


Unit 4: The Dot Product

1. Lecture 1.040

The Dot Product

Physical Motivation
"work = force x distance"




$$w = [|\vec{F} \cos \theta| |\vec{d}|]$$

$$= |\vec{F}| |\vec{d}| \cos \theta$$

"General" Definition
 $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$

Geometrically:



Use the Law of Cosines to eliminate $\cos \theta$

$$|\vec{A} - \vec{B}|^2 = |\vec{A}|^2 + |\vec{B}|^2 - 2|\vec{A}| |\vec{B}| \cos \theta$$

$$= |\vec{A}|^2 + |\vec{B}|^2 - 2\vec{A} \cdot \vec{B}$$

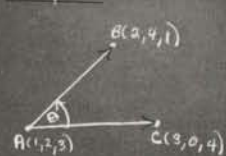
Therefore,
 $\vec{A} \cdot \vec{B} = \frac{|\vec{A} - \vec{B}|^2 - |\vec{A}|^2 - |\vec{B}|^2}{2}$

This result is independent of any coordinate system.

a.*

In Cartesian Coordinates,
if $\vec{A} = a_1 \vec{i} + a_2 \vec{j} + a_3 \vec{k}$
and $\vec{B} = b_1 \vec{i} + b_2 \vec{j} + b_3 \vec{k}$
then
 $\vec{A} \cdot \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Example 1



Find the value of θ .

$$\vec{AB} = \vec{i} + 2\vec{j} - 2\vec{k}$$


$$\vec{AC} = 2\vec{i} - 2\vec{j} + \vec{k}$$

$$|\vec{AB}| = \sqrt{1^2 + 2^2 + (-2)^2} = 3$$

$$|\vec{AC}| = \sqrt{2^2 + (-2)^2 + 1^2} = 3$$

$$\vec{AB} \cdot \vec{AC} = 2 - 4 - 2 = -4$$


$$\vec{AB} \cdot \vec{AC} = |\vec{AB}| |\vec{AC}| \cos \theta$$

$$\therefore \cos \theta = \frac{-4}{9}$$


$$\theta = 180^\circ - \cos^{-1} \frac{4}{9}$$

b.

Example 2
Projections



Let $\vec{u}_B = \frac{\vec{B}}{|\vec{B}|}$
(the unit vector in direction of \vec{B})
Then
 $|\vec{A}| \cos \theta = |\vec{A}| |\vec{u}_B| \cos \theta$
 $= \vec{A} \cdot \vec{u}_B$

Summary
 $\frac{\vec{A} \cdot \vec{B}}{|\vec{B}|}$ is the projection of \vec{A} in the direction of \vec{B}

Example 3
Suppose that both \vec{A} and \vec{B} are unit vectors
Then
 $\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta$
 $= \cos \theta$

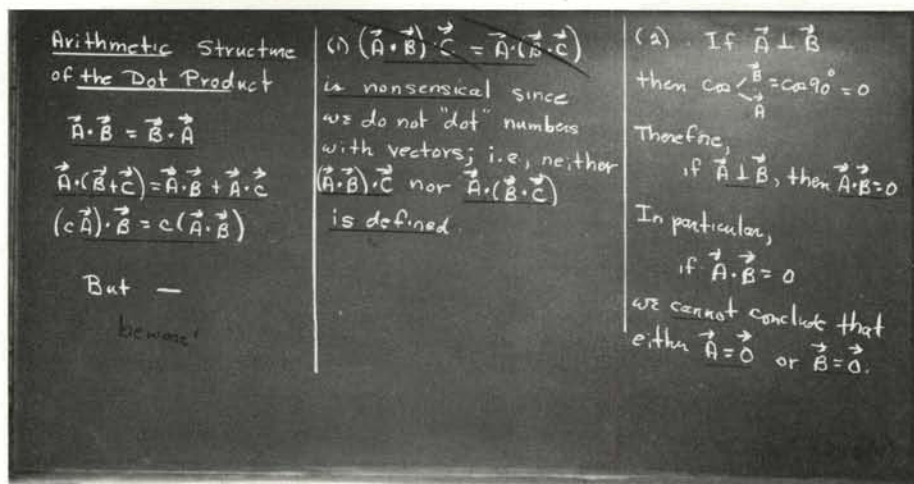
Therefore, given any vector \vec{A} , we let $\vec{u}_A = \frac{\vec{A}}{|\vec{A}|}$
Then
 $\vec{u}_A \cdot \vec{i} = \cos \theta = \cos \alpha$
 $\vec{u}_A \cdot \vec{j} = \cos \theta = \cos \beta$
 $\vec{u}_A \cdot \vec{k} = \cos \theta = \cos \gamma$

Directional Cosines

c.

*See note on next page.

Lecture 1.040 continued



d.

Note: Last equation on first board (a) should read:

$$\vec{A} \cdot \vec{B} = \frac{|\vec{A}|^2 + |\vec{B}|^2 - |\vec{A} - \vec{B}|^2}{2}$$

2. Read Thomas 12.6.

3. Exercises:

1.4.1(L)

- a. Let $\vec{A} = \vec{i} + \vec{j} + \vec{k}$ and let $\vec{B} = 2\vec{i} + 3\vec{j} + 4\vec{k}$. Find the relationship between x , y , and z if $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$, where $\vec{C} = x\vec{i} + y\vec{j} + z\vec{k}$.
- b. Use vector arithmetic to generalize the result of part (a) by showing that if $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ and we know that $\vec{A} \neq \vec{0}$ and $\vec{B} \neq \vec{C}$, then \vec{A} is perpendicular to $\vec{B} - \vec{C}$.
- c. With \vec{A} and \vec{B} as in part (a), use the result of part (b) to interpret geometrically the set of all vectors \vec{C} for which $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$.
- d. In terms of structure, explain why the result of this exercise is not a contradiction of the cancellation law of numerical arithmetic.

1.4.2

- a. Let $P_0(2,3,4)$ be in the plane M and let $\vec{V} = 5\vec{i} + 8\vec{j} + 6\vec{k}$ be perpendicular to M . Determine the relationship between x , y , and z if $P(x,y,z)$ is to lie in the plane M . (This relation is called the equation of the plane in Cartesian coordinates.)
- b. Is $(1,1,8)$ in the plane M ? If not, where is it relative to the plane? Explain.

1.4.3

Lines are drawn from $A(1,2,3)$ to both $B(3,3,5)$ and $C(4,4,9)$. Determine the measure of $\sphericalangle BAC$ (i.e., the angle between \vec{AB} and \vec{AC}).

1.4.4(L)

Find the point at which the line through $A(2,3,4)$ perpendicular to the line $y = -3x + 2$ meets the line $y = -3x + 2$.

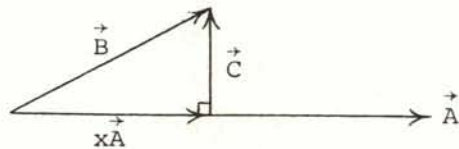
1.4.5(L)

Let $\vec{v}_1 = 3\vec{i} + 2\vec{j} + 5\vec{k}$ and let $\vec{v}_2 = 2\vec{i} + 7\vec{j} + 3\vec{k}$. Determine the value of $(3\vec{v}_1 + 4\vec{v}_2) \cdot (4\vec{v}_1 - 3\vec{v}_2)$.

Study Guide
Block 1: Vector Arithmetic
Unit 4: The Dot Product

1.4.6(L)

- a. In the diagram below, \vec{A} and \vec{B} are given vectors. Determine $x\vec{A}$ and \vec{C} in terms of \vec{A} and \vec{B} by using vector arithmetic.
- b. Do the same as in (a) but use geometry instead of arithmetic.



1.4.7

Find the length of the projection of $\vec{i} + 5\vec{j} + 6\vec{k}$ onto the vector $2\vec{i} + 6\vec{j} + 3\vec{k}$.

1.4.8(L)

Use vector methods to show that the perpendicular distance from the point $A(x_0, y_0)$ to the line $ax + by + c = 0$ is given by

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$$

1.4.9

The line L is determined by the two points in space, $(1, 2, 3)$ and $(2, 4, 5)$. Find the perpendicular distance from the point $(3, 5, 9)$ to L .

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Resource: Calculus Revisited: Multivariable Calculus
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