Study Guide

BLOCK 2:
VECTOR CALCULUS

Pretest

1. (a) Suppose $\vec{f}(t)$ denotes a vector function of the scalar variable t. How are $\vec{f}(t)$ and $\vec{f}'(t)$ related if the magnitude of $\vec{f}(t)$ is constant?

   (b) Describe the direction of the acceleration of a particle if the particle moves in a circular path with constant speed.

2. A particle moves in the plane in such a way that its polar equation of motion is $\vec{R} = t\,\vec{i} + (t^2+1)\vec{j}$.

   (a) What are its normal and tangential components of acceleration at any time t?

   (b) What is the curvature of its path at any time t?

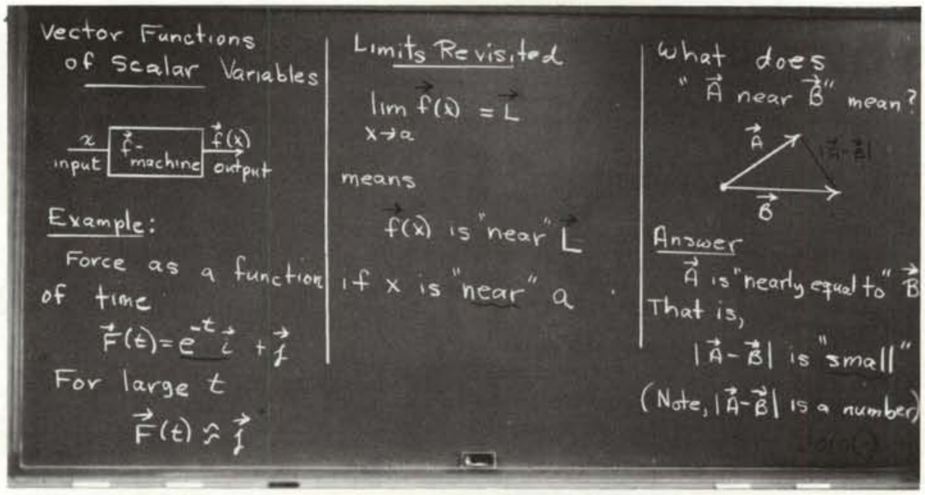
3. The polar equation for the curve C_1 is $r = \cos 2\theta$, while the curve C_2 is described by the polar equation $r = 1+\cos\theta$. Find all points at which C_1 and C_2 intersect.

4. Find the area of the region enclosed between the two curves C_1 and C_2 where C_1 has the polar equation $r = \sin\theta$ and C_2 has the polar equation $r = \cos\theta$.

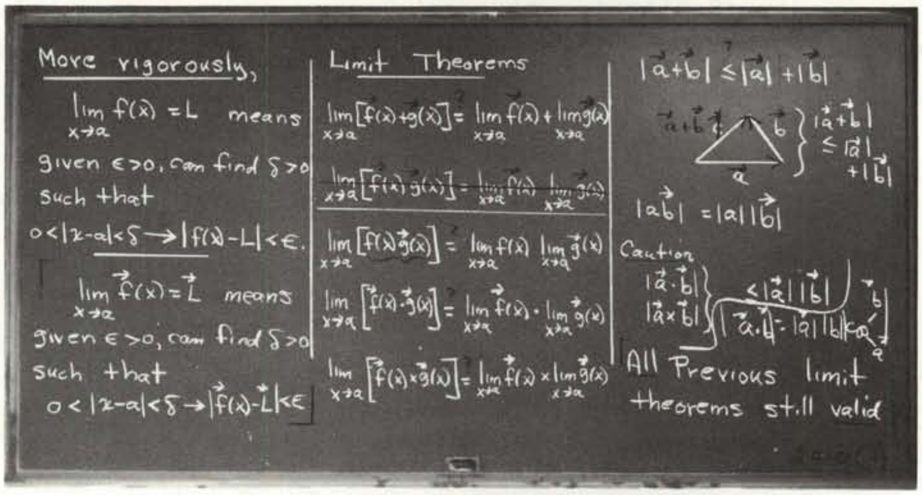
5. A particle moves according to the polar equation $r = 1+\cos\theta$, $\theta = e^t$, where t is in seconds and r in feet. What are the $\vec{u}_r$ and $\vec{u}_\theta$ components of acceleration of this particle at the instant $t = \ln\frac{\pi}{2}$?

Unit 1: Differentiation of Vector Functions

1. Lecture 2.010



a.



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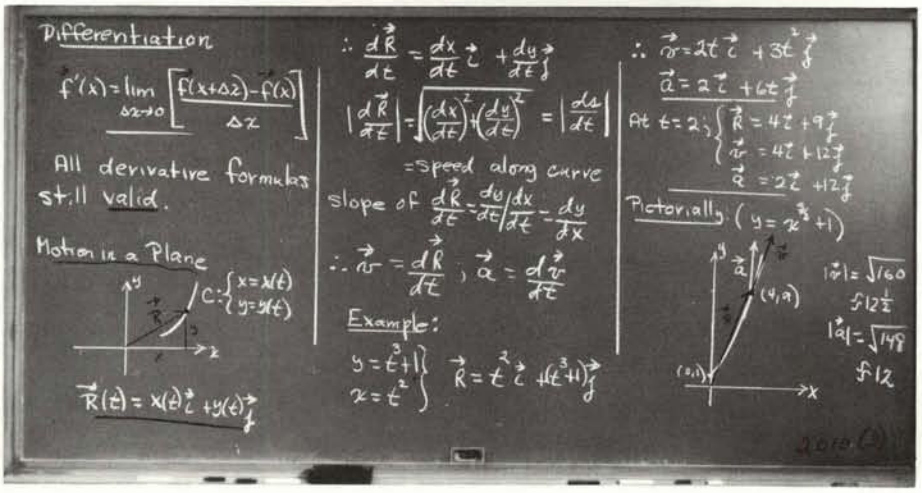

c.

2. Read Supplementary Notes, Chapter 3.

3. Read Thomas, Sections 14.1 and 14.2.

4. Exercises:

2.1.1(L)

a. If $\vec{c}$ is any constant vector, prove $\lim_{x \to a} \vec{c} = \vec{c}$.

b. If $\lim_{x \to a} \vec{f}(x) = \vec{L}$ then $\lim_{x \to a} [\vec{c} \cdot \vec{f}(x)] = \vec{c} \cdot \vec{L}$.

c. If $\vec{f}$ is a differentiable function of x and if $h(x) = \vec{c} \cdot \vec{f}(x)$, then h is also a differentiable function of x and $h'(x) = \vec{c} \cdot \vec{f}'(x)$.

2.1.2

(Note: the aim of this exercise is to have you see that the product rule for differentiating a cross product is similar to the usual product rule except that the order of the factors is crucial; i.e., $\vec{f}'(t) \times \vec{g}(t) = -\vec{g}(t) \times \vec{f}'(t)$.)

Let $\vec{f}(t) = t\vec{i} + t^2\vec{j} + (2t + 1)\vec{k}$ and let $\vec{g}(t) = t^3\vec{i} + 3t\vec{j} + (t^2 + 1)\vec{k}$.

a. Let $\vec{h}(t) = \vec{f}(t) \times \vec{g}(t)$. Compute $\vec{h}(t)$ and then differentiate this result to obtain $\vec{h}'(t)$.

b. Find $\vec{g}'(t)$ and $\vec{f}'(t)$ and then compute $[\vec{f}(t) \times \vec{g}'(t)] + [\vec{f}'(t) \times \vec{g}(t)]$.

c. How do the answers to a. and b. compare?

d. Compute $[\vec{f}(t) \times \vec{g}'(t)] + [\vec{g}(t) \times \vec{f}'(t)]$.

e. Are the answers to b. and d. the same? Why?

2.1.3

a. Mimic the corresponding scalar procedure to prove that if $\vec{R}$ is a differentiable function of t at $t = t_1$ then

$$\Delta R = \left(\frac{d\vec{R}}{dt}\right)_{t=t_1} \Delta t + \vec{k}\,\Delta t, \text{ where } \lim_{\Delta t \to 0} \vec{k} = \vec{0}.$$

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2.1.3 continued

b. Use a. to prove the following version of the chain rule: If $\vec{R}$ is a differentiable function of x and x is a differentiable function of u, then $\vec{R}$ is also a differentiable function of u and

$\frac{d\vec{R}}{du} = \left(\frac{d\vec{R}}{dx}\right)\left(\frac{dx}{du}\right)$.

c. Let $\vec{R} = (t + 1)\vec{i} + t^2\vec{j} + t^3\vec{k}$ and let $t = u^4$. Compute $\frac{d\vec{R}}{du}$ in two different ways, first by replacing t by u^4 in the formula for $\vec{R}$, and then by the chain rule. Show that the two answers thus obtained are equal.

2.1.4(L)

A particle moves along the curve C in the xy-plane where the equation of motion is given parametrically by $x = x(t)$ and $y = y(t)$, where both x and y are differentiable functions of t. Let $\vec{R}(t)$ denote the vector from the origin to the point on C which corresponds to t.

a. Show why $\frac{d\vec{R}}{dt}$ agrees with the physical notion of speed along a curve.

b. What can we deduce about the path if $\vec{R}$ is 1 - 1, i.e., if $\vec{R}(t_1) = \vec{R}(t_2)$ implies that $t_1 = t_2$?

2.1.5

A particle moves along the curve according to the equation of motion $x = e^t$ and $y = e^{2t}$.

a. Find the velocity and the acceleration of the particle as functions of t (where, of course, t denotes time).

b. At t = 0, what is the velocity, speed, and acceleration of the particle?

c. What is the slope of the path at t = 0?

2.1.6(L)

a. Assume that $\vec{f}$ is a differentiable function of t. By differentiating $\vec{f}(t) \cdot \vec{f}(t)$, show that if the magnitude of $\vec{f}(t)$ is constant then $\vec{f}(t) \cdot \vec{f}'(t) \equiv 0$.

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2.1.6(L) continued

b. Apply the result of a. to see what can be concluded about the motion of a particle if the particle moves with constant speed.

c. Prove that if a particle moves in a circular path with constant speed, the acceleration is always along the radius drawn to the given point.

2.1.7

A particle moves so that its equation of motion in vector form is given by

$$\vec{R} = \left[\frac{\sin^{-1}t}{2} + \frac{t\sqrt{1 - t^2}}{2}\right]\vec{i} + \frac{1}{2}t^2\vec{j},\ 0 \leqslant t < 1.$$

a. Show that the particle moves with constant speed.

b. Compute $\vec{v}$ and $\vec{a}$, and verify that $\vec{v} \cdot \vec{a} = 0$ (as it should be when the speed is constant).

c. Since the magnitude of the speed is constant, must the magnitude of the acceleration also be constant? Explain.

d. Find the velocity and the acceleration of the particle when $t = \frac{3}{5}$.

2.1.8(L)

a. By writing all functions in terms of $\vec{i}$ and $\vec{j}$ components, show that if $\vec{F}'(t) = \vec{f}(t)$ then the set of all functions whose derivative with respect to t is f, is given by $\{\vec{F}(t) + \vec{c}:\vec{c}$ is an arbitrary vector constant$\}$.

b. A particle moves in the xy-plane in such a way that its acceleration at any time t is given by

$$\vec{a} = (8\cos 2t)\vec{i} + (8\sin 2t)\vec{j}.$$

Moreover, at t = 0, both $\vec{R}$ and $\vec{v}$ are $\vec{0}$. Find $\vec{R}$ as a function of t.

c. Where is the particle at $t = \frac{\pi}{2}$?