by Eq. 2, requires that

$$\delta w = -a^\alpha \left\langle\left\langle \tilde{\phi}^\alpha \delta D_n^\alpha \right\rangle\right\rangle + a^\beta \left\langle\left\langle \tilde{\phi}^\beta \delta D_n^\beta \right\rangle\right\rangle \quad (4)$$

The  $\langle\langle \rangle\rangle$  indicate averages over the respective surfaces of excitation. The areas ( $a^\alpha, a^\beta$ ) are in particular

$$a^\beta = \begin{cases} (2\pi)^2/k_y k_z & \text{Cartesian} \\ [(2\pi)^2/mk] \binom{\alpha}{\beta} & \text{cylindrical} \\ (4\pi/m) \binom{\alpha^2}{\beta^2} & \text{spherical} \end{cases} \quad (5)$$

In writing Eq. 2 as Eq. 4, contributions of surfaces other than the  $\alpha$  and  $\beta$  surfaces cancel because of the spatial periodicity. It is assumed that  $(k_y, k_z)$ ,  $(m, k)$  and  $m$  are real numbers.

The transfer relations developed in Sec. 2.16 take the general form

$$\begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix} = \begin{bmatrix} -A_{11} & A_{12} \\ -A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \tilde{D}_n^\alpha \\ \tilde{D}_n^\beta \end{bmatrix} \quad (6)$$

The coefficients  $A_{ij}$  are real. Hence, for the purpose of deducing properties of  $A_{ij}$ , there is no loss in generality in taking  $(\tilde{D}_n^\alpha, \tilde{D}_n^\beta)$  and hence  $(\tilde{\phi}^\alpha, \tilde{\phi}^\beta)$  as being real. Then, Eq. 4 takes the form

$$\delta w = C[-a^\alpha \tilde{\phi}^\alpha \delta \tilde{D}_n^\alpha + a^\beta \tilde{\phi}^\beta \delta \tilde{D}_n^\beta] \quad (7) \quad \times$$

where  $C$  is  $1/2$  in the Cartesian and cylindrical cases and is a positive constant in the spherical case.

With the assumption that  $w = w(\tilde{D}_n^\alpha, \tilde{D}_n^\beta)$ , the incremental energy can also be written as

$$\delta w = \frac{\partial w}{\partial \tilde{D}_n^\alpha} \delta \tilde{D}_n^\alpha + \frac{\partial w}{\partial \tilde{D}_n^\beta} \delta \tilde{D}_n^\beta \quad (8)$$

where  $(\tilde{D}_n^\alpha, \tilde{D}_n^\beta)$  constitute independent electrical "terminal" variables. Thus, from Eqs. 7 and 8,

$$-a^\alpha \tilde{\phi}^\alpha = \frac{\partial w}{\partial \tilde{D}_n^\alpha} ; a^\beta \tilde{\phi}^\beta = \frac{\partial w}{\partial \tilde{D}_n^\beta} \quad (9)$$

A reciprocity condition is obtained by taking derivatives of these expressions with respect to  $\tilde{D}_n^\beta$  and  $\tilde{D}_n^\alpha$ , respectively, and eliminating the energy function. In view of the transfer relations, Eq. 6,

$$a^\alpha A_{12} = a^\beta A_{21} \quad (10)$$

Thus, in the planar layer where the areas  $a^\alpha$  and  $a^\beta$  are equal, the mutual coupling terms  $A_{12} = A_{21}$ . That the relations are related by Eq. 10 in the spherical case is easily checked, but the complicated expressions for the cylindrical case simplify the mutual terms (footnote to Table 2.16.2).

The energy can be evaluated by integrating Eq. 7 using the "constitutive" laws of Eq. 6. The integration is first carried out with  $\tilde{D}_n^\beta = 0$ , raising  $\tilde{D}_n^\alpha$  to its final value. Then, with  $\tilde{D}_n^\alpha = \tilde{D}_n^\alpha$ ,  $\tilde{D}_n^\beta$  is raised to its final value

$$w = C \left[ \frac{1}{2} a^\alpha A_{11} (\tilde{D}_n^\alpha)^2 - a^\beta A_{21} \tilde{D}_n^\alpha \tilde{D}_n^\beta + \frac{1}{2} a^\beta A_{22} (\tilde{D}_n^\beta)^2 \right] \quad (11)$$

With either excitation alone,  $w$  must be positive and so from this relation it follows that

$$A_{11} > 0, \quad A_{22} > 0 \quad (12)$$

These conditions are also met by the relations found in Sec. 2.16.

### 2.18 Solenoidal Fields, Vector Potential and Stream Function

Irrotational fields, such as the quasistatic electric field, are naturally represented by a scalar potential. Not only does this reduce the vector field to a scalar field, but the potential function evaluated on such surfaces as those of "perfectly" conducting electrodes becomes a lumped parameter terminal variable, e.g., the voltage.

Solenoidal fields, such as the magnetic flux density  $\vec{B}$ , are for similar reasons sometimes represented in terms of a vector potential  $\vec{A}$ :

$$\vec{B} = \nabla \times \vec{A} \quad (1)$$

Thus,  $\vec{B}$  automatically has no divergence. Unfortunately, the vector field  $\vec{B}$  is represented in terms of another vector field  $\vec{A}$ . However, for important two-dimensional or symmetric configurations, a single component of  $\vec{A}$  is all required to again reduce the description to one involving a scalar function. Four commonly encountered cases are summarized in Table 2.18.1.

The first two are two-dimensional in the usual sense. The field  $\vec{B}$  lies in the x-y (or r- $\theta$ ) plane and depends only on these coordinates. The associated vector potential has only a z component. The third configuration, like the second, is in cylindrical geometry, but with  $\vec{B}$  independent of  $\theta$  and hence with  $\vec{A}$  having only an  $i_\theta$  component. The fourth configuration is in spherical geometry with symmetry about the z axis and the vector potential directed along  $\phi_0$ .

Like the scalar potential used to represent irrotational fields, the vector potential is closely related to lumped parameter variables. If  $\vec{B}$  is the magnetic flux density, it is convenient for evaluation of the flux linkage  $\lambda$  (Eq. 2.12.1). For an incompressible flow, where  $\vec{B}$  is replaced by the fluid velocity  $\vec{v}$ , the vector potential is conveniently used to evaluate the volume rate of flow. In that application,  $A$  and  $\Lambda$  become "stream functions."

The connection between the flux linked and the vector potential follows from Stokes's theorem, Eq. 2.6.3. The flux  $\Phi_\lambda$  through a surface  $S$  enclosed by a contour  $C$  is

$$\Phi_\lambda = \int_S \vec{B} \cdot \vec{n} da = \int_S \nabla \times \vec{A} \cdot \vec{n} da = \oint_C \vec{A} \cdot d\vec{\ell} \quad (2)$$

In each of the configurations of Table 2.18.1, Eq. 2 amounts to an evaluation of the surface integral. For example, in the Cartesian two-dimensional configuration, contributions to the integration around a contour  $C$  enclosing a surface having length  $\ell$  in the z direction, only come from the legs running in the z direction. Along these portions of the contour, denoted by (a) and (b), the coordinates (x,y) are

constant. Hence, the flux through the surface is simply  $\ell$  times the difference  $A(a) - A(b)$ , as summarized in Table 2.18.1.

In the axisymmetric cylindrical and spherical configurations,  $r$  and  $r \sin \theta$  dependences are respectively introduced, so that evaluation of  $\Lambda$  essentially gives the flux linked. For example, in the spherical configuration, the flux linked by a surface having inner and outer radii  $r \cos \theta$  evaluated at (a) and (b) is simply

$$\Phi_\lambda = \oint_C \frac{\Lambda(r, \theta)}{r \sin \theta} \vec{i}_\phi \cdot d\vec{\ell} = \frac{\Lambda}{r \sin \theta} 2\pi(r \sin \theta) \Big|_b^a = 2\pi[\Lambda(a) - \Lambda(b)] \quad (3)$$

Used in fluid mechanics to represent incompressible fluid flow,  $\Lambda$  is the Stokes's stream function. Note that the flux is positive if directed through the surface in the direction of  $\vec{n}$ , which is specified in terms of the contour  $C$  by the right-hand rule.

### 2.19 Vector Potential Transfer Relations for Certain Laplacian Fields

Even in dealing with magnetic fields in regions where  $\vec{J}_f = 0$ , if the flux linkages are of interest, it is often more convenient to develop a model in terms of transfer relations specified in terms of a vector rather than scalar potential. The objective in this section is to summarize these relations for the first three configurations identified in Table 2.18.1.

Table 2.18.1. Important configurations having solenoidal field  $\vec{B}$  represented by single components of vector potential  $\vec{A}$ .

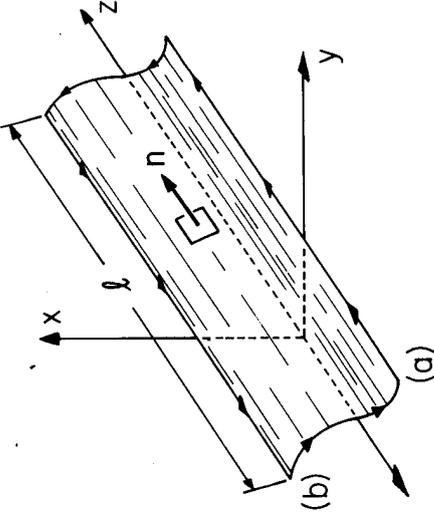
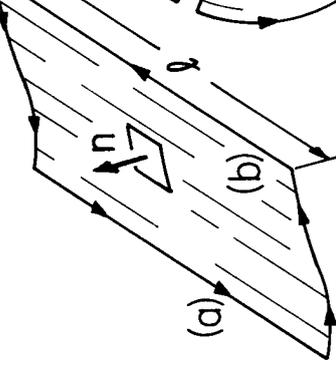
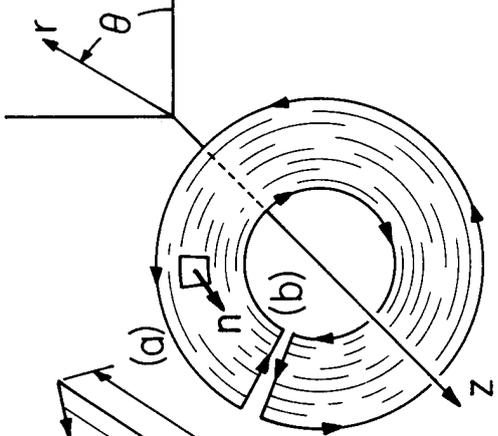
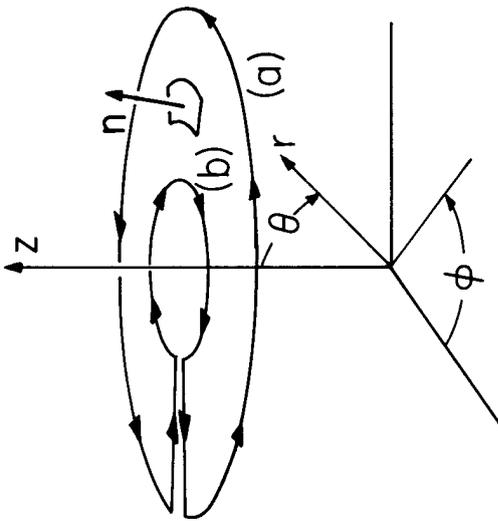
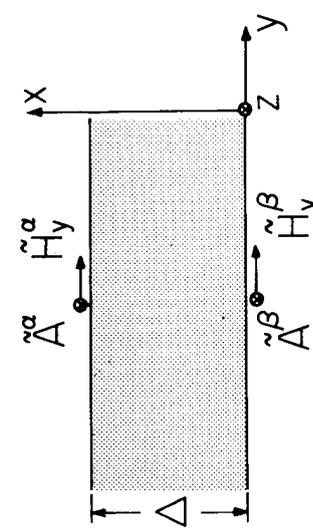
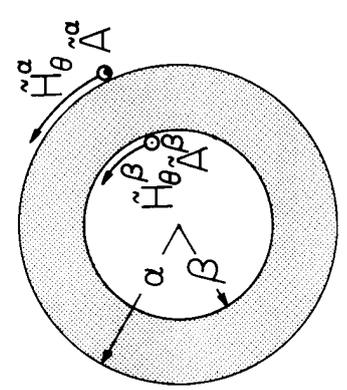
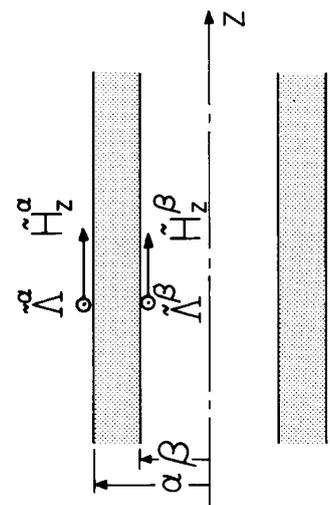
			
Two-dimensional Cartesian	Polar	Axisymmetric cylindrical	Axisymmetric spherical
$\vec{A} = A(x, y) \hat{i}_z \quad (a)$ $\vec{B} = \frac{\partial A}{\partial y} \hat{i}_x - \frac{\partial A}{\partial x} \hat{i}_y \quad (b)$ $\phi_\lambda = \ell [A(a) - A(b)] \quad (c)$	$\vec{A} = A(r, \theta) \hat{i}_z \quad (d)$ $\vec{B} = \frac{1}{r} \frac{\partial A}{\partial \theta} \hat{i}_r - \frac{\partial A}{\partial r} \hat{i}_\theta \quad (e)$ $\phi_\lambda = \ell [A(a) - A(b)] \quad (f)$	$\vec{A} = \frac{\Lambda(r, z)}{r} \hat{i}_\theta \quad (g)$ $\vec{B} = -\frac{1}{r} \frac{\partial \Lambda}{\partial z} \hat{i}_r + \frac{1}{r} \frac{\partial \Lambda}{\partial r} \hat{i}_z \quad (h)$ $\phi_\lambda = 2\pi [\Lambda(a) - \Lambda(b)] \quad (i)$	$\vec{A} = \frac{\Lambda(r, \theta)}{r \sin \theta} \hat{i}_\phi \quad (j)$ $\vec{B} = \frac{1}{r \sin \theta} \left[ \frac{1}{r} \frac{\partial \Lambda}{\partial \theta} \hat{i}_r - \frac{\partial \Lambda}{\partial r} \hat{i}_\theta \right] \quad (k)$ $\phi_\lambda = 2\pi [\Lambda(a) - \Lambda(b)] \quad (l)$

Table 2.19.1. Vector potential transfer relations for two-dimensional or symmetric Laplacian fields.

Two-dimensional Cartesian	Polar	Axisymmetric cylindrical
		
$\vec{A} = \hat{i}_z \operatorname{Re} \tilde{A}(x) \exp(-jky)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{H}_y^\alpha \end{bmatrix} = \frac{k}{\mu} \begin{bmatrix} -\coth(k\Delta) \frac{1}{\sinh(k\Delta)} \\ \frac{-1}{\sinh(k\Delta)} \coth(k\Delta) \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} \quad (a)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} = \frac{\mu}{k} \begin{bmatrix} -\coth(k\Delta) \frac{1}{\sinh(k\Delta)} \\ \frac{-1}{\sinh(k\Delta)} \coth(k\Delta) \end{bmatrix} \begin{bmatrix} \tilde{H}_y^\alpha \\ \tilde{H}_y^\beta \end{bmatrix} \quad (b)$	$\vec{A} = \hat{i}_z \operatorname{Re} \tilde{A}(r) \exp(-jm\theta)$ $\begin{bmatrix} \tilde{H}_\theta^\alpha \\ \tilde{H}_\theta^\beta \end{bmatrix} = \frac{1}{\mu} \begin{bmatrix} f_m(\beta, \alpha) g_m(\alpha, \beta) \\ g_m(\beta, \alpha) f_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} \quad (c)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} = \mu \begin{bmatrix} F_m(\beta, \alpha) G_m(\alpha, \beta) \\ G_m(\beta, \alpha) F_m(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{H}_\theta^\alpha \\ \tilde{H}_\theta^\beta \end{bmatrix} \quad (d)$	$\vec{A} = \hat{i}_\theta \operatorname{Re} \tilde{A}(r) \exp(-jkz); \quad \tilde{A} = \tilde{A}r$ $\begin{bmatrix} \tilde{H}_z^\alpha \\ \tilde{H}_z^\beta \end{bmatrix} = -\frac{k^2}{\mu} \begin{bmatrix} F_o(\beta, \alpha) G_o(\alpha, \beta) \\ G_o(\beta, \alpha) F_o(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} \quad (e)$ $\begin{bmatrix} \tilde{A}^\alpha \\ \tilde{A}^\beta \end{bmatrix} = -\left(\frac{\mu}{k}\right) \begin{bmatrix} f_o(\beta, \alpha) g_o(\alpha, \beta) \\ g_o(\beta, \alpha) f_o(\alpha, \beta) \end{bmatrix} \begin{bmatrix} \tilde{H}_z^\alpha \\ \tilde{H}_z^\beta \end{bmatrix} \quad (f)$
$\begin{bmatrix} \hat{A}^\alpha \\ \hat{A}^\beta \end{bmatrix} = \frac{j}{k} \begin{bmatrix} \hat{B}_x^\alpha \\ \hat{B}_x^\beta \end{bmatrix}$	$\begin{bmatrix} \hat{A}^\alpha \\ \hat{A}^\beta \end{bmatrix} = \frac{j}{m} \begin{bmatrix} \alpha \hat{B}_r^\alpha \\ \beta \hat{B}_r^\beta \end{bmatrix}$	$\begin{bmatrix} \hat{A}^\alpha \\ \hat{A}^\beta \end{bmatrix} = \frac{-j}{k} \begin{bmatrix} \hat{B}_r^\alpha \\ \hat{B}_r^\beta \end{bmatrix}$
<p>For <math>f_m, g_m, F_m, G_m</math> see Table 2.16.2, <math>k = 0, m \neq 0</math></p>		<p>For <math>F_o, G_o, f_o, g_o</math> see Table 2.16.2, <math>m = 0, k \neq 0</math></p>

With  $\vec{B}$  represented in terms of  $\vec{A}$  by Eq. 2.18.1, Ampère's law (Eq. 2.3.23) requires that in a region of uniform permeability  $\mu$ ,

$$\nabla \times \nabla \times \vec{A} = \mu \vec{J}_f \quad (1)$$

For a given magnetic flux density  $\vec{B}$ ,  $\text{curl } \vec{A}$  is specified. But to make  $\vec{A}$  unique, its divergence must also be specified. Here, the divergence of  $\vec{A}$  is defined as zero. Thus, the vector identity  $\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$  reduces Eq. 1 to the vector Poisson's equation:

$$\nabla^2 \vec{A} = -\mu \vec{J}_f; \quad \nabla \cdot \vec{A} = 0 \quad (2)$$

The vector Laplacian is summarized in Appendix A for the three coordinate systems of Table 2.18.1. Even though the region described in the following developments is one where  $\vec{J}_f = 0$ , the source term on the right has been carried along for later reference.

**Cartesian Coordinates:** In the Cartesian coordinate system of Table 2.18.1 it is the z component of Eq. 2 that is of interest. The z component of the vector Laplacian is the same operator as for the scalar Laplacian. Thus, the situation is analogous to that outlined by Eqs. 2.16.11 to 2.16.16 with  $\phi \rightarrow A$ . With solutions of the form  $A = \text{Re } A(x,t) \exp(-jky)$  so that  $\gamma \rightarrow k_y \equiv k$ , the appropriate linear combination of solutions is

$$\vec{A} = \tilde{A}^\alpha \frac{\sinh kx}{\sinh k\Delta} - \tilde{A}^\beta \frac{\sinh k(x - \Delta)}{\sinh k\Delta} \quad (3)$$

Because  $\vec{H} = \vec{B}/\mu$ , the associated tangential field intensity is given by Eq. (b), Table 2.18.1,

$$H_y = -\frac{1}{\mu} \frac{\partial A}{\partial x} \quad (4)$$

Expressed in terms of Eq. 3 and evaluated at the surfaces  $x = \alpha$  and  $x = \beta$ , respectively, Eq. 4 gives the first transfer relations, Eq. (a), of Table 2.19.1. Inversion of these relations gives Eqs. (b).

**Polar Coordinates:** In cylindrical coordinates with no z dependence, it is again the z component of Eq. 2 that is pertinent. The configuration is summarized in Table 2.18.1. Solutions take the form  $A = \text{Re } A(r,t) \exp(-jm\theta)$  and are analogous to Eq. 2.16.21 with  $\phi$  replaced by A:

$$\vec{A} = \tilde{A}^\alpha \frac{[(\frac{\beta}{r})^m - (\frac{r}{\beta})^m]}{[(\frac{\beta}{\alpha})^m - (\frac{\alpha}{\beta})^m]} + \tilde{A}^\beta \frac{[(\frac{r}{\alpha})^m - (\frac{\alpha}{r})^m]}{[(\frac{\beta}{\alpha})^m - (\frac{\alpha}{\beta})^m]} \quad (5)$$

The tangential field is then evaluated from Eq. (e), Table 2.18.1:

$$H_\theta = -\frac{1}{\mu} \frac{\partial A}{\partial r} \quad (6)$$

Evaluation at the respective surfaces  $r = \alpha$  and  $r = \beta$  gives the transfer relations, Eqs. (c) of Table 2.19.1. Inversion of these relations gives Eqs. (d).

**Axisymmetric Cylindrical Coordinates:** By contrast with the two-dimensional configurations so far considered, where the vector Laplacian of  $A_z$  is the same as the scalar Laplacian, the vector nature of Eq. 2 becomes apparent in the axisymmetric cylindrical configuration. The  $\theta$  component of Eq. 2 is the scalar Laplacian of  $A_\theta$  plus  $(-A_\theta/r^2)$  (see Appendix A). With  $A_\theta \equiv A$ ,

$$\frac{\partial^2 A}{\partial r^2} + \frac{1}{r} \frac{\partial A}{\partial r} - \frac{A}{r^2} + \frac{\partial^2 A}{\partial z^2} = -\mu J_\theta \quad (7)$$

Even though solutions do not have a  $\theta$  dependence, so that

$$A = \text{Re } \tilde{A}(r,t) e^{-jkz} \quad (8)$$

equation 7 reduces to a form of Bessel's equation to which solutions are Bessel's and Hankel's functions of order unity:

$$\frac{\partial^2 \tilde{A}}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{A}}{\partial r} - (k^2 + \frac{1}{r^2}) \tilde{A} = -\mu \tilde{J} \quad (9)$$

(Compare Eq. 9 to Eq. 2.16.19.) It follows that solutions are of the form of Eq. 2.16.25 with  $\tilde{\Phi} \rightarrow \tilde{A}$  and  $m = 1$ :

$$\tilde{\Lambda} \equiv r\tilde{A} = \tilde{A}^\alpha \frac{H_1(jk\beta)[rJ_1(jkr)] - J_1(jk\beta)[rH_1(jkr)]}{H_1(jk\beta)J_1(jk\alpha) - J_1(jk\beta)H_1(jk\alpha)} + \tilde{A}^\beta \frac{J_1(jk\alpha)[rH_1(jkr)] - H_1(jk\alpha)[rJ_1(jkr)]}{J_1(jk\alpha)H_1(jk\beta) - H_1(jk\alpha)J_1(jk\beta)} \quad (10)$$

The tangential field intensity follows from Eq. 10 and Eq. (h) of Table 2.18.1:

$$H_z = \frac{1}{\mu r} \frac{\partial \Lambda}{\partial r} \quad (11)$$

In performing the differentiation, observe from Eq. 2.16.26d that whether  $R_m$  is  $J_m$  or  $H_m$ ,

$$\frac{d}{dr} [rR_1(jkr)] = jkrR_0(jkr) \quad (12)$$

Evaluation of  $H_z$  at the respective surfaces  $r = \alpha$  and  $r = \beta$  gives the transfer relations, Eqs. (e) of Table 2.19.1. Inversion of these relations gives Eqs. (f).

## 2.20 Methodology

As descriptions of subregions composing a heterogeneous system, transfer relations (illustrated for quasistatic fields in Sec. 2.16) are building blocks for describing complicated interactions. By appropriate identification of variables, the same relations can be used to describe different regions.

As an example, three planar regions are shown in Fig. 2.20.1. The symbols in parentheses denote positions adjacent to the surfaces demarking subregions. At the surfaces, variables can be discontinuous. Hence it is necessary to distinguish variables evaluated on adjacent sides of a boundary. The transfer relations describe the fields within the subregions and not across the boundaries.

The transfer relations of Table 2.16.1 can be applied to the upper region by identifying  $(\alpha) \rightarrow (d)$ ,  $(\beta) \rightarrow (e)$ ,  $\Delta \rightarrow a$  and  $\epsilon$  or  $\mu \rightarrow \epsilon_a$  or  $\mu_a$ . Similarly, for the middle region,  $(\alpha) \rightarrow (f)$ ,  $(\beta) \rightarrow (g)$ ,  $\Delta \rightarrow b$ , and  $\epsilon$  or  $\mu \rightarrow \epsilon_b$  or  $\mu_b$ . Boundary conditions and jump relations across the surfaces then provide coupling conditions on the surface variables. Once the surface variables have been self-consistently determined, the field distributions within the region can be evaluated using the bulk distributions evaluated in terms of the surface coefficients. With appropriate surface amplitudes and  $x \rightarrow x'$ , where the latter is defined for each region in Fig. 2.20.1, Eq. 2.16.15 describes the potential distribution.

This approach will be used not only in other geometries but in representing mechanical and electromechanical processes.

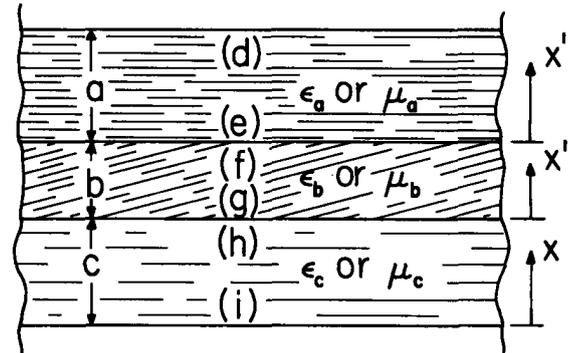


Fig. 2.20.1. Convention used to denote surface variables.

Problems for Chapter 2

For Section 2.3:

Prob. 2.3.1 Perfectly conducting plane parallel plates are shorted at  $z = 0$  and driven by a distributed current source at  $z = -\ell$ , as shown in Fig. P2.3.1.

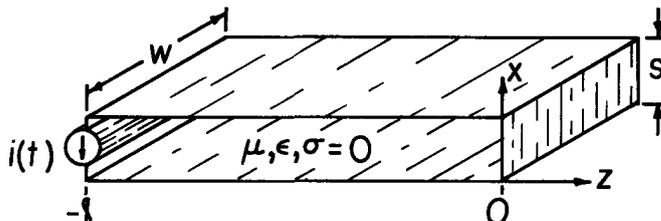


Fig. P2.3.1

- (a) Apply the normalization of Eq. 4b to Maxwell's equations used to represent the fields between the plates. There is no material between the plates, so magnetization, polarization and conduction between the plates are ignorable.
- (b) Simplify these equations by assuming that  $\vec{E} = \vec{E}_x(z, t)\hat{i}_x$  and  $\vec{H} = \vec{H}_y(z, t)\hat{i}_y$ .
- (c) The driving current is  $i(t) = \text{Re } I_0 \exp j\omega t$ . Find  $\vec{E}_x$ ,  $\vec{H}_y$ , the surface current and surface charge on the lower plate to second order.
- (d) Convert the results of (c) to dimensional expressions.
- (e) Solve for the exact fields and expand in  $\beta$  to check the results of (d).

Prob. 2.3.2 The parallel plates of Prob. 2.3.1 are now driven along their left edges by a voltage source  $v(t)$ . They are open along their right edges. Carry out the steps analogous to those of Prob. 2.3.1. A normalization that makes the EQS limit the zero order approximation is appropriate.

Prob. 2.3.3 Perfectly conducting plane parallel electrodes in the planes  $x = a$  and  $x = 0$  "sandwich" and make electrical contact with a layer of material having conductivity  $\sigma$  and thickness  $a$ . These plates are driven along their edges so that the surface current is  $\text{Re } K \exp(j\omega t)\hat{i}_z$  in the lower plate at  $z = -\ell$  and the negative of this in the upper plate. The edges of the plates at  $z = 0$  are "open-circuit." In the conductor, fields take the form  $\vec{E}_x(z, t)$ ,  $\vec{H}_y(z, t)$ .

- (a) Show that all of Maxwell's equations are satisfied if

$$\frac{d^2 \hat{H}_y}{dz^2} + k^2 \hat{H}_y = 0; \quad k \equiv \sqrt{\omega^2 \mu_0 \epsilon_0 - j\omega \mu_0 \sigma}; \quad \hat{E}_x = \frac{-1}{(\sigma + j\omega \epsilon_0)} \frac{d\hat{H}_y}{dz}$$

- (b) Show that

$$\vec{H}_y = \text{Re } \hat{K} \frac{e^{-jkz} - e^{jkz}}{e^{jkl} - e^{-jkl}} e^{j\omega t}; \quad \vec{E}_x = \frac{\text{Re } \hat{K} jk (e^{-jkz} + e^{jkz}) e^{j\omega t}}{(\sigma + j\omega \epsilon_0) (e^{jkl} - e^{-jkl})}$$

- (c) In Fig. 2.3.1,  $\tau \rightarrow 1/\omega$  and provided  $\tau_e \neq \tau_m$ , there are two possibilities:

- (i)  $\omega \tau_{em} \ll 1$  and  $\omega \tau_m \ll 1$ . Show that in this case  $k\ell \ll 1$  and

$$\vec{E}_x \rightarrow \text{Re } \frac{\hat{K} e^{j\omega t}}{(\sigma + j\omega \epsilon_0) \ell}$$

so that the system is equivalent to a capacitor shorted by a resistor (what values?).

- (ii)  $\omega \tau_{em} \ll 1$ ,  $\omega \tau_e \ll 1$ . Show that in this case  $k \rightarrow (-1 + j)/\delta_m$ , where the skin depth

$\delta_m \equiv \sqrt{2/\omega\mu\sigma}$ , and that  $H_y$  is the superposition of "skin-effect" waves decaying in the direction of phase propagation.

(d) Now, consider the EQS model from the outset. Under what conditions are the laws (Eqs. 23a - 27a) valid? Show that the solution for  $E_x$  is consistent with part (c).

(e) Consider the magnetoquasistatic laws (Eqs. 23b - 27b) from the outset and show that the result is consistent with part (c). For what conditions are these laws valid?

**Prob. 2.3.4** Given the EQS laws, Eqs. 23a - 25a, together with conduction and polarization constitutive laws and the material motions,  $\vec{E}$ ,  $\vec{P}$  and  $\rho_f$  can be determined. This is generally possible because the constitutive laws do not typically involve  $\vec{H}$ . Then, if  $\vec{H}$  is required, Eqs. 26a and 26b, together with a magnetization constitutive law, can be used. It is clear that these relations uniquely define  $\vec{H}$ , because they stipulate both  $\nabla \times \vec{H}$  and  $\nabla \cdot \vec{H}$ . Consider now the analogous question of uniquely determining  $\vec{E}$  in an MQS system. In such a system the conduction and magnetization constitutive laws respectively take the form

$$\vec{J}_f = \sigma(\vec{r}, t)(\vec{E} + \vec{v} \times \mu_0 \vec{H}) \quad ; \quad \vec{M} = \vec{M}(\vec{H}, \vec{v})$$

and Eqs. 23b - 25b together with a knowledge of the material motion can be used to find  $\vec{H}$  and  $\vec{M}$ . Show that  $\vec{E}$  is then uniquely specified and that recourse to Gauss' Law is made only to make an "after the fact" evaluation of the charge density.

For Section 2.4:

**Prob. 2.4.1** A material suffers a rigid-body rotation about the z axis with constant angular velocity  $\Omega$ . The particle at the position  $(r_0, \theta_0)$  when  $t = 0$  is found at

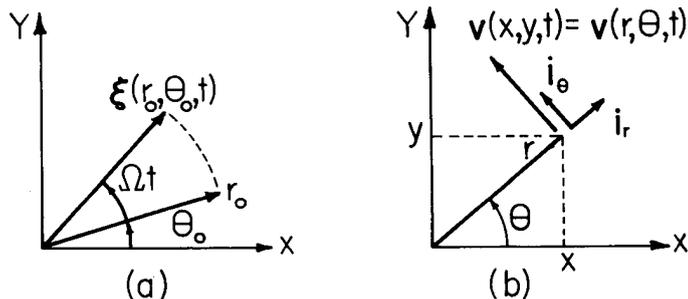
$$\vec{\xi}(r_0, \theta_0, t) = r_0 \cos(\Omega t + \theta_0) \vec{i}_x + r_0 \sin(\Omega t + \theta_0) \vec{i}_y$$

at a subsequent time  $t$ . This Lagrangian description is pictured in Fig. P2.4.1. Use Eqs. 2.4.1 and 2.4.2 to show that the velocity and acceleration are respectively

$$\vec{v} = r_0 \Omega [-\sin(\Omega t + \theta_0) \vec{i}_x + \cos(\Omega t + \theta_0) \vec{i}_y]$$

$$\vec{a} = -\Omega^2 \vec{\xi}$$

Fig. P2.4.1. Specific example in which rigid-body steady rotation is represented in (a) Lagrangian coordinates and (b) Eulerian coordinates.



**Prob. 2.4.2** One incentive for using an Eulerian representation is that motions which are time dependent in Lagrangian coordinates can become independent of time. To illustrate, consider the alternative representation of the rigid body rotation of Prob. 2.4.1.

The material velocity at a given point  $(r, \theta)$  or  $(x, y)$  is

$$\vec{v} = \vec{i}_\theta \Omega r = \Omega(-r \sin \theta \vec{i}_x + r \cos \theta \vec{i}_y) = \Omega(-y \vec{i}_x + x \vec{i}_y)$$

i.e., the velocity is independent of time. Clearly the acceleration is not obtained by taking the partial derivative with respect to time, as might be suggested by the misuse of Eq. 2.4.2. Use Eq. 2.4.4 to find  $\vec{a}$  and compare to the result of Prob. 2.4.1.

For Section 2.5:

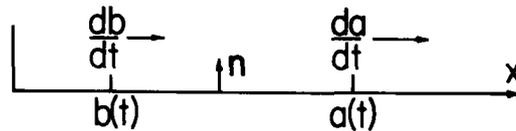
Prob. 2.5.1 A scalar function takes the traveling-wave form  $\phi = \text{Re} \hat{\phi}(x,y) \exp^{j(\omega t - kz)}$  in the frame of reference  $(\vec{r}, t)$ . The primed frame moves in the  $z$  direction relative to the unprimed frame with the velocity  $U$ . Use the convective derivative to find the rate of change of  $\phi$  for an observer moving with the velocity  $U \hat{i}_z$ . Compute this same time rate of change by expressing  $\phi = \phi(x', y', z', t')$  and finding  $\partial/\partial t'$ . Use these results to deduce the transformation  $\omega' = \omega - kU$ . If  $\omega' = 0$ ,  $\omega = kU$ . Explain in physical terms.

Prob. 2.5.2 A vector function  $\vec{A}(x,y,z,t)$  can also be evaluated as  $\vec{A}(x',y',z',t')$  where the prime coordinates are related to the unprimed ones by Eq. 2.5.1. Show that Eq. 2.5.2b holds.

For Section 2.6:

Prob. 2.6.1 The one-dimensional form of Leibnitz' rule pertains to taking an integral between endpoints (b) and (a) which are themselves a function of time, as sketched in Fig. P2.6.1.

Fig. P2.6.1. One-dimensional form of Leibnitz' rule specifies how derivative can be taken of the integral between time-varying endpoints.



Define  $\vec{A} = f(x,t) \hat{i}_x$  and use Eq. 2.6.4 with a suitable surface to show that, for the one-dimensional case, Leibnitz' rule becomes

$$\frac{d}{dt} \int_{b(t)}^{a(t)} f(x,t) dx = \int_b^a \frac{\partial f}{\partial t} dx + f(a,t) \frac{da}{dt} - f(b,t) \frac{db}{dt}$$

Prob. 2.6.2 The following steps lead to a derivation of the generalized Leibnitz rule, Eq. 2.6.4 where  $S$  is pictured as  $S_2$ , and  $S_1$  at the times  $t + \Delta t$  and  $t$ , respectively. The vector function  $\vec{A}$  depends on both space and time. However, for convenience, the spatial dependence is not explicitly indicated in the following. By definition:

$$\frac{d}{dt} \int_S \vec{A} \cdot \vec{n} da = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left( \int_{S_2} \vec{A}(t+\Delta t) \cdot \vec{n} da - \int_{S_1} \vec{A}(t) \cdot \vec{n} da \right) \quad (1)$$

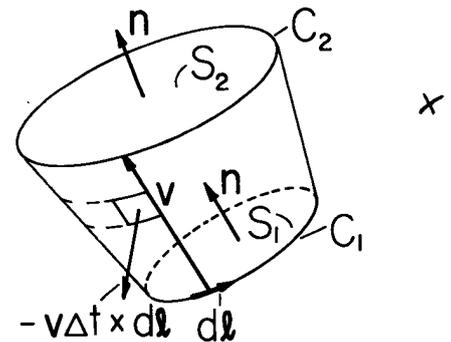
so the first integral in brackets on the right must be evaluated to first order in  $\Delta t$ . To that end,

(a) Apply Gauss theorem to the volume  $V$  swept out by  $S$  during the time  $\Delta t$ . Note that  $\vec{n}$  is the normal to the open surface  $S$  and show that to first order in  $\Delta t$ ,

$$\int_V \nabla \cdot \vec{A} dV = \int_{S_2} \vec{A}(t) \cdot \vec{n} da - \int_{S_1} \vec{A}(t) \cdot \vec{n} da - \Delta t \oint_{C_1} \vec{A} \cdot \vec{v} \times d\vec{\ell} \quad (2)$$

(b) Argue that also to first order in  $\Delta t$ ,

Fig. P2.6.2



$$\int_{S_2} \vec{A}(t+\Delta t) \cdot \vec{n} da \approx \int_{S_2} \vec{A}(t) \cdot \vec{n} da + \int_{S_1} \frac{\partial \vec{A}}{\partial t}(t) \Delta t \cdot \vec{n} da + \dots \quad (3)$$

(c) Finally, show that the volume element  $dV$ , called for in evaluating the left side of Eq. 2, is  $dV = \Delta t \vec{v} \cdot \vec{n} da$ .

(d) Combine these results to evaluate the right-hand side of Eq. 1 and deduce Eq. 2.6.4.

**Prob. 2.6.3** It is sometimes necessary to evaluate the time rate of change of a line integral of a vector variable having time-varying end points. The problem is to evaluate the derivative

$$\frac{d}{dt} \int_{\vec{a}(t)}^{\vec{b}(t)} \vec{A} \cdot d\vec{\ell} = \lim_{\Delta t \rightarrow 0} \left[ \frac{\int_{\vec{a}(t+\Delta t)}^{\vec{b}(t+\Delta t)} \vec{A}(t+\Delta t) \cdot d\vec{\ell} - \int_{\vec{a}(t)}^{\vec{b}(t)} \vec{A}(t) \cdot d\vec{\ell}}{\Delta t} \right]$$

Here  $\vec{a}$  and  $\vec{b}$  denote time-dependent vector positions in space. What is meant by the line integration is indicated by Fig. P2.6.3.

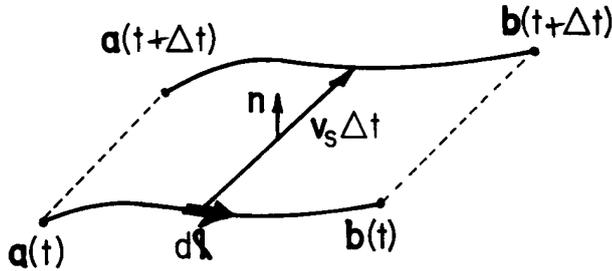


Fig. P2.6.3. Time-varying contour of line integration.

The contour of integration at the time  $t$  is instantaneously sketched. At that instant each point on the contour has a velocity  $\vec{v}_s$  so that in a time  $\Delta t$  the contour has moved by an amount  $\vec{v}_s \Delta t$ . By definition, the velocity of the end point is  $\vec{v}_s$  evaluated at the end point.

The theorem to be derived shows how the integration can be carried out after the time derivative has been taken. Thus it is analogous to the generalized Leibnitz rule for differentiation of a surface integral having time-varying geometry. The desired theorem states that

$$\frac{d}{dt} \int_{\vec{a}(t)}^{\vec{b}(t)} \vec{A} \cdot d\vec{\ell} = \int_{\vec{a}(t)}^{\vec{b}(t)} \frac{\partial \vec{A}}{\partial t} \cdot d\vec{\ell} + \vec{A}(\vec{b}, t) \cdot \vec{v}_s(\vec{b}, t) - \vec{A}(\vec{a}, t) \cdot \vec{v}_s(\vec{a}, t) + \int_{\vec{a}}^{\vec{b}} (\nabla \times \vec{A}) \times \vec{v}_s \cdot d\vec{\ell}$$

Show that this rule can be derived following steps motivated by those used in the derivation of the generalized Leibnitz rule for a time-varying surface integration.

**For Section 2.8:**

**Prob. 2.8.1** To illustrate how the steady-state motion of dipoles results in a  $\vec{J}_p$  and hence an induced magnetic field, consider a slab of material extending to infinity in the  $y$  and  $z$  directions between infinitely permeable surfaces at  $x = \pm a$ . The slab has a thickness  $2a$ , moves in the  $y$  direction with uniform velocity  $U$  and supports the polarization  $\vec{P} = -(\rho_0 a / \pi) \sin(\pi x / a) \vec{i}_x$ , where  $\rho_0$  is a given constant. Fields are in the steady state and there is no free current density.

- Observe that Ampere's law, Eq. 2.2.2, and the boundary conditions are satisfied by making  $\vec{H} = \vec{P} \times \vec{v}$ . What is  $\vec{H}$ ?
- Compute  $\vec{J}_p$  and then use Ampere's law to find  $\vec{H}$  in much the same way as if  $\vec{J}_p$  were a free current density.
- Find  $\rho_p$  and show that in this case  $\vec{J}_p$  is simply the result of polarization charge in motion

**For Section 2.9:**

**Prob. 2.9.1** To someone not appreciating the importance of keeping field transformations consistent with the fundamental laws, it might appear that Faraday's law written in the Chu formulation (Eq. 2.2.1) would imply that a magnetized and conducting material set into motion would automatically support an electric field that would drive a free current density. In fact, there is an  $\vec{E}$ , but no  $\vec{J}_f$ . Consider as a specific case a magnetized slab, having  $\vec{M} = -(\rho_0 a / \pi \mu_0) \sin(\pi x / a) \vec{i}_x$ , extending to infinity in the  $y$  and  $z$  directions, having boundaries at  $x = \pm a$  in the  $x$  direction and suffering a uniform  $y$ -

Prob. 2.9.1 (continued)

directed translation with velocity  $U$ . Perfectly conducting walls bound the slab at  $x = \pm a$ . Steady state conditions prevail.

- (a) Find the  $\vec{H}$  induced by the given magnetization.
- (b) Use Faraday's law to deduce  $\vec{E}$ .
- (c) Now, if the material also has a conductivity  $\sigma$ , so that an observer at rest in the conductor can apply Ohm's law in the form  $\vec{J}_f' = \sigma \vec{E}'$ , because  $\vec{J}_f = \vec{J}_f'$  but  $\vec{E}' = \vec{E} + \vec{v} \times \mu_0 \vec{H}$  (Eqs. 2.5.11 and 2.5.12),  $\vec{J}_f = \sigma(\vec{E} + \vec{v} \times \mu_0 \vec{H})$ . Show that in fact  $\vec{J}_f = 0$ .

For Section 2.11:

Prob. 2.11.1 A plane parallel capacitor with electrodes at potentials  $v_1$  and  $v_2$  is used to impose a field on a third electrode that is grounded and free to move either longitudinally or transversely with displacements  $(\xi_1, \xi_2)$ . The electrodes, shown in Fig. P2.11.1, have depth  $d$  into paper. Ignore fringing fields and find the capacitance matrix relating the charges  $(q_1, q_2)$  to the voltages  $(v_1, v_2)$ .

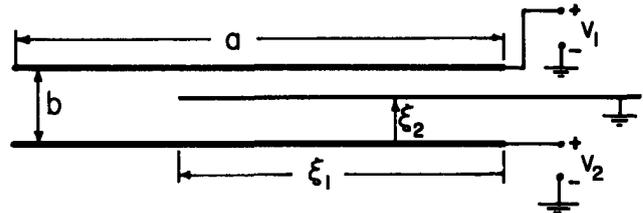


Fig. P2.11.1

For Section 2.12:

Prob. 2.12.1 A pair of perfectly conducting coaxial one-turn coils have the shape of circular cylinders of radius  $a$  and  $\xi$ , each with a length  $d \gg a$ . Currents  $i_1$  and  $i_2$  are fed to the coils through parallel electrodes having a spacing that is negligible compared to other dimensions of interest. Determine the inductance matrix, Eq. 2.12.5, relating  $(\lambda_1, \lambda_2)$  to  $(i_1, i_2)$ .

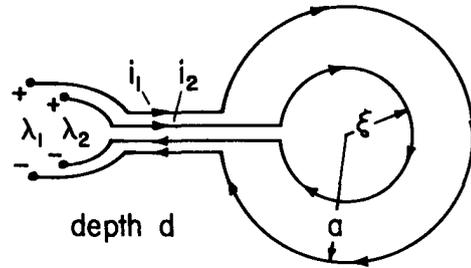


Fig. P2.12.1

For Section 2.13:

Prob. 2.13.1 For the system of Prob. 2.11.1, find the total coenergy storage  $w'(v_1, v_2, \xi_1, \xi_2)$  by integrating Eq. 2.13.10.

Prob. 2.13.2 The dielectric slab shown in Fig. P2.13.2 is composed of material having the constitutive law  $\vec{D} = \epsilon_0 \vec{E} + \vec{E}/\alpha_1 \sqrt{\alpha_2^2 + E^2}$ . The slab has depth  $d$  into the paper. Under the assumption that  $\rho_f = 0$  in the dielectric and that its edges remain well removed from the fringing fields, find the dependence of the coenergy on  $(v, \xi)$ .

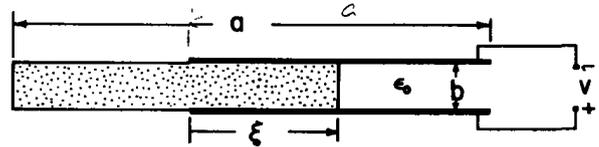


Fig. P2.13.2

For Section 2.14:

Prob. 2.14.1 For the system described in Prob. 2.12.1,

- (a) Find the energy,  $w = w(\lambda_1, \lambda_2, \xi)$ ,
- (b) the coenergy  $w' = w'(i_1, i_2, \xi)$ .

For Section 2.15:

Prob. 2.15.1 Show that the Fourier coefficients given by Eq. 2.15.8 follow from the procedure outlined in the paragraph following Eq. 2.15.7.

**Prob. 2.15.2** A function  $\phi(z,t)$  is a square-wave function of  $z$  with magnitude  $V_0(t)$ . That is,  $\phi = V_0(t)$ ,  $-\ell/4 < z < \ell/4$  and  $\phi = -V_0(t)$ ,  $\ell/4 < z < 3\ell/4$ . Show that the Fourier coefficients are

$$\tilde{\phi}_m = 0, \quad m \text{ even and } \tilde{\phi}_m = 4V_0(t) \sin\left(\frac{k_m \ell}{4}\right) / (k_m \ell), \quad m \text{ odd}$$

**Prob. 2.15.3** A function  $\phi(z,t)$  is zero except in the interval  $-\ell/2 < z < \ell/2$ , where it is  $V_0(t)$ . Show that its Fourier transform is  $\tilde{\phi}(k,t) = \ell V_0(t) \sin(k\ell/2) / (k\ell/2)$ .

**Prob. 2.15.4** Carry out the spatial average of the product of two Fourier series, as called for in completing Eq. 2.15.17.

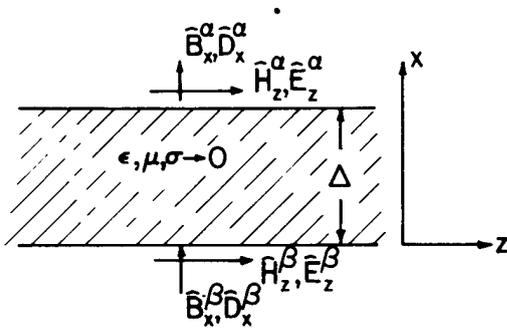
**For Section 2.16:**

**Prob. 2.16.1** Start with Eq. 2.16.14 and the relation between potential and flux, Eq. 2.16.5 and deduce the transfer relations of Table 2.16.1 for a planar layer.

**Prob. 2.16.2** Start with Eqs. 2.16.20, 2.16.21 and 2.16.25 and deduce the transfer relations of Table 2.16.2. Use the properties of the Bessel functions as  $r \rightarrow 0$  and  $r \rightarrow \infty$  to deduce the limiting cases of Eqs. c and d.

**Prob. 2.16.3** Start with Eq. 2.16.36 and deduce the transfer relations of Table 2.16.3. Evaluate the appropriate limits to arrive at Eqs. c and d.

**Prob. 2.16.4** A region of free space is bounded by fictitious parallel planes at  $x = \Delta$  and  $x = 0$ , as shown in Fig. P2.16.4.



Fields take the form

$$\vec{E} = \text{Re } \hat{E}(x) e^{j(\omega t - kz)}; \\ \vec{H} = \text{Re } \hat{H}(x) e^{j(\omega t - kz)}$$

so that there is no dependence on  $y$  and the time dependence is explicitly taken as  $\exp(j\omega t)$ . The objective is to obtain transfer relations between tangential and perpendicular field components at the  $\alpha$  and  $\beta$  surfaces without the quasistatic approximation.

Fig. P2.16.4

- With fields taking the given form, show that all components of  $\vec{E}$  and  $\vec{H}$  can be written in terms of the axial components of  $\hat{E}_z$  and  $\hat{H}_z$ . (This follows from Ampere's and Faraday's laws). Also show that  $\hat{E}_z$  and  $\hat{H}_z$  satisfy the wave equation.
- Write  $\hat{E}_z$  and  $\hat{H}_z$  in terms of the amplitudes  $\hat{E}_z^\alpha$ ,  $\hat{E}_z^\beta$  and  $\hat{H}_z^\alpha$ ,  $\hat{H}_z^\beta$  defined as these quantities evaluated on the respective surfaces.
- Show that the transfer relation for the layer is

$$\begin{bmatrix} \epsilon \hat{E}_x^\alpha \\ \epsilon \hat{E}_x^\beta \\ \mu \hat{H}_x^\alpha \\ \mu \hat{H}_x^\beta \end{bmatrix} = \begin{bmatrix} j \frac{\epsilon k}{\gamma} \coth(\gamma \Delta) & -j \frac{\epsilon k}{\gamma} \frac{1}{\sinh(\gamma \Delta)} & 0 & 0 \\ j \frac{\epsilon k}{\gamma} \frac{1}{\sinh(\gamma \Delta)} & -j \frac{\epsilon k}{\gamma} \coth(\gamma \Delta) & 0 & 0 \\ 0 & 0 & j \frac{\mu k}{\gamma} \coth(\gamma \Delta) & -j \frac{\mu k}{\gamma} \frac{1}{\sinh(\gamma \Delta)} \\ 0 & 0 & j \frac{\mu k}{\gamma} \frac{1}{\sinh(\gamma \Delta)} & -j \frac{\mu k}{\gamma} \coth(\gamma \Delta) \end{bmatrix} \begin{bmatrix} \hat{E}_z^\alpha \\ \hat{E}_z^\beta \\ \hat{H}_z^\alpha \\ \hat{H}_z^\beta \end{bmatrix}$$

where the other components of  $\vec{E}$  and  $\vec{H}$  are found from

$$\hat{H}_y = \frac{\omega \epsilon}{k} \hat{E}_x, \quad \hat{E}_y = \frac{-\omega \mu}{k} \hat{H}_x, \quad \text{and } \gamma \equiv \sqrt{k^2 - \omega^2 \mu \epsilon}$$

Prob. 2.16.4 (continued)

- (d) Show that in the quasistatic limit the relation reduces to the electroquasistatic and magnetoquasistatic transfer relations of Table 2.16.1 with appropriate identification of variables for the electric and magnetic relations.
- (e) To make a connection with TE and TM modes in a plane parallel plate waveguide, let the  $\alpha$  and  $\beta$  surfaces be perfectly conducting electrodes. Thus, the boundary conditions are

$$\hat{E}_z^\alpha = \hat{E}_z^\beta = 0 \quad \text{TM modes}$$

$$\hat{B}_x^\alpha = \hat{B}_x^\beta = 0 \quad \text{TE modes}$$

where the transverse magnetic and transverse electric modes can be separated because of the form taken by the transfer relations. Use these relations to argue that fields within that satisfy these homogeneous boundary conditions must also satisfy the dispersion equations

$$\omega^2 \mu \epsilon = k^2 + \left(\frac{n\pi}{\Delta}\right)^2 \quad ; \quad n = 1, 2, 3 \dots$$

Prob. 2.16.5 A planar region, shown in Table 2.16.1, is filled by an inhomogeneous dielectric, with a permittivity that depends on  $x$ :

$$\epsilon(x) = \epsilon_\beta \exp 2\eta x, \quad \eta \equiv \ln(\epsilon_\alpha / \epsilon_\beta) / 2\Delta$$

The free charge density is zero.

- (a) Show that the potential distribution is

$$\tilde{\phi} = \tilde{\phi}^\alpha e^{-\eta(x-\Delta)} \frac{\sinh \lambda x}{\sinh \lambda \Delta} - \tilde{\phi}^\beta e^{-\eta x} \frac{\sinh \lambda(x-\Delta)}{\sinh \lambda \Delta}$$

where

$$\lambda \equiv \sqrt{k^2 + \eta^2}$$

- (b) Show that the transfer relations are

$$\begin{bmatrix} \tilde{D}_x^\alpha \\ \tilde{D}_x^\beta \end{bmatrix} = \epsilon_\beta \lambda \begin{bmatrix} \left(\frac{\eta}{\lambda} - \coth \lambda \Delta\right) e^{\eta 2\Delta} & \frac{e^{\eta \Delta}}{\sinh \lambda \Delta} \\ \frac{-e^{\eta \Delta}}{\sinh \lambda \Delta} & \frac{\eta}{\lambda} + \coth \lambda \Delta \end{bmatrix} \begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix}$$

Prob. 2.16.6 A planar region, shown in Table 2.16.1, is filled by an anisotropic material having the constitutive law  $D_i = \epsilon_{ij} E_j$ . The permittivity coefficients are uniform throughout. Determine the transfer relations in the form of Eqs. (a) of Table 2.16.1.

For Section 2.17:

Prob. 2.17.1 In developing conditions on coefficients in the transfer relations with the potentials expressed as functions of the "flux" variables, it is natural to use the energy function as exemplified in this section. The coenergy function is more convenient in dealing with the potentials as the independent variables. For the transfer relations of Sec. 2.16 written in the form

$$\begin{bmatrix} \tilde{D}_n^\alpha \\ \tilde{D}_n^\beta \end{bmatrix} = \begin{bmatrix} -B_{11} & B_{12} \\ -B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} \tilde{\phi}^\alpha \\ \tilde{\phi}^\beta \end{bmatrix}$$

derive conditions analogous to those of Eqs. 2.17.10 and 2.17.12.

Prob. 2.17.2 Use the reciprocity condition, Eq. 2.17.10 to show

$$kx[H_m(jkx) J'_m(jkx) - J_m(jkx) H'_m(jkx)] = \text{constant}$$

Use Eqs. 2.16.22 and 2.16.23 to establish that the constant is  $2/\pi$ . Thus, the numerators of the functions  $g_m$  and  $G_m$  in the cases  $k \neq 0$  of Table 2.16.2 are considerably simplified from what is obtained by direct evaluation.

Prob. 2.17.3 With Eq. 2.17.7, it is assumed that the excitations on the  $\alpha$  and  $\beta$  surfaces are in spatial phase, and that the  $A_{ij}$  are real. By allowing the excitations to have arbitrary phase, it is possible to learn more about these coefficients. In general, the expression replacing Eq. 2.17.7 in Cartesian or cylindrical geometry is

$$\delta w = \frac{1}{2} C \operatorname{Re}[-a^{\alpha\tilde{\alpha}} \tilde{\Phi}_r \delta(\tilde{D}_n^\alpha)^* + a^{\beta\tilde{\beta}} \tilde{\Phi}_i \delta(\tilde{D}_n^\beta)^*]$$

Because  $\operatorname{Re} \tilde{u} \delta \tilde{V} = \tilde{u}_r \delta \tilde{V}_r + \tilde{u}_i \delta \tilde{V}_i$ , this expression becomes

$$\delta w = \frac{1}{2} C [-a^{\alpha\tilde{\alpha}} \tilde{\Phi}_r \delta \tilde{D}_{nr}^\alpha - a^{\alpha\tilde{\alpha}} \tilde{\Phi}_i \delta \tilde{D}_{ni}^\alpha + a^{\beta\tilde{\beta}} \tilde{\Phi}_r \delta \tilde{D}_{nr}^\beta + a^{\beta\tilde{\beta}} \tilde{\Phi}_i \delta \tilde{D}_{ni}^\beta]$$

That is, the real and imaginary parts of the excitations on each surface are independent variables. Use the fact that the energy is a state variable:  $w = w(\tilde{D}_{nr}^\alpha, \tilde{D}_{ni}^\alpha, \tilde{D}_{nr}^\beta, \tilde{D}_{ni}^\beta)$  and show that

$$-a^{\alpha\tilde{\alpha}} \tilde{\Phi}_r = \frac{\partial w}{\partial \tilde{D}_r^\alpha} ; \quad -a^{\alpha\tilde{\alpha}} \tilde{\Phi}_i = \frac{\partial w}{\partial \tilde{D}_i^\alpha} ; \quad a^{\beta\tilde{\beta}} \tilde{\Phi}_r = \frac{\partial w}{\partial \tilde{D}_r^\beta} ; \quad a^{\beta\tilde{\beta}} \tilde{\Phi}_i = \frac{\partial w}{\partial \tilde{D}_i^\beta}$$

From these relations, derive reciprocity relations between the derivatives of  $(\tilde{\Phi}_r^\alpha, \tilde{\Phi}_i^\alpha, \tilde{\Phi}_r^\beta, \tilde{\Phi}_i^\beta)$  with respect to  $(\tilde{D}_{nr}^\alpha, \tilde{D}_{ni}^\alpha, \tilde{D}_{nr}^\beta, \tilde{D}_{ni}^\beta)$ . Assume that the  $A_{ij}$  can have real and imaginary parts, and show from these reciprocity relations that  $A_{11}$  and  $A_{22}$  must be real and that  $a^{\alpha A_{12}} = a^{\beta A_{21}^*}$ .

Prob. 2.17.4 Use the results of Prob. 2.17.1 to show that the transfer relations of Prob. 2.16.5 satisfy the reciprocity relations.

For Section 2.18:

Prob. 2.18.1 For the axisymmetric cylindrical case of Table 2.18.1, show that Eq. (h) follows from Eq. (g) and that Eq. 2.18.2 can be used to deduce the expression for the total flux, Eq. (i).

Prob. 2.18.2 Show that Eq. (k) of Table 2.18.1 follows from Eq. (j).

For Section 2.19:

Prob. 2.19.1 Derive Eqs. (e) and (f) of Table 2.19.1.