Unit 4: (Optional) The Directional Derivative in n-Dimensional Vector Spaces

1. Preface:

As you may already have noticed, the concept of continuity and differentiability of real-valued functions of n real variables is quite difficult even for $n = 2$. For many students, it will be sufficiently difficult to absorb even the material of Units 2 and 3 which dealt almost exclusively (except for a few illustrations) with the case $n = 2$. Moreover, in many ordinary physical applications, the case $n = 2$ or even $n = 3$ usually suffices.

On the other hand, there are times when we are dealing with functions of more than three independent variables, in which case the gradient and the directional derivative are still important even though they do not lend themselves to the simple interpretation that we obtain geometrically for $n = 2$ or $n = 3$. Thus, it is possible that the student who feels he has mastered the material of the preceeding units might like to see a more general treatment of derivatives in n-dimensional space.

Moreover, while the approach in the text was good and allowed us to utilize our intuition in terms of the geometric approach, the fact is that we have deviated from the spirit in which we were playing the game of mathematics. In particular, it seems that in terms of our game, we are almost bound to try to define $f'(\underline{a})$ by

$$f'(\underline{a}) = \lim_{\Delta\underline{x}\to\underline{0}}\left[\frac{f(\underline{a}+\Delta\underline{x})-f(\underline{a})}{\Delta\,\underline{x}}\right]$$

Our aim of this unit is to show what would have happened had we tried such an approach and how the results we would have obtained are related to those obtained in the text.

While it is true that one can proceed with a standard elementary treatment of functions of several variables without recourse to this unit, it is our feeling that the reader can increase his depth and his grasp of the subject by trying to master this unit. Thus, it is our recommendation that unless you have definite reasons for not doing so, you try to work through this unit. If it is too difficult for you or if you later prefer to abandon it in order to continue on with the mainstream of the course, feel free to do so. No problems on the quiz will be drawn from this unit, of course.

2. Read, Supplementary Notes, Chapter 5.

3. Exercises:

3.4.1

a. Describe the linear algebraic equation for which $\frac{15}{(4,3,2,1)}$ is the solution set.

b. Describe the vector, $\underline{\frac{15}{(4,3,2,1)}}$.

c. Generalize parts (a) and (b) to cover the more general case $\frac{c}{(a_1,a_2,a_3,a_4)}$.

3.4.2

Show that if $\underline{x}$ and $\underline{y}$ are both members of $\frac{c}{\underline{v}}$ that $\underline{x}+\underline{y}$ need not be. Under what condition(s) will $\underline{x}+\underline{y}$ also belong to $\frac{c}{\underline{v}}$?

3.4.3

a. In the case that (1) $\underline{a}=(a_1,a_2)$ or (2) $\underline{a}=(a_1,a_2,a_3)$ show what $\frac{c}{\underline{a}}$ means from a geometrical point of view.

b. Interpret Exercise 3.4.2 in terms of part (a)(1).

3.4.4

In the case $\underline{a}=(a_1,a_2)$, explain geometrically how the set $\frac{c}{\underline{a}}$ is related to the vector $\underline{\frac{c}{\underline{a}}}$.

3.4.5

For any vector space of dimension n, define a function f by $f(\underline{x}) = \|\underline{x}\|^2$.

a. Compute $f'_{\underline{u}}(\underline{a})$.

b. With n = 5 and f as above, $\underline{u}$ the unit vector in the direction of (1,3,4,-1,2) and $\underline{a}$ the point (n-tuple) (1,2,5,3,1), compute $f'_{\underline{u}}(\underline{a})$.

c. With f and $\underline{a}$ as in (b), compute $f'_{\underline{u}}(\underline{a})$ if $\underline{u}$ is the unit vector in the direction (2,1,2,1,2).

d. Check the result of part (a) by translating the problem into n-tuple notation and then finding the directional derivative as described in the text.

3.4.6

Using the result of Exercise 3.4.5, part (a), deduce that $\|f'_{\underline{u}}(\underline{a})\| \leqslant 2\|\underline{a}\|$ and that equality holds when $\underline{u}$ is in the direction of $\underline{a}$. From this, deduce the value of $f'(\underline{a})$.

3.4.7

Another definition of differentiability in n-dimensional space is the following: We say that f is differentiable at $\underline{a}$ if there exists a neighborhood $N(\underline{a})$ of $\underline{a}$ such that for each point $\underline{a}+\underline{h}$ in $N(\underline{a})$, there is a constant $\underline{C}$, which depends on f and $\underline{a}$ but not on $\underline{h}$, such that

$$f(\underline{a}+\underline{h})-f(\underline{a}) = \underline{C}\cdot\underline{h}+k\|\underline{h}\|, \text{ where } \lim_{\|h\|\to 0} k = 0.$$

Use this definition of the derivative to show that if f is differentiable at $\underline{a}$ then C must be $f'(\underline{a})$.