Dynamic Portfolio Choice I

Static Approach to Dynamic Portfolio Choice

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Outline

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Constant Portfolio Weights

- Consider a market with N assets.
- Gross asset returns, R_t^n, $n = 1, 2, ..., N$ are IID over time.
- Consider self-financing portfolio rules with constant weights

$$\text{Portfolio Rule: } \omega = (\omega_1, \omega_2, ..., \omega_N)$$

- Focus on constant weights is natural in an IID environment.
- Under rule ω, portfolio value W_t changes as

$$W_t = W_{t-1} \sum_{n=1}^{N} \omega_n R_t^n$$

- Single out a particular rule $\omega^\star$, such that

$$\omega^\star = \arg\max_{\omega} \mathrm{E}\left[\ln\left(\sum_{n=1}^{N} \omega_n R_t^n\right)\right]$$

Constant Portfolio Weights

- Compare the long-run performance of the portfolio following the rule $\omega^\star$ to the one following any other constant rule $\widetilde{\omega}$.
- Denote the corresponding portfolio values by $W^\star$ and $\widetilde{W}$. Assume both start at 1 at $t = 0$.

$$\ln \frac{W_T^\star}{\widetilde{W}_T} = \ln \frac{\prod_{t=1}^T \sum_{n=1}^N \omega_n^\star R_t^n}{\prod_{t=1}^T \sum_{n=1}^N \widetilde{\omega}_n R_t^n} = \sum_{t=1}^T \ln\left(\sum_{n=1}^N \omega_n^\star R_t^n\right) - \sum_{t=1}^T \ln\left(\sum_{n=1}^N \widetilde{\omega}_n R_t^n\right)$$

- By LLN,

$$\frac{1}{T}\sum_{t=1}^T \ln\left(\sum_{n=1}^N \omega_n R_t^n\right) \to \mathrm{E}\left[\ln\left(\sum_{n=1}^N \omega_n R_t^n\right)\right] \equiv \mathcal{G}(\omega)$$

- By definition of $\omega^\star$,

$$\mathcal{G}(\omega^\star) - \mathcal{G}(\widetilde{\omega}) > 0$$

Constant Portfolio Weights

- We have established that

$$\frac{1}{T} \ln \frac{W_T^\star}{\widetilde{W}_T} \to \mathcal{G}(\omega^\star) - \mathcal{G} > 0$$

- Conclusion: in the long run, the portfolio rule $\omega^\star$ produces higher portfolio value than any other constant weight rule with probability one!
- Is the portfolio rule maximizing the expected log of return the best choice for any long-horizon investor? Is there role for individual preferences?

20-20 Hindsight

- This discussion is based on the universal portfolio of Thomas Cover.
- Define the geometric rate of return on a portfolio between 0 and T as

$$\frac{1}{T} \ln \frac{W_T}{W_0}$$

- Consider a market with N stocks paying no dividends.
- Suppose that, after observing returns on N stocks between 0 and T, we select the stock with the highest geometric rate of return.
- Is it possible for a portfolio to match performance of the best-performing stock? Yes, in the long run.
- Consider an equal-weighted, buy-and-hold portfolio. It has the same asymptotic geometric rate of return (as $T \to \infty$) as the best performing (*ex post*) stock!
- Universal portfolio is an equal-weighted allocation to all possible portfolios with positive constant weights. It can beat the best performing stock and matches the best (*ex post*) constant-weight rule asymptotically.
- Should we follow the equal-weight, buy-and-hold rule, or the universal portfolio?

Where is the Catch

- It is hard to have a meaningful discussion of which portfolio rules are preferable without an explicitly specified objective.
- Both of the portfolio rules described above may have nice asymptotic properties, but at any *finite* time point T they may produce return distributions with too much risk.
- Consistent decision making under uncertainty can be based on the concept of expected utility.
- Expected utility is not a dogma: it is based on behavioral assumptions.
- Expected utility assumes rational, consistent choices.
- Empirically, people often violate expected utility axioms.
- No surprise there, people often behave irrationally.

The Framework

- Define preference over random payoffs (gambles, lotteries), e.g.,

$$[\$1000_{\underbrace{(0.5)}_{prob.}}, \$0_{\underbrace{(0.5)}_{prob.}}] \quad \text{vs.} \quad [\$2000_{\underbrace{(0.3)}_{prob.}}, \$200_{\underbrace{(0.7)}_{prob.}}]$$

- Preferences are over outcomes only, e.g., do not depend on the mechanism by which cash flows are generated. Can apply to portfolio choice.
- $\widetilde{x} \succ \widetilde{y}$ (prefer $\widetilde{x}$ to $\widetilde{y}$), $\widetilde{x} \sim \widetilde{y}$ (indifferent between $\widetilde{x}$ and $\widetilde{y}$)
- Expected Utility Theory is a mathematical representation of preferences

$$\widetilde{x} \succ \widetilde{y} \quad \Leftrightarrow \quad \mathrm{E}[U(\widetilde{x})] > \mathrm{E}[U(\widetilde{y})]$$

- When we evaluate a random payoff $\widetilde{x}$, we care only about the numerical distribution of cash flows: $\mathrm{E}[U(\widetilde{x})]$.
- Properties of preferences are captured by the shape of the utility function $U(x)$.

Outline

1. Expected Utility
2. Risk Aversion
3. Derivatives and Portfolio Choice

Risk Aversion

- We commonly assume that people prefer more to less:

$$\widetilde{x} + \varepsilon \succeq \widetilde{x} \quad \text{for all } \widetilde{x},\ \varepsilon \geqslant 0$$

This means that the utility $U(x)$ is non-decreasing:

$$U'(x) \geqslant 0$$

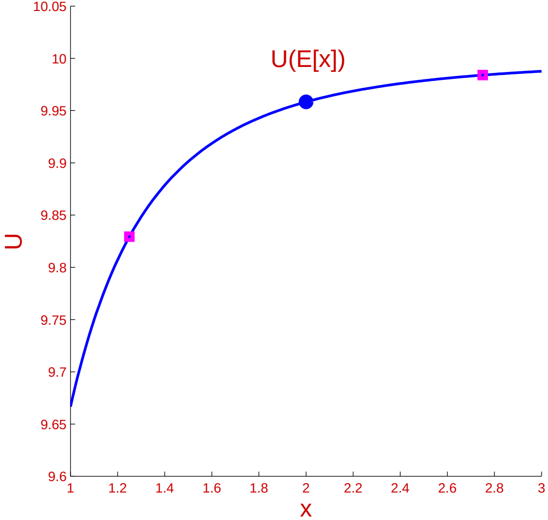
- We also assume aversion to risk: prefer $\overline{x} = \mathrm{E}[\widetilde{x}]$ to $\widetilde{x}$, i.e.,

$$U(\overline{x}) \geqslant \mathrm{E}[U(\widetilde{x})]$$

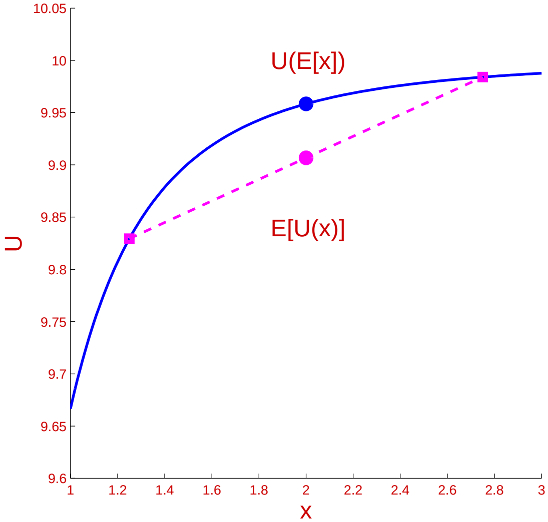
This means that $U(x)$ is concave

$$U''(x) \leqslant 0$$

Risk Aversion



Risk Aversion



Coefficient of Relative Risk Aversion $\gamma(W)$

- Start with initial wealth W. Compare two gambles, with payoffs

$$W(1+x) \quad \text{and} \quad W(1+x_{CE}), \qquad x_{CE} \text{ is a constant}$$

- What value of x_{CE} makes the agent indifferent?
- Small gamble x. Use Taylor expansion around $W(1+\overline{x})$:

$$U(W(1+x)) \approx U(W(1+\overline{x})) + U'(W(1+\overline{x}))W(x-\overline{x}) + \frac{1}{2}U''(W(1+\overline{x}))W^2(x-\overline{x})^2 + \cdots$$

$$U(W(1+x_{CE})) \approx U(W(1+\overline{x})) + U'(W(1+\overline{x}))W(x_{CE}-\overline{x})$$

- Indifference implies

$$\mathrm{E}\left[U(W(1+x))\right] = U(W(1+x_{CE}))$$

$$U'(W(1+\overline{x}))W\overline{x} + \frac{1}{2}U''(W(1+\overline{x}))W^2\mathrm{Var}(x) \approx U'(W(1+\overline{x}))Wx_{CE}$$

Coefficient of Relative Risk Aversion $\gamma(W)$

Certainty-Equivalent Return

$$x_{CE} \approx \overline{x} - \frac{1}{2}\gamma(W(1+\overline{x}))\,\mathrm{Var}(x), \quad \text{where} \;\; \gamma(W) = -\frac{U''(W)W}{U'(W)}$$

Examples of Utility Functions

- Linear utility
$$U(W) = a + bW, \quad b > 0$$
implies that $\gamma(W) = 0$.
- Payoffs are compared by their expected value, linear utility implies risk neutrality.
- Exponential utility
$$U(W) = -\exp(-aW), \quad a > 0$$
- Assume $W \sim N(\mu, \sigma^2)$. Then
$$\mathrm{E}\left[U(W)\right] = -\exp\left(-a\mu + \frac{a^2\sigma^2}{2}\right)$$
- Payoffs are compared by
$$\mu - a\frac{\sigma^2}{2}$$
- Increasing relative risk aversion
$$\gamma(W) = aW$$

Examples of Utility Functions

- Constant relative risk aversion (CRRA) utility exhibits

$$\gamma(W) = \gamma$$

- Using the definition $\gamma(W) = -U''(W)W/U'(W)$, recover the utility function

$$U(W) = \begin{cases} \frac{1}{1-\gamma} W^{1-\gamma}, & \gamma \neq 1 \\ \ln W, & \gamma = 1 \end{cases}$$

- CRRA utility is a very popular choice because of its implications for portfolio strategies.

Outline

Binomial Setting

- Consider a market with constant interest rate r and a stock paying no dividends, with price following a binomial tree:

$$S_t = S_{t-1} \times \begin{cases} u, & \text{with probability } p \\ d, & \text{with probability } 1-p \end{cases}$$

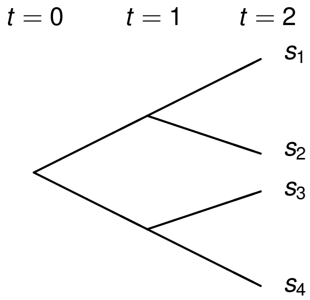
- Recall that the state-price density in this market is given by

$$\frac{\pi_{t+1}(u)}{\pi_t} = \frac{1}{p(1+r)} \frac{1+r-d}{u-d}$$
$$\frac{\pi_{t+1}(d)}{\pi_t} = \frac{1}{(1-p)(1+r)} \frac{u-(1+r)}{u-d}$$

- We want to find the portfolio maximizing expected utility

$$\mathrm{E}_0\left[U(W_T)\right]$$

Main Idea



- Imagine that any possible state-contingent claim with payoff at $T = 2$ can be traded. Then portfolio choice is a simple static optimization problem: choose the claim $W_T^\star(s)$ producing the highest expected utility subject to the budget constraint.
- Any state-contingent claim can be replicated by trading dynamically in the stock and the bond, therefore our optimal choice for $W_T^\star(s)$ can be generated by dynamic trading.

CRRA Utility

Formulation

- Suppose our objective is to maximize

$$\mathrm{E}_0\left[\frac{1}{1-\gamma}W_T^{1-\gamma}\right]$$

starting with W_0.

- We look for the best state-contingent wealth allocation $W_T(s)$ that costs W_0 at $t=0$.
- Recall that the time-0 value of any cash flow can be computed using the SPD as

$$W_0 = \sum_s \mathrm{Prob}_0(s)\pi_T(s)W_T(s)$$

- Solve the static problem

$$\max \sum_s \mathrm{Prob}_0(s)\frac{1}{1-\gamma}W_T(s)^{1-\gamma} \quad \text{s.t.} \quad \sum_s \mathrm{Prob}_0(s)\pi_T(s)W_T(s) = W_0$$

CRRA Utility

Solution of the Static Problem

- Solve the static problem

$$\max_{\{W_T(s)\}} \sum_s \mathrm{Prob}_0(s) \frac{1}{1-\gamma} W_T(s)^{1-\gamma} \quad \text{s.t.} \quad \sum_s \mathrm{Prob}_0(s) \pi_T(s) W_T(s) = W_0$$

- Relax the constraint with a Lagrange multiplier λ

$$\max_{\{W_T(s)\}} \sum_s \mathrm{Prob}_0(s) \frac{1}{1-\gamma} W_T(s)^{1-\gamma} - \lambda \left(\sum_s \mathrm{Prob}_0(s) \pi_T(s) W_T(s) - W_0 \right)$$

- First-order optimality conditions

$$W_T^\star(s)^{-\gamma} = \lambda \pi_T(s) \;\Rightarrow\; W_T^\star(s) = (\lambda \pi_T(s))^{-1/\gamma}$$

- Find the multiplier from

$$\sum_s \mathrm{Prob}_0(s) \, (\lambda \pi_T(s))^{-1/\gamma} \, \pi_T(s) = W_0$$

CRRA Utility

Solution of the Static Problem

- We conclude that the optimal choice of time-T state-contingent cash flow is

$$W_T^\star(s) = \frac{W_0}{\sum_s \text{Prob}_0(s)\pi_T(s)^{1-1/\gamma}}\pi_T(s)^{-1/\gamma}$$

- The recombining binomial tree model has a special property that the SPD is a function of the terminal stock price.
- If the terminal stock price equals

$$S_T = S_0 u^{(\text{\# Up moves})} d^{(T - \text{\# Up moves})}$$

then the SPD in the same state equals

$$\pi_T = \pi_1(u)^{(\text{\# Up moves})}\pi_1(d)^{(T - \text{\# Up moves})}$$

$$\pi_1(u) = \frac{1}{p(1+r)}\frac{1+r-d}{u-d}, \quad \pi_1(d) = \frac{1}{(1-p)(1+r)}\frac{u-(1+r)}{u-d}$$

CRRA Utility

Dynamic Trading Strategy

- We were able to express the optimal state-contingent portfolio value at the terminal date, $W_T^\star(s)$, as a function of the terminal stock price, denoted as

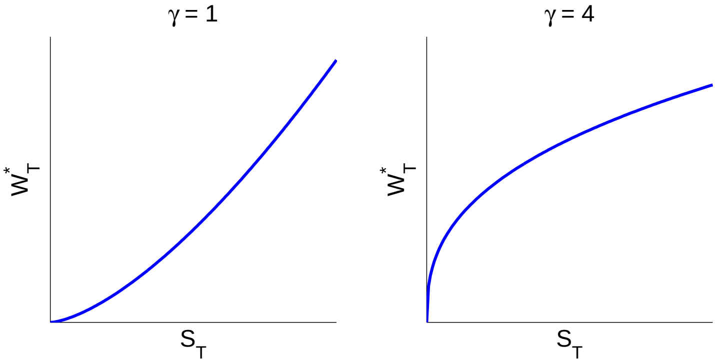
$$W_T^\star(s) = H(S_T(s))$$

- The optimal portfolio must replicate the European derivative security with terminal payoff $H(S_T)$.
- We know how to construct the optimal trading strategy in the stock and the bond: it is the replicating strategy for the above derivative. Can compute it by backward induction.

CRRA Utility

State-Contingent Allocation

- Qualitatively, want to achieve higher wealth in states with lower SPD (higher stock price).
- Illustrate by plotting the analytical solution from B-S framework (below).



Discussion

- In the binomial setting, finding an optimal dynamic portfolio strategy reduces to figuring out which derivative security we would like to buy with initial wealth W_0.
- The static problem is easy to solve using a Lagrange multiplier.
- Since any derivative can be replicated by dynamic trading in the stock and the bond, we know how to construct the optimal dynamic strategy for any utility function (e.g., proceeding backwards on the tree).

Black-Scholes Framework

- Black-Scholes framework is a continuous-time limit of a recombining binomial tree, and dynamic portfolio choice is equally simple.
- The advantage of continuous time is that we can obtain transparent closed-form solution.
- We use the B-S framework to recover the famous Merton's solution to the dynamic portfolio choice problem.
- Assume interest rate r and the stock price process

$$\frac{dS_t}{S_t} = \mu\, dt + \sigma\, dZ_t$$

- We are performing calculations under the physical probability measure, so $Z_t = Z_t^{\mathbf{P}}$.

CRRA Utility

Static Problem

- In analogy with the binomial-tree setting, we replace the dynamic problem with a static problem

$$\max_{\{W_T\}} \mathrm{E}_0 \left[\frac{1}{1-\gamma} W_T^{1-\gamma} \right] \quad \text{subject to} \quad \mathrm{E}_0 \left[\pi_T W_T \right] = W_0$$

- Keep in mind that W_T and π_T are both functions of the state, which is the entire trajectory of the Brownian motion between 0 and T.
- Relax the constraint using a Lagrange multiplier λ:

$$\max_{\{W_T\}} \mathrm{E}_0 \left[\frac{1}{1-\gamma} W_T^{1-\gamma} - \lambda \left(\pi_T W_T - W_0 \right) \right]$$

- First-order optimality conditions

$$\left(W_T^\star \right)^{-\gamma} = \lambda \pi_T \;\Rightarrow\; W_T^\star = \left(\lambda \pi_T \right)^{-1/\gamma}$$

CRRA Utility

Static Problem

- We find the multiplier from the static budget constraint:

$$\mathrm{E}_0\left[\pi_T W_T^\star\right] = W_0 \;\Rightarrow\; \mathrm{E}_0\left[\pi_T \left(\lambda \pi_T\right)^{-1/\gamma}\right] = W_0$$

- Find

$$\lambda^{-1/\gamma} = \frac{W_0}{\mathrm{E}_0\left[\pi_T^{1-1/\gamma}\right]}$$

- We need to derive the dynamic strategy replicating the optimal state-contingent claim $W_T^\star$.
- We first derive the process for the optimal portfolio value, $W_t^\star$, and then figure out how to delta-hedge it using the stock.

CRRA Utility

Dynamic Strategy

- If the optimal portfolio value at T is given by $W_t^\star$, by DCF formula, portfolio value at earlier times must be equal to

$$W_t^\star = \mathrm{E}_t \left[\frac{\pi_T}{\pi_t} W_T^\star \right], \quad W_T^\star = (\lambda \pi_T)^{-1/\gamma}$$

- Recall that in the Black-Scholes model, the price of risk is constant, $\eta = (\mu - r)/\sigma$, and the SPD is given by

$$\pi_t = e^{-rt} e^{-\left(\eta^2/2\right) t - \eta Z_t}$$

- We can compute $W_t^\star$:

$$W_t^\star = \mathrm{E}_t \left[\lambda^{-1/\gamma} \pi_T^{-1/\gamma} \frac{\pi_T}{\pi_t} \right] = \mathrm{E}_t \left[\lambda^{-1/\gamma} \left(\frac{\pi_T}{\pi_t} \right)^{1-1/\gamma} \right] \pi_t^{-1/\gamma}$$
$$= F(t) \pi_t^{-1/\gamma}$$

for some function of time $F(t)$, which we could compute explicitly.

Dynamic Strategy

- To figure out the dynamic trading strategy, we relate the SPD to the stock price, just like we did for binomial tree.
- The SPD is given by

$$\pi_t = e^{-rt} e^{-\left(\eta^2/2\right) t - \eta Z_t}$$

- The stock price equals

$$S_t = S_0 e^{(\mu - \sigma^2/2) t + \sigma Z_t}$$

- We conclude that

$$\pi_t = G(t) S_t^{-\eta/\sigma}$$

for some function $G(t)$. We can compute $G(t)$ explicitly but do not need to.

- The optimal portfolio follows

$$W_t^\star = F(t) G(t)^{-1/\gamma} S_t^{\eta/(\gamma\sigma)}$$

Dynamic Strategy

- Let $\theta_t^\star$ denote the optimal number of stock shares in the dynamic portfolio.
- Define $\phi_t^\star$ to be the weight of the stock in the optimal portfolio:

$$\phi_t^\star = \frac{\theta_t^\star S_t}{W_t^\star}$$

- Delta-hedging rule tells us how many stock shares to include in the replicating portfolio

$$\theta_t^\star = \frac{\partial W_t^\star}{\partial S_t}$$

- Using

$$W_t^\star = F(t)G(t)^{-1/\gamma} S_t^{\eta/(\gamma\sigma)}$$

we conclude that

$$\phi_t^\star = \frac{\eta}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2}$$

Dynamic Strategy

Merton's Solution

In the Black-Scholes setting with CRRA utility function, the weight of the stock in the optimal portfolio is

$$\phi_t^\star = \frac{\mu - r}{\gamma\sigma^2}$$

- Merton's solution says that the optimal portfolio weights are constant, independent of the problem horizon T.
- We call such solution myopic. It is optimal to behave as if the horizon of the problem is very short.
- The optimal weight of the stock is increasing in the risk premium, and decreasing in relative risk aversion and stock return volatility.
- For a general utility function, $U(W_T)$, we would not obtain the same solution, optimal portfolio strategy would not be myopic.
- CRRA utility is special: constant relative risk aversion. This, combined with the fact that returns on all assets in the B-S model are IID, leads to a myopic optimal portfolio.

Merton's Solution

- Merton's solution easily generalizes to a multi-variate case.
- If an investor has a CRRA utility function, interest rate is constant, r, and returns on N risky assets follow

$$\frac{dS_t^i}{S_t^i} = \mu_i \, dt + \sum_{j=1}^{N} \Sigma_{ij} \, dZ_t^j,$$

then the vector of optimal portfolio weights on the N stocks is given by

$$\phi_t^\star = \frac{1}{\gamma} \left(\Sigma\Sigma'\right)^{-1} (\mu - r\mathbf{1})$$

where $\mu = (\mu_1, ..., \mu_N)'$, $\mathbf{1} = (1, 1, ..., 1)'$, and

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} & \dots & \Sigma_{1N} \\ \Sigma_{21} & \Sigma_{22} & \dots & \Sigma_{2N} \\ & & \dots & \\ \Sigma_{N1} & \Sigma_{N2} & \dots & \Sigma_{NN} \end{bmatrix}$$

Summary

- Need a coherent objective to formulate an optimal dynamic trading strategy: Expected Utility Theory.
- In a setting in which all state-contingent claims can be replicated by dynamic trading (e.g., binomial tree, Black-Scholes model) can connect optimal portfolio choice to option pricing.
- In the Black-Scholes setting with CRRA preferences, the optimal portfolio strategy is myopic, given by the Merton's solution.

References

- T. Cover, 1991, “Universal Portfolios,” *Mathematical Finance* 1, 1-29.

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