Stochastic Calculus

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15.450 Recitation 3

- Defining properties of Brownian motion Z_t are

1. $Z_0 = 0$.
2. It has continuous paths.
3. It has independent increments: if $t_1 < t_2 \leq s_1 < s_2$, then $Z_{t_2} - Z_{t_1}$ and $Z_{s_2} - Z_{s_1}$ are independent random variables.
4. Its increments are normally distributed: $Z_{t_2 - t_1} \sim N(0, t_2 - t_1)$.

- Brownian motion is the continuous time analogue of random walk in discrete time. Brownian motion is the basic building block that we use to model uncertatinty or random movements in our continous time finance models.

Ito Processes

- An Ito process X_t is of the form

$$X_t = \int_0^t \mu_s ds + \int_0^t \sigma_s dZ_s$$

  or in differential form

$$dX_t = \mu_t dt + \sigma_t dZ_t$$

- Heuristically speaking, over a very small time interval Δ, the increment $X_{t+\Delta} - X_t$ has normal distribution with mean $\mu_t \Delta$ and variance $\sigma_t^2 \Delta$. Call μ_t the instantaneous drift and σ_t the instantaneous volatility or diffusion coefficient.

Ito's Lemma

- Suppose X_t is an Ito process. Ito's Lemma states that $f(t, X_t)$ is an Ito process as well and shows how to compute the drift and diffusion coefficient of $df(t, X_t)$.
- Ito's Lemma: Suppose

$$dX_t = \mu_t dt + \sigma_t dZ_t$$

then

$$d(f(t, X_t)) = \left(\frac{\partial f(t, X_t)}{\partial t} + \frac{\partial f(t, X_t)}{\partial X_t} \mu_t + \frac{1}{2} \frac{\partial^2 f(t, X_t)}{\partial X_t^2} \sigma_t^2 \right) dt + \frac{\partial f(t, X_t)}{\partial X_t} \sigma_t dZ_t$$

- Remember the rule

$$(dt)^2 = o(dt)$$
$$dt \cdot dZ_t = o(dt)$$
$$(dZ_t)^2 = dt$$

In other words, the first two are of smaller order than dt, and we can ignore them in Ito's lemma.

Example of Ito's Lemma

- Suppose

$$\frac{dS_t}{S_t} = rdt + \sigma dZ_t$$

and define $Y_t = f(t, S_t) = e^{-rt} S_t$. Calculate dY_t.

- Use's Ito's lemma:

$$\begin{aligned} dY_t &= \left(\frac{\partial f(t,S_t)}{\partial t} + \frac{\partial f(t,S_t)}{\partial S_t} rS_t + \frac{1}{2} \frac{\partial^2 f(t,S_t)}{\partial S_t^2} \sigma^2 S_t^2 \right) dt + \frac{\partial f(t,S_t)}{\partial S_t} \sigma S_t dZ_t \\ &= \left(-re^{-rt} S_t + re^{-rt} S_t + 0 \right) dt + e^{-rt} \sigma S_t dZ_t \\ &= \sigma Y_t dZ_t \end{aligned}$$

- Why does Y_t have zero drift?

Another Example

- Suppose η_t is some stochastic process. Define

$$\xi_t = \exp\left(-\int_0^t \eta_s dZs - \int_0^t \frac{\eta_s^2}{2} ds\right)$$

Calculate $d\xi_t$.

- Define $X_t = \int_0^t \eta_s dZs + \int_0^t \frac{\eta_s^2}{2} ds$. Note that

$$dX_t = \frac{\eta_t^2}{2} dt + \eta_t dZ_t$$

and $\xi_t = \exp(-X_t)$.

- Apply Ito's Lemma and we get

$$\begin{aligned} d\xi_t &= -\xi_t dX_t + \xi_t (dX_t)^2 \\ &= -\xi_t \eta_t dZ_t \end{aligned}$$

- Practice and make sure you can do those calculations!

Example of Multivariate Ito's Lemma

- Suppose

$$\frac{dX_t}{X_t} = \mu_x dt + \sigma_x dZ_t^1$$
$$\frac{dY_t}{Y_t} = \mu_y dt + \sigma_y dZ_t^2$$

where

$$dZ_t^1 \cdot dZ_t^2 = \rho\, dt$$

- Define

$$M_t = \frac{X_t}{Y_t}$$

What is the instantaneous volatility of $\frac{dM_t}{M_t}$?

Continued

- Let $M_t = f(X_t, Y_t) = \frac{X_t}{Y_t}$. Then by Ito's Lemma,

$$dM_t = \frac{\partial f(X_t, Y_t)}{\partial X_t} dX_t + \frac{\partial f(X_t, Y_t)}{\partial Y_t} dY_t$$
$$+ \frac{1}{2}\left(\frac{\partial^2 f(X_t, Y_t)}{\partial X_t^2}(dX_t)^2 + 2\frac{\partial^2 f(X_t, Y_t)}{\partial X_t \partial Y_t} dX_t \cdot dY_t + \frac{\partial^2 f(X_t, Y_t)}{\partial Y_t^2}(dY_t)^2\right)$$

- Carry out the calculation and arrive at

$$dM_t = M_t\left(\mu_x - \mu_y - \rho\sigma_x\sigma_y + \sigma_y^2\right)dt + M_t\sigma_x dZ_t^1 - M_t\sigma_y dZ_t^2$$
$$\frac{dM_t}{M_t} = \left(\mu_x - \mu_y - \rho\sigma_x\sigma_y + \sigma_y^2\right)dt + \sigma_x dZ_t^1 - \sigma_y dZ_t^2$$

- Note that

$$\left(\frac{dM_t}{M_t}\right)^2 = \left(\sigma_x dZ_t^1 - \sigma_y dZ_t^2\right)^2$$
$$= \left(\sigma_x^2 + \sigma_y^2 - \rho\sigma_x\sigma_y\right)dt$$

so that instantaneous volatility of $\frac{dM_t}{M_t}$ is

$$\sqrt{\sigma_x^2 + \sigma_y^2 - \rho\sigma_x\sigma_y}$$

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