

Study Guide

BLOCK 2:
VECTOR CALCULUS

Pretest

- (a) Suppose $\vec{f}(t)$ denotes a vector function of the scalar variable t . How are $\vec{f}(t)$ and $\vec{f}'(t)$ related if the magnitude of $\vec{f}(t)$ is constant?

(b) Describe the direction of the acceleration of a particle if the particle moves in a circular path with constant speed.
- A particle moves in the plane in such a way that its polar equation of motion is $\vec{R} = t \vec{i} + (t^2+1)\vec{j}$.

(a) What are its normal and tangential components of acceleration at any time t ?

(b) What is the curvature of its path at any time t ?
- The polar equation for the curve C_1 is $r = \cos 2\theta$, while the curve C_2 is described by the polar equation $r = 1+\cos \theta$. Find all points at which C_1 and C_2 intersect.
- Find the area of the region enclosed between the two curves C_1 and C_2 where C_1 has the polar equation $r = \sin \theta$ and C_2 has the polar equation $r = \cos \theta$.
- A particle moves according to the polar equation $r = 1+\cos \theta$, $\theta = e^t$, where t is in seconds and r in feet. What are the \vec{u}_r and \vec{u}_θ components of acceleration of this particle at the instant $t = \ln \frac{\pi}{2}$?

Unit 1: Differentiation of Vector Functions

1. Lecture 2.010

Vector Functions of Scalar Variables


input x → machine $f(x)$ → output

Example:
Force as a function of time
 $\vec{F}(t) = e^{-t} \vec{i} + t \vec{j}$
For large t
 $\vec{F}(t) \approx t \vec{j}$

Limits Revisited

$\lim_{x \rightarrow a} f(x) = \vec{L}$
means
 $f(x)$ is "near" \vec{L} if x is "near" a

What does " \vec{A} near \vec{B} " mean?
Answer
 \vec{A} is "nearly equal" to \vec{B}
That is,
 $|\vec{A} - \vec{B}|$ is "small"
(Note, $|\vec{A} - \vec{B}|$ is a number)



a.

More rigorously,

$\lim_{x \rightarrow a} f(x) = \vec{L}$ means
given $\epsilon > 0$, can find $\delta > 0$ such that
 $0 < |x - a| < \delta \Rightarrow |f(x) - \vec{L}| < \epsilon$.


$\lim_{x \rightarrow a} f(x) = \vec{L}$ means
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 $0 < |x - a| < \delta \Rightarrow |f(x) - \vec{L}| < \epsilon$

Limit Theorems

$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$
 $\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$
 $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
 $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x)$

$|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$
 $|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$
 $|\vec{a}| = |\vec{a}| |\vec{1}|$

Caution
 $|\vec{a} \cdot \vec{b}| < |\vec{a}| |\vec{b}|$
All Previous limit theorems still valid



b.

Differentiation

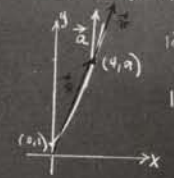
$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

All derivative formulas still valid.

Motion in a Plane
 $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$
 $\vec{R}(t) = x(t)\vec{i} + y(t)\vec{j}$

$\therefore \frac{d\vec{R}}{dt} = \frac{dx}{dt} \vec{i} + \frac{dy}{dt} \vec{j}$
 $\left| \frac{d\vec{R}}{dt} \right| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \left| \frac{ds}{dt} \right|$
= speed along curve
slope of $\frac{d\vec{R}}{dt} = \frac{dy/dx}{dx/dt} = \frac{dy}{dx}$
 $\therefore \vec{v} = \frac{d\vec{R}}{dt}, \vec{a} = \frac{d\vec{v}}{dt}$
Example:
 $y = t^3 + 1, x = t^2$
 $\vec{R} = t^2 \vec{i} + (t^3 + 1) \vec{j}$

$\therefore \vec{v} = 2t \vec{i} + 3t^2 \vec{j}$
 $\vec{a} = 2 \vec{i} + 6t \vec{j}$
At $t = 2$:
 $\vec{R} = 4\vec{i} + 9\vec{j}$
 $\vec{v} = 4\vec{i} + 12\vec{j}$
 $\vec{a} = 2\vec{i} + 12\vec{j}$

Pictorially, ($y = x^3 + 1$)

 $|\vec{v}| = \sqrt{160} = 5\sqrt{12}$
 $|\vec{a}| = \sqrt{148} = 2\sqrt{37}$

c.

Study Guide

Block 2: Vector Calculus

Unit 1: Differentiation of Vector Functions

2. Read Supplementary Notes, Chapter 3.

3. Read Thomas, Sections 14.1 and 14.2.

4. Exercises:

2.1.1(L)

- a. If \vec{c} is any constant vector, prove $\lim_{x \rightarrow a} \vec{c} = \vec{c}$.
- b. If $\lim_{x \rightarrow a} \vec{f}(x) = \vec{L}$ then $\lim_{x \rightarrow a} [\vec{c} \cdot \vec{f}(x)] = \vec{c} \cdot \vec{L}$.
- c. If \vec{f} is a differentiable function of x and if $h(x) = \vec{c} \cdot \vec{f}(x)$, then h is also a differentiable function of x and $h'(x) = \vec{c} \cdot \vec{f}'(x)$.

2.1.2

(Note: the aim of this exercise is to have you see that the product rule for differentiating a cross product is similar to the usual product rule except that the order of the factors is crucial; i.e., $\vec{f}'(t) \times \vec{g}(t) = -\vec{g}'(t) \times \vec{f}(t)$.)

Let $\vec{f}(t) = t\vec{i} + t^2\vec{j} + (2t + 1)\vec{k}$ and let $\vec{g}(t) = t^3\vec{i} + 3t\vec{j} + (t^2 + 1)\vec{k}$.

- a. Let $\vec{h}(t) = \vec{f}(t) \times \vec{g}(t)$. Compute $\vec{h}(t)$ and then differentiate this result to obtain $\vec{h}'(t)$.
- b. Find $\vec{g}'(t)$ and $\vec{f}'(t)$ and then compute $[\vec{f}(t) \times \vec{g}'(t)] + [\vec{f}'(t) \times \vec{g}(t)]$.
- c. How do the answers to a. and b. compare?
- d. Compute $[\vec{f}(t) \times \vec{g}'(t)] + [\vec{g}'(t) \times \vec{f}(t)]$.
- e. Are the answers to b. and d. the same? Why?

2.1.3

- a. Mimic the corresponding scalar procedure to prove that if \vec{R} is a differentiable function of t at $t = t_1$ then

$$\Delta R = \left(\frac{d\vec{R}}{dt} \right)_{t=t_1} \Delta t + \vec{k} \Delta t, \text{ where } \lim_{\Delta t \rightarrow 0} \vec{k} = \vec{0}.$$

(continued on next page)

2.1.3 continued

- b. Use a. to prove the following version of the chain rule:
If \vec{R} is a differentiable function of x and x is a differentiable function of u , then \vec{R} is also a differentiable function of u and
- $$\frac{d\vec{R}}{du} = \left(\frac{d\vec{R}}{dx}\right) \left(\frac{dx}{du}\right).$$
- c. Let $\vec{R} = (t + 1)\vec{i} + t^2\vec{j} + t^3\vec{k}$ and let $t = u^4$. Compute $\frac{d\vec{R}}{du}$ in two different ways, first by replacing t by u^4 in the formula for \vec{R} , and then by the chain rule. Show that the two answers thus obtained are equal.

2.1.4(L)

A particle moves along the curve C in the xy -plane where the equation of motion is given parametrically by $x = x(t)$ and $y = y(t)$, where both x and y are differentiable functions of t . Let $\vec{R}(t)$ denote the vector from the origin to the point on C which corresponds to t .

- a. Show why $\frac{d\vec{R}}{dt}$ agrees with the physical notion of speed along a curve.
- b. What can we deduce about the path if \vec{R} is 1 - 1, i.e., if $\vec{R}(t_1) = \vec{R}(t_2)$ implies that $t_1 = t_2$?

2.1.5

A particle moves along the curve according to the equation of motion $x = e^t$ and $y = e^{2t}$.

- a. Find the velocity and the acceleration of the particle as functions of t (where, of course, t denotes time).
- b. At $t = 0$, what is the velocity, speed, and acceleration of the particle?
- c. What is the slope of the path at $t = 0$?

2.1.6(L)

- a. Assume that \vec{f} is a differentiable function of t . By differentiating $\vec{f}(t) \cdot \vec{f}(t)$, show that if the magnitude of $\vec{f}(t)$ is constant then $\vec{f}(t) \cdot \vec{f}'(t) = 0$.

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Study Guide
Block 2: Vector Calculus
Unit 1: Differentiation of Vector Functions

2.1.6(L) continued

- b. Apply the result of a. to see what can be concluded about the motion of a particle if the particle moves with constant speed.
- c. Prove that if a particle moves in a circular path with constant speed, the acceleration is always along the radius drawn to the given point.

2.1.7

A particle moves so that its equation of motion in vector form is given by

$$\vec{R} = \left[\frac{\sin^{-1}t}{2} + \frac{t\sqrt{1-t^2}}{2} \right] \vec{i} + \frac{1}{2} t^2 \vec{j}, \quad 0 \leq t < 1.$$

- a. Show that the particle moves with constant speed.
- b. Compute \vec{v} and \vec{a} , and verify that $\vec{v} \cdot \vec{a} = 0$ (as it should be when the speed is constant).
- c. Since the magnitude of the speed is constant, must the magnitude of the acceleration also be constant? Explain.
- d. Find the velocity and the acceleration of the particle when $t = \frac{3}{5}$.

2.1.8(L)

- a. By writing all functions in terms of \vec{i} and \vec{j} components, show that if $\vec{F}'(t) = \vec{f}(t)$ then the set of all functions whose derivative with respect to t is f , is given by $\{\vec{F}(t) + \vec{c}; \vec{c}$ is an arbitrary vector constant $\}$.
- b. A particle moves in the xy -plane in such a way that its acceleration at any time t is given by

$$\vec{a} = (8\cos 2t)\vec{i} + (8\sin 2t)\vec{j}.$$

Moreover, at $t = 0$, both \vec{R} and \vec{v} are $\vec{0}$. Find \vec{R} as a function of t .

- c. Where is the particle at $t = \frac{\pi}{2}$?

MIT OpenCourseWare
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Resource: Calculus Revisited: Multivariable Calculus
Prof. Herbert Gross

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